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# PROCEEDINGS

OF

# THE LONDON MATHEMATICAL SOCIETY

## SECOND SERIES

## VOLUME 3

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1905

LONDON
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.

*Made and Printed by the Replika Process in Great Britain by*
*Percy Lund, Humphries & Co. Ltd.*
*3 Amen Corner, London, E.C.4*
*and at Bradford*

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# RECORDS OF PROCEEDINGS AT MEETINGS

## SESSION NOVEMBER, 1904—JUNE, 1905.

### *Thursday, November 10th*, 1904.

ANNUAL GENERAL MEETING.

Prof. H. LAMB, President, in the Chair.

Present nineteen members and two visitors.

Dr. G. H. Hallett was elected a member.

The President referred to the losses sustained by the Society by the deaths of the late Prof. Pirie and Mr. R. W. H. T. Hudson.

The Treasurer presented his Report. On the motion of the President, seconded by Prof. Hill, the Report was received.

Dr. Leathem was appointed Auditor.

Prof. Love, as Secretary. reported that the number of members at the beginning of the previous Session was 260. During the year the Society had lost 4 members by death and 1 by resignation. The number of new members elected was 14. The number of members at the beginning of the Session was 269.

In connexion with the election of a Council and Officers for the ensuing Session, the President stated that Prof. Burnside was retiring from the office of Honorary Secretary. He expressed the appreciation of the Society of the services which Prof. Burnside had rendered during his tenure of the office, and referred to the advantage which had accrued to the Society through having had Prof. Burnside as a Secretary.

The Council and Officers for the ensuing Session were elected. The list is as follows:—President, Prof. A. R. Forsyth; Vice-Presidents, Prof. W. Burnside, Prof. E. B. Elliott, Prof. H. Lamb; Treasurer, Prof. J. Larmor; Secretaries, Prof. A. E. H. Love and Mr. J. H. Grace; other members of the Council, Mr. A. Berry, Mr. J. E. Campbell, Dr. J. W. L. Glaisher, Dr. E. W. Hobson, Major P. A. MacMahon, Mr. G. B. Mathews, Mr. A. E. Western, Mr. E. T. Whittaker, Mr. A. Young.

Prof. Forsyth having taken the Chair, the retiring President, Prof.

Lamb, delivered an address on "Deep-Water Waves." The Address was ordered to be printed in the *Proceedings* of the Society.

The following papers were communicated:—

- *Note on the Application of the Method of Images to Problems of Vibrations: Prof. Volterra.
- *On the Zeroes of certain Classes of Integral Taylor's Series (two papers): Mr. G. H. Hardy.
- *The Linear Difference Equation of the First Order: Rev. E. W. Barnes.
- Remarks on Alternants and Continuous Groups: Dr. H. F. Baker.
- *Expansions of the Elliptic and Zeta Functions of $\frac{2}{3}K$ in powers of $q$: Dr. J. W. L. Glaisher.
- *Examples of Perpetuants: Mr. J. E. Wright.
- Two Simple Results in the Attraction of Uniform Wires obtained by Quaternions, with, for comparison, their Verification by the Geometry of the Complex: Prof. R. W. Genese.
- *On the Reducibility of Covariants of Binary Quantics of Infinite Order: Mr. P. W. Wood.
- A Theorem relating to Quotient Groups: Prof. G. A. Miller.
- †On certain Classes of Syzygies: Mr. A. Young.

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### *Thursday, December 8th,* 1904.

Prof. A. R. FORSYTH, President, in the Chair.

Present fifteen members and a visitor.

Messrs. H. W. Chapman, I. O. Griffiths, L. N. G. Filon were elected members.

The following papers were communicated:—

- *On a Deficient Multinomial Expansion: Major P. A. MacMahon.
- †The Application of Basic Numbers to Bessel's and Legendre's Functions (Second Paper): Rev. F. H. Jackson.
- *On Groups of Order $p^{\alpha} q^{\beta}$ (Second Paper): Prof. W. Burnside.
- †On the Failure of Convergence of Fourier's Series: Dr. E. W. Hobson.
- †An Extension of Borel's Exponential Method of Summation of Divergent Series applied to Linear Differential Equations: Mr. E. Cunningham.
- On the Linear Differential Equation of the Second Order: Prof. A. C. Dixon.

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* Printed in *Proceedings*, Ser. 2, Vol. 2. † Printed in this volume.

*Thursday, January 12th,* 1905.

Prof. A. R. FORSYTH, President, in the Chair.

Present seventeen members and a visitor.

Mr. O. Glauert was elected a member.

Messrs. J. H. Jeans and H. W. Chapman were admitted into the Society.

The President stated that the Report of the Auditor had been received. On the motion of the President, seconded by the Rev. F. H. Jackson, the Report of the Treasurer (presented in November) and the Report of the Auditor were adopted, and the thanks of the Society were given to the Treasurer and Auditor.

The following papers were communicated:—

Basic Generalisations of well known Analytic Functions: Rev. F. H. Jackson.

*Current Flow in Rectangular Conductors: Mr. H. Fletcher Moulton.

*On the Kinematics and Dynamics of a Granular Medium in Normal Piling: Mr. J. H. Jeans.

*Generational Relations for the Abstract Group simply isomorphic with the Group $LF[2, p^n]$: Dr. W. H. Bussey.

*On Alternants and Continuous Groups: Dr. H. F. Baker.

*A Generalization of the Legendre Polynomial: Mr. H. Bateman.

*A Class of Expansions in Oscillating Functions: Prof. A. C. Dixon.

Isogonal Transformation and the Diameter Transformation: Mr. H. L. Trachtenberg.

*Thursday, February 9th,* 1905.

Prof. A. R. FORSYTH, President, in the Chair.

Present sixteen members and a visitor.

Mr. E. Cunningham was elected a member.

Dr. L. N. G. Filon was admitted into the Society.

The President referred to the loss sustained by the Society by the death of Mr. R. Tucker, who was Honorary Secretary for thirty-five years, and moved a resolution of condolence with Mr. Tucker's surviving relatives. This was seconded by Dr. Glaisher and carried unanimously.

The following papers were communicated:—

*On the General Theory of Transfinite Numbers and Order Types: Dr. E. W. Hobson.

*On the Function $\sum_{n=1}^{\infty} \frac{x^n}{n^s}$: Mr. G. H. Hardy.

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* Printed in this volume.

*On the Reducibility of Covariants of Binary Quantics of Infinite Order. Part II.: Mr. P. W. Wood.

*The Maclaurin Sum-Formula and the Asymptotic Expansion of Integral Functions of Finite Non-zero Order: Rev. E. W. Barnes.

### *Thursday, March 9th*, 1905.

Prof. A. R. FORSYTH, President, and, temporarily, Dr. E. W. HOBSON, in the Chair.

Present eighteen members and a visitor.

Dr. W. H. Eccles and Mr. J. F. Cameron were elected members.

Dr. W. H. Eccles and Mr. H. Bateman were admitted into the Society.

A cabinet photograph of the late Mr. Tucker was placed in the Society's Album by Prof. Hill.

The following papers were communicated:—

*On the Projection of two Triangles on to the same Triangle: Prof. M. J. M. Hill, Dr. L. N. G. Filon, and Mr. H. W. Chapman.

*The Weddle Quartic Surface: Mr. H. Bateman.

*On the Complete Reduction of any Transitive Permutation Group; and on the Arithmetical Nature of the Coefficients in its Irreducible Components: Prof. W. Burnside.

*On the Theory of the Logarithmic Potential: Prof. T. J. I'A. Bromwich.

*Alternative Expressions for Perpetuant Types: Mr. P. W. Wood.

Prof. Forsyth made an informal communication:

"On the Theory of Geodesics."

### *Thursday, April 13th*, 1905.

Dr. E. W. HOBSON in the Chair.

Present nine members and a visitor.

The following papers were communicated:—

*On Irreducible Jacobians of Degree Six: Mr. P. W. Wood.

*Ordinary Inner Limiting Sets in the Plane or Higher Space: Dr. W. H. Young.

†Note on a case of $F(\alpha, \beta, \gamma, \delta, \epsilon, 1)$: Rev. F. H. Jackson.

Informal communications were made as follows:—

†Fermat's Numbers and the Converse of Fermat's Theorem: Mr. A. E. Western.

On the Strains that accompany Bending: Prof. A. E. H. Love.

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* Printed in this volume.

† Abstracted in this volume under the heading "Notes and Corrections."

### *Thursday, May 11th,* 1905.

Prof. A. R. FORSYTH, President, in the Chair.

Present eighteen members and two visitors.

Mr. W. M. Roberts was admitted into the Society.

The following papers were communicated:—

*On the Intersections of two Conic Sections: Mr. J. A. H. Johnston.

On a System of Conics yielding Operators which annihilate a Cubic, and its bearing on the Reduction of the Cubic to a Sum of Four Cubes: Mr. H. G. Dawson.

Informal communications were made as follows:—

†High Pellian Factorisations: Lt.-Col. A. Cunningham.

The Stability of a Loaded Column: Prof. A. E. H. Love.

### *Thursday, June 8th,* 1905.

Prof. A. R. FORSYTH, President, and, temporarily, Prof. W. BURNSIDE, Vice-President, in the Chair.

Present eleven members.

Mr. V. Ramaswami Aiyar was elected a member.

The President announced that the De Morgan Medal for 1905 had been awarded to Dr. H. F. Baker for his researches in Pure Mathematics.

The following papers were communicated:—

*On the Conditions of Reducibility of any Group of Linear Substitutions: Prof. W. Burnside.

*On Criteria for the Finiteness of the Order of a Group of Linear Substitutions: Prof. W. Burnside.

‡On a Class of many-valued Functions, defined by a Definite Integral: Mr. G. H. Hardy.

Informal communications were made as follows:—

The First Principles of Cauchy's Theory of Functions: Mr. G. H. Hardy.

Differential Equations whose Integrals are expressible by Partial Quadratures: Prof. A. R. Forsyth.

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* Printed in this volume.

† Abstracted in this volume under the heading "Notes and Corrections."

‡ Printed in this volume with the title "On a Class of Analytic Functions."

# LIBRARY

## *Presents.*

Between January and October, 1905, the following presents were made to the Library:—

From the Right Hon. Sir James Stirling:—

C. G. J. Jacobi.—"Gesammelte Werke," 7 vols., and Supplement; Berlin, 1881–1891.
H. Resal.—"Méchanique Générale," 7 vols.; Paris, 1873–1889.

From La Société Hollandaise des Sciences:—

"Œuvres Complètes de Christiaan Huygens," vol. 10; Harlem, 1905.

From the respective Authors or Publishers:—

Anthony, E.—"Decimal Coinage and the Metric System"; London, 1905.
Dini, U,—"Studii sulle Equazioni diff. lineari."
Kiepert, L.—"Grundriss der Diff.- u. Integral-Rechnung," Teil I., 10e Auflage, ed. Dr. M. Stegemann; Hannover, 1905.
Kiseljak, M.—"Zahlentheorie eines speziellen Systems von komplexen Grössen"; Bonn, 1905.
Lucas, A. Santos.—"Quelques Mots sur Mathématiques en Portugal."
Picard, E.—"Sur le Développement d'Analyse"; Paris, 1905.
Zeuthen, H. G.—"Théorème de Pythagore," and "Gebrauch u. Missbrauch historischer Benennungen."

Breslauer Naturforscher-Versammlung, Verhandlungen, 1905.
Hamburg: Math. Gesellschaft, Mittheilungen, bd. 4, heft 5, 1905.
Indian Engineering, vol. 36, nos. 26, 27, 1904; vol. 37, 1905; vol. 38, nos. 1–12, 1905.
La Plata, Buenos Aires: Demografia, años 1900, 1902.
London: Mathematical Gazette, vol. 3, nos. 50–52, 1905.
London: Educational Times, vol. 58, nos. 526–530, 1905.
London: Educational Times, Math. Questions and Solutions, vol. 7, 1905.
Nautical Almanac for 1908 (presented by the Admiralty).
Paris: L'Enseignement Math., ann. 7, nos. 1-5, 1905.
Saõ Paulo, Brazil: "Revista da Soc. Scientifica," no. 1, 1905.
South African Association for the Advancement of Science, Report, 1904.
Tokyo: Mathematico-Physical Soc., Sūgaku, vol. 2, nos. 13–20, 1905.
Warsaw: Wiadomości Matem., tom 8, zeszyt 4–6, 1904; tom 9, zeszyt 1, 2, 1905.

## *Exchanges.*

Between January and October, 1905, the following exchanges were received:—

American Journal of Mathematics, vol. 27, nos. 1, 2, 3, 1905.
American Mathematical Society, Transactions, vol. 6, nos. 1, 2, 3, 1905.

American Mathematical Society, Bulletin, vol. 11, nos. 4–10, 1905; vol. 12, no. 1, 1905.

American Philosophical Society, Proceedings, vol. 43, no. 178, 1904; vol. 44, nos. 179, 180, 1905.

Amsterdam: Nieuw Archiev, deel 6, stuk 4, 1905; deel 7, stuk 1, 1905.

Amsterdam: Revue Semestrielle, tome 13, pts. 1, 2, 1905.

Amsterdam: Wiskundige Opgaven, deel 9, stuk 3, 1905.

Belgique: Académie Royale des Sciences, Annuaire, 1905.

Belgique: Académie Royale des Sciences, Bulletin, 1904, no. 12; 1905, nos. 1–5.

Berlin: Jahrbuch über die Fortschritte, bd. 33, heft 3, 1905; bd. 34, heft 1, 1905.

Berlin: Journal für die Mathematik, bd. 128, hefte 3, 4, 1905; bd. 129, hefte 1, 2, 1905; bd. 130, hefte 1, 2, 1905.

Berlin: Sitzungsberichte der K. Preuss. Akademie, 1904, nos. 41–55; 1905, nos. 1–38.

Bordeaux: Société des Sciences, Mémoires, tome 2, cah. 2, 1904.

Bordeaux: Société des Sciences, Procès-Verbaux, 1904.

Cambridge Philosophical Society, Proceedings, vol. 13, pts. 1, 2, 1905.

Cambridge, Mass.: Annals of Mathematics, vol. 6, nos. 2–4, 1905.

Catania: Accademia Gioenia, Atti, Ser. 4, vol. 17, 1904.

Catania: Accademia Gioenia, Bollettino, fasc. 80–83, 1904; fasc. 84–86, 1905.

Coimbra: Jornal de Sciencias Mathematicas, vol. 15, no. 5, 1905.

Edinburgh: Mathematical Society, Proceedings, vol. 21, 1903.

Edinburgh: Royal Society, Proceedings, vol. 24, nos. 4–6, 1903; vol. 25, no. 1, 1904.

Edinburgh: Royal Society, Transactions, vol. 40, pt. 3, 1903.

Erlangen: Physik.-medicin. Societät, Sitzungsberichte, bd. 36, 1904.

France: Société Mathématique, Bulletin, tome 32, fasc. 4, 1904; tome 33, fasc. 1–3, 1905.

Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1904, heft 6; 1905, hefte 1–3.

Göttingen: Königl. Gesell. der Wissenschaften, Mittheilungen, 1904, heft 2; 1905, heft 1.

La Haye: Archives Néerlandaises, tome 10, liv. 1–4, 1905.

Leipzig: Beiblätter zu den Annalen der Physik, bd. 29, hefte 1–19, 1905.

Leipzig: K. Sächsische Gesell., Math. Klasse, Berichte, 1904, no. 5; 1905, nos. 1, 2.

Leipzig: K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 29, nos. 3, 4, 1905.

Livorno: Periodico di Matematica, anno 20, fasc. 4–6, 1905; anno 21, fasc. 1, 1905.

Livorno: Periodico di Matematica, Supplemento, anno 8, fasc. 3–9, 1905.

London: Royal Society, Proceedings, vol. 74, nos. 503–506, 1905; vol. 75, pt. 4, 1905; vol. 76, Series A, nos. 507–512, 1905; vol. 76, Series B, nos. 507–512, 1905.

London: Royal Society, Transactions, Series A, vol. 204, 1905.

London: Royal Society, Evolution Committee, Report 2, 1905.

London: Physical Society, Proceedings, vol. 19, pts. 5, 6, 1905.

London: Institute of Actuaries, Journal, vol. 37, pts. 5, 6, 1903; vol. 38, pts. 1–3, 1903–4.

London: Institution of Naval Architects, Transactions, vol. 45, 1903.

London: Nat. Physical Laboratory, Report, 1904; Collected Researches, vol. 1, 1905.

London: Nature, vol. 71, nos. 1837–1852, 1905; vol. 72, 1853–1863, 1905.

Manchester Literary and Philosophical Society, Memoirs, vol. 47, pts. 3–6, 1903; vol. 48, pt. 1, 1903.

Marseille: Annales de la Faculté des Sciences, tome 14, 1904.

Milano: Reale Istituto Lombardo, Memorie, vol. 20, fasc. 3, 4, 1905.

Milano: Reale Istituto Lombardo, Rendiconti, vol. 37, fasc. 17–20, 1904; vol. 38, fasc. 1–4, 1905.

Napoli: Accademia delle Scienze, Rendiconti, vol. 10, fasc. 8–12, 1904; vol. 11, fasc. 1–3, 1905; Indice, 1737–1903.

Odessa: Société des Naturalistes, tome 26, 1904; tome 27, 1905.

Palermo: Rendiconti del Circolo Matematico, tomo 19, 1905; tomo 20, fasc. 1, 1905.

Paris: Bulletin des Sciences Mathématiques, tome 29, Jan.–Août, 1905.

Roma: Reale Accademia dei Lincei, Rendiconti, vol. 13, sem. 2, fasc. 12, 1904; vol. 14, sem. 1, and sem. 2, fasc. 1–5, 1905.

Roma: Reale Accademia dei Lincei, Rendiconto dell'Adunanza Solenne, vol. 3, 1905.
Stockholm: Acta Mathematica, bd. 29, pts. 2, 3, 1905.
Toulouse: Faculté des Sciences, Annales, tome 6, fasc. 3, 4, 1904; tome 7, fasc. 1, 2, 1905.
Wien: Monatshefte für Mathematik, jahr. 16, 1905.
Zurich: Vierteljahrsschrift, 1905.

## *International Catalogue of Scientific Literature.*

In the year April, 1904, to March, 1905 (inclusive), the following exchanges were sent in the first instance to Prof. Love to be indexed for the International Catalogue of Scientific Literature:—

"Proceedings of the Edinburgh Mathematical Society," Vol. XXII., 1904, and Index of Vols. I.-XX.

"Proceedings of the Royal Society of Edinburgh," Vol. XXV., Nos. 2–6, 1904–5.

"Transactions of the Institution of Naval Architects," London, 1904.

"Journal of the Institute of Actuaries," Vol. XXXVIII., Pts. 4–6, and Vol. XXXIX., Pt. 1, 1904–5.

"Proceedings of the Manchester Literary and Philosophical Society," Vol. XLVIII., Pts. 2, 3, and Vol. XLIX., Pt. 1, 1904–5.

The following also were sent especially for the purposes of the Catalogue:—

"Mathematical Gazette," Nos. 45–50; London, 1904–5.

"Educational Times," Nos. 516–527; London, 1904–5.

"Journal of the Royal Statistical Society," Vol. XLVII., Pts. 1–4; London, 1904.

"Transactions of the Royal Society of Edinburgh," Vol. XLI., Pt. 1, 1904.

"Transactions of the Insurance and Actuarial Society of Glasgow," Series 5, Nos. 15, 16, 1904.

# OBITUARY NOTICE

## ROBERT TUCKER

[For this notice the Council is indebted to Prof. M. J. M. Hill.]

It was with a deep sense of responsibility that the writer of this notice undertook the task, committed to him by the Council of the London Mathematical Society. The Council desired that an adequate presentment of the life and work of one who counted for so much in the fortunes of the Society should be prepared for the *Proceedings*. Without the kindly co-operation of several members of the Society, and of others acquainted with the work of our late honoured Secretary, no approximation to the desired ideal would have been possible. Fortunately, the most ready help has been rendered by those to whom the writer has applied for imformation.

### MR. TUCKER'S LIFE.*

Robert Tucker was born at Walworth, in Surrey, on April 26th, 1832. His immediate forefathers were men of the Isle of Wight who had probably migrated from Devon or Dorset; and his near kinsmen were, for the most part, of military profession, and therein did good service for the King. His father, Robert Tucker (who died in his son's early infancy), was in the Commissariat branch of the Army during the Peninsular War. His mother, Fanny Tucker (husband and wife being second cousins), was daughter of Richard Tucker, who was for many years in His Majesty's Customs. Uncles on both sides followed careers of equal loyalty. His father's brother, Richard Tucker, served also in the Commissariat in the Peninsular War, and afterwards was manciple at the Charterhouse.

Robert Tucker's early training commenced at a school in Newport. Later on he was transferred to the "Woodard" School at New Shoreham, of which the temporary head was his cousin, Henry Jacobs, Michel Fellow of Queen's College, Oxford, and afterwards Dean of Christchurch, New Zealand.

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* This account is taken in substance from an article in the *University College Magazine*, by one of Mr. Tucker's former pupils, Mr. Geoffrey T. Bennett, who has kindly placed the article at the writer's disposal.

It was with this cousin's assistance and advice that he became a candidate for a sizarship at St. John's College, Cambridge. The first attempt was unsuccessful; but a second, in the following year, was rewarded, and the Johnian freshman entered upon his University career in the Michaelmas term of 1851.

St. John's in those days was, doubtless, not greatly different from the St. John's of to-day; yet, at least, some present features must be counted new, and some few old ones have disappeared.

The *Eagle*, the faithful chronicle of all good Johnians since 1858, was not then fledged. In the matter of college examinations, the *viva voce* and the "Seven Devils" had yet to disappear. The latter, thus profanely nick-named, was nothing worse than an examination paper in algebra, consisting of seven problem-puzzles of the most trying description, in which the data mostly translated themselves into simultaneous equations in an appalling number of unknowns. The mathematical "vivas" must have been truly a terror. Imagine being asked to expand tan $x$ in series to the first few terms as a mental exercise! Such is one of the recollections of the subject of this notice, nor did his success stop at the first term of the expansion. How many of us would get beyond the cheerful $x$ with which we should promptly start?

The reward of steady work did not fail: three years later he was promoted to a Foundation Scholarship, and in the Tripos list of 1855 he came out thirty-fifth Wrangler. He next turned his attention to Hebrew, gaining a prize for his effort, and afterwards to Moral Science, in which he was beaten only by his friend Leonard Courtney, second Wrangler of the same year, and afterwards Member of Parliament for Bodmin. At this time he had the intention of undertaking the Voluntary Theological Tripos, which was soon afterwards abolished. His plans, however, were changed, and he took his first mastership under the Rev. J. R. Pears, of Windlesham, Bagshot, who was the Head Master of a preparatory school for Haileybury. After eighteen months spent here, there followed a short period of mastership at a private school (Mr. John Ogle's) in St. John's Wood. In February, 1859, he took his M.A., and went as a master to the school of the Rev. J. A. Wall at Portarlington. Here he taught, with success, for five years. Among the pupils he recalled with pride the name of W. M. J. Morgan, who afterwards took a brilliant degree at Trinity College, Dublin, and became Head Master of the Royal School, Armagh. In 1864 he returned to the Isle of Wight for a year's rest. After this he took a mastership at Brighton College, where he had part charge of the Civil and Military Department. In September, 1865, University College School had need of a successor to the late

George Campbell De Morgan as Mathematical Master. The Head Master of the school, Professor Key, influenced chiefly by the warm support of Mr. Tucker's candidature by Isaac Todhunter, appointed him to the post which he was to fill so well until July, 1899. He is remembered with respect and affection by all those pupils who had the good fortune to come within the sphere of his influence. As to his success as a teacher, let the Tripos lists of the University of Cambridge bear evidence.

In the year 1865 there was founded the London Mathematical Society, and in October Mr. Tucker was elected a member. This was followed, in November, 1867, by his appointment to the Honorary Secretaryship of the Society, which office he held until 1902, and to his work therein a special section of this notice is devoted below.

In April, 1866, he married Elizabeth Byles, the only daughter of William Byles, of Freshwater, by whom he had three daughters.

In 1871 the Association for the Improvement of Geometrical Teaching was formed. The impulse to which the origin of this Society was due was given in a correspondence in *Nature*, in which Mr. Tucker took part; and its success was in no small measure due to his influence and geometrical enthusiasm. This Association he first served as local Secretary for London, and subsequently as Honorary Secretary and Vice-President.

In the year 1896 he was elected an Honorary Fellow of the College of Preceptors.

To his duties as master were added, for a long period of years, those of examiner at South Kensington, the College of Preceptors, and the Scotch Education Department. It is difficult to record this without a groan. How much of the energy of the teaching profession is expended upon a system which an unenlightened public opinion insists upon demanding! Will the members of that profession never earn a living wage for their professional services? Mr. Tucker, at any rate, did not. Though he resigned his mastership after a period of service and at an age which entitled him to rest, it was necessary for those who knew the obligations of mathematical science to his unselfish devotion to make an application to the Crown for a Civil List pension. In this application they were happily successful—a pension of £40 a year being obtained in the year 1900. Though small in amount, it must have relieved to some extent the anxiety which weighed upon him.

The members of the Society will learn with sorrow that, when he left London for Worthing in 1902, he was obliged to sell his fine mathematical library, and that the necessity of parting with it affected him deeply. He died at Worthing on January 29th, 1905, after a long period of illness.

### AS SECRETARY.

The first meeting of the Society which Mr. Tucker attended was held on the 20th November, 1865, when he made the acquaintance of the then President, Augustus de Morgan, of Sylvester, and of others who have made valuable contributions to mathematics. On the death of the younger De Morgan in 1867 he succeeded him as Secretary to the Society. He told the Society that from this latter date, until he vacated office in 1902, he was absent from only three meetings of the Society. He edited the *Proceedings* from No. 12 to No. 766. Comparing this editorial work as it was when he began with what it had become when he laid down his office, we may note that Vols. I. and II. contain 110 and 256 pages respectively, and extend over four years (1865–9), whilst Vol. XXIX. (1897–8) contains 741 pages and extends over one Session only. Only one meeting of the Society failed through the absence of a *quorum* during the whole period of his tenure of office.

It is impossible to estimate the value and the extent of his services to the Society, especially during its early years. No one was so anxious as he to obtain new members for the Society, and for it no one worked with more unselfish devotion. The large amount of time which he devoted to the Minutes of the Society, to the editing of the *Proceedings*, and to the correspondence with authors of papers, involved great personal sacrifice to one so ill-paid for his professional work. To his unfailing promptness and courtesy in conducting that correspondence is largely due the successful establishment of the Society upon its present assured foundation.

In his Presidential Address on November 10th, 1898, Prof. Elliott said of Mr. Tucker: "Nearly all the papers which the Society has ever received have caused him correspondence. Sixteen Presidents have, like myself, found their office free from anxiety because of his and his colleagues' assiduity. There has been no limit to the burdens he would willingly take upon himself in his absolutely unselfish devotion to the interests of the Society. Such a use of what might have been the leisure of half a life-time has put mathematical science under an obligation for which no gratitude would be excessive. May he long be good enough, and have the health and strength, to add to this load of obligation!"

His tenure of office was unique. In most cases the secretary of a scientific society, after a few years of service, passes on to one of the higher offices, and then withdraws gradually from active co-operation in its work. He held this office for thirty-five years without loss or diminution of interest in its work.

In 1902, when he ceased to live in London, he resigned the office of Secretary, and the Society unanimously passed, on the 18th November, on the motion of the President, seconded by Dr. Glaisher, the following resolution:—

"That the thanks of the London Mathematical Society be offered to Mr. Robert Tucker for the eminent services which he has rendered to the Society during the thirty-five years in which he has held the office of Honorary Secretary."

This resolution but faintly indicates the warmth and the depth of the feeling expressed in their speeches by the proposer and seconder of the resolution.

On the 8th January, 1903, a testimonial was presented to Mr. Tucker by the members of the Society. In returning thanks he said that he had served under nineteen Presidents of the Society, of whom the first six were gone, and three later ones as well, and he concluded with these words:—

"There is one man who stands out as a brilliant mathematician and kind friend—I mean W. K. Clifford. He and H. J. S. Smith made our meetings delightful to look back upon. To the former I was glad to be able to render such service as it was in my power to show. His wish that I should edit his papers was totally unexpected, and would have been impossible of execution had it not been for the promise of aid from others.

"The resignation of the Secretaryship to which I have devoted half my life will at first exercise a depressing effect upon me; but the rest I shall obtain, and the knowledge that I shall have acquired that my labours, though falling short of what they might have been, have been valued, will cheer me in my retirement.

"And now I conclude with a quotation which I used in my preface to Clifford's papers: 'If I have done well, it is that which I desired; but, if slenderly and meanly, it is that which I could attain unto.'"

### HIS SCIENTIFIC WORK.

From 1863 onwards he was a constant contributor to the mathematical columns of the *Educational Times*. His memorial and biographical notices of Gauss, Sylvester, Chasles, Spottiswoode, and Hirst may be found in *Nature*. There are others in the *Academy*, the *Saturday Review*, and the *Journal of Education*. He edited, as has been already mentioned, the late Prof. Clifford's papers.

His own contributions to mathematics, a list of which is given at the end of this notice, deal with the Modern Geometry of the Triangle and investigations arising out of this subject. Of these I select the three which attracted most attention for special notice, viz., the investigations dealing with the Triplicate-Ratio Circle, the group of circles now known as Tucker Circles,* and the Harmonic Quadrilateral.

## THE TRIPLICATE-RATIO OR FIRST LEMOINE CIRCLE.†

**If through the insymmedian point of a triangle parallels be drawn to the three sides, the six points in which they meet the sides will be concyclic.**

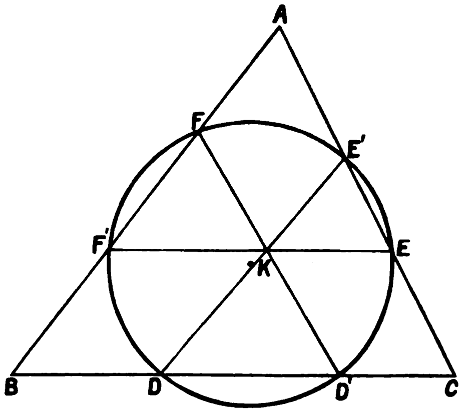
This circle was first discovered by Lemoine in 1873. Ten years later Mr. Tucker (*Quarterly Journal*, Vol. xix., 1883, pp. 344–346), unaware of M. Lemoine's researches, rediscovered the circle, and obtained some new properties, viz.:—

Calling the triangle $ABC$, the insymmedian point $K$, and the points in which the circle meets the sides $D$, $D'$; $E$, $E'$; $F$, $F'$, Mr. Tucker showed that the triangles cut off from $ABC$ by $E'F$, $F'D$, $D'E$ are together equal to the triangle $DEF$ or $D'E'F'$; that the following six angles are equal: $DFB$, $EDC$, $FEA$, $D'E'C$, $F'D'B$, $E'F'A$.

If each of these angles be denoted by $\omega$, then

$$\angle AFE = \angle AE'F' = B+C-\omega,$$
$$\angle BDF = \angle BF'D' = C+A-\omega,$$
$$\angle CED = \angle CD'E' = A+B-\omega.$$

The following points are concyclic:—$B$, $C$, $E'$, $F$; $C$, $A$, $F'$, $D$; $A$, $B$, $D'$, $E$.



The circle is called the Triplicate-Ratio Circle because $DD'$, $EE'$, $FF'$ are proportional to the cubes of $BC$, $CA$, $AB$.

## THE TUCKER CIRCLES.‡

**If two triangles $ABC$, $A_1B_1C_1$ be similar and similarly situated, and have $K$ the insymmedian point for centre of similitude, then the six points $D$, $D'$, $E$, $E'$, $F$, $F'$ in which the sides of $A_1B_1C_1$ meet the sides of $ABC$ are concyclic.**

The circle on which these six points lie is called a Tucker Circle. Its centre is at the mid-point $T$ of the straight line joining the circumcentres $O$, $O_1$ of the triangles $ABC$ and $A_1B_1C_1$.

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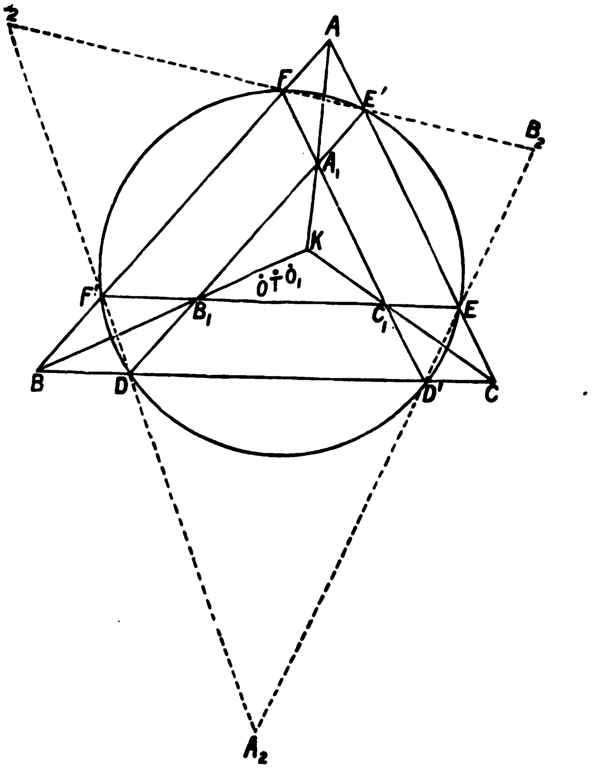
* For Mr. Tucker's detailed account see Vol. xiv. of the *Proceedings of the London Mathematical Society*, pp. 316–319.

† *Quarterly Journal of Pure and Applied Mathematics*, Vol. xix., 1883, pp. 342–348.

‡ *Ibid.*, Vol. xx., 1884, pp. 57–59.

The triangles $FDE$, $E'F'D'$ are directly similar to $ABC$ and congruent to each other. If $E'F$, $FD$, $D'E$ be produced to meet and form a triangle $A_2B_2C_2$, then $T$ is the incentre of the triangle $A_2B_2C_2$ and the radius of the incircle is half the sum of the radii of the circumcircles of $ABC$ and $A_1B_1C_1$. When the triangle $A_1B_1C_1$ becomes the triangle $ABC$ the Tucker circle $DD'EE'FF'$ becomes the circumcircle of $ABC$.



When the triangle $A_1B_1C_1$ reduces to the point $K$, *i.e.*, when the parallels $B_1C_1$, $C_1A_1$, $A_1B_1$ to the sides of the triangle $ABC$ pass through $K$, the Tucker circle $DD'EE'FF'$ becomes the triplicate-ratio or first Lemoine circle.

## THE HARMONIC QUADRILATERAL.[*] [†]

The first systematic study of harmonic quadrilaterals was made by Mr. Tucker. In his article "Some Properties of a Quadrilateral in a Circle the Rectangles under whose opposite Sides are equal," read to the London Mathematical Society on 12th February, 1885, and printed in *Mathematical Questions from the Educational Times*, Vol. XLIV., pp. 125–135, 1886, he states that in his attempt to extend the properties of the Brocard points and circle of a triangle to a quadrilateral, he was brought to a stand at the outset by the fact that the equality of the angles does not involve the similarity of the figures for figures of a higher order than the triangle.

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[*] *Mathematical Questions from the Educational Times*, Vol. XLIV., 1866, pp. 125–135.

[†] This account is due to Dr. Mackay.

Limiting his consideration to quadrilaterals which can be circumscribed by a circle, he arrived at several beautiful results, of which the most important are the following:—

Let $ABCD$ be a quadrilateral inscribed in a circle. Let its diagonals meet at $E$. Suppose that a point $P$ exists such that

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA.$$

The condition for the existence of such a point is that the rectangles contained by opposite sides of the quadrilateral must be equal.

[It was pointed out by Prof. Neuberg that the vertices of a quadrilateral of this kind were four harmonic points on the circumference of the circle; and they are now generally described as harmonic quadrilaterals.]

Mr. Tucker further showed that, if there were one such point $P$ for the quadrilateral, there would be another point $P'$, just as there are two Brocard points for any triangle. He proved further that the four points in which $PA$ meets $P'B$, $PB$ meets $P'C$, $PC$ meets $P'D$, $PD$ meets $P'A$, and the points $P$, $P'$ lie on the circumference of a circle whose diameter is $OE$, where $O$ is the centre of the circle $ABCD$. Next, if through $E$ parallels be drawn to the sides of the quadrilateral, these parallels will meet the sides in eight points which lie on a circle concentric with the previous one.

Lastly, he shows that the symmedian points $\rho_1$, $\rho_2$ of $ABD$, $BCD$ lie on $AC$; the symmedian points $\sigma_1$, $\sigma_2$ of $ABC$, $ADC$ lie on $BD$; the lines $O\rho_1$, $O\rho_2$, $O\sigma_1$, $O\sigma_2$ are the diameters of the Brocard circles of the triangles $ABD$, $BCD$, $ABC$, $ACD$ respectively; the centres of the four Brocard circles lie two and two on straight lines, parallel to $AC$, $BD$; the circles themselves intersect two and two on the diagonals $AC$, $BD$ at their mid-points, that is, where the Brocard circle of the quadrilateral meets the diagonals.

Mr. Tucker's researches were taken up by Messrs. Neuberg and Tarry, Dr. Casey, and the Rev. T. C. Simmons, and there now exists a tolerably extensive theory of harmonic polygons.

## LIST OF MR. TUCKER'S PAPERS.

(I.) From the *Proceedings of the London Mathematical Society*:—

1. On Radial Curves, Vol. I., 1866, No. 5.
2. On certain Fractions, Vol. X., pp. 225, 226.
3. On Two Concentric Circles, Vol. XIV., pp. 316–321.
4. [Geometry of Triangle], Vol. XV., pp. 279–281.
5. Conjugate Tucker Circles and Orthocentroidal Circle, Vol. XVII., pp. 420–430.
6. Uni-Brocardal Triangles, Vol. XVIII., pp. 393–399.
7. Isoscelians, Vol. XIX., pp. 163–170; Vol. XXII., p. 178.
8. A Group of Isostereans, Vol. XIX., pp. 218–220.
9. Isoscelian Hexagrams, Vol. XXI., pp. 4–29.
10. Some Properties of Co-normal Points on a Parabola, Vol. XXI., pp. 442–451.
11. A Property of the Circumcircle, Vol. XXII., pp. 470–472; Vol. XXV., pp. 315–318.
12. On a Group of Triangles inscribed in a given Triangle $ABC$ whose Sides are Parallel to the Connectors of any Point $P$ with $A$, $B$, $C$, Vol. XXIV., pp. 131–143.
13. A Geometrical Note, Vol. XXIV., pp. 162–166.
14. Some Properties of Two Tucker Circles, Vol. XXV., pp. 389–394.
15. Note on Isoscelians, Vol. XXXIII., pp. 87–90.
16. Two In-triangles which are similar to the Pedal Triangle, Vol. XXXIII., pp. 91–97.
17. The Brocardal Properties of some Associated Triangles, Vol. XXXIII., pp. 404, 405.

(II.) From the *Messenger of Mathematics* (First Series):—

18. The Trigonometrical Ratios of $A+B$ established by means of Ptolemy's Theorem, Euc. VI., D, Vol. III., 1866, p. 123.
19. Construction for Axes of an Ellipse, Vol. III., 1866, pp. 151, 152.
20. Curves whose Evolutes are similar to themselves, Vol. III., 1866, pp. 191, 192.
21. Trigonometrical Ratios of Compound Angles, Vol. IV., 1868, pp. 31–33.

(III.) From the *Messenger of Mathematics* (Second Series):—

22. Note on a Formula in Trigonometry, Vol. iv., 1875, p. 101.
23. To find the Latus Rectum of a Parabola, Vol. iv., 1875, p. 189.
24. Note on Mr. H. M. Taylor's Six-Point Circle, Vol. xii., 1883, pp. 181, 182.
25. Graphical Construction for the Cube of any Number, Vol. xiv., 1885, p. 114.
26. The "Sine-Triple-Angle" Circle, Vol. xvi., 1887, pp. 125, 126.
27. The "Cosine" Orthocentres of a Triangle and a Cubic through them, Vol. xvii., 1888, pp. 97–103.

(IV.) From the *Quarterly Journal of Pure and Applied Mathematics*:—

28. The Radial of an Ellipse, Vol. xviii., 1882, pp. 311, 312.
29. The Triplicate-Ratio Circle, Vol. xix., 1883, pp. 342–348.
30. A Group of Circles,* Vol. xx., 1884, pp. 57–59.
31. The Symmedian Point Axis of an Associated System of Triangles, Vol. xx., 1885, pp. 167–169.

(V.) From the *Proceedings of the Edinburgh Mathematical Society*:—

32. Geometrical Notes, Vol. xi., 1893, pp. 57–60.
33. Two Circular Notes, Vol. xii., 1894, pp. 17–22.
34. Geometrical Notes, Vol. xii., 1894, pp. 51–54.
35. Two Triplets of Circum-hyperbolas, Vol. xii., 1894, pp. 69–75.
36. Three Parabolas connected with a Plane Triangle, Vol. xii., 1894, pp. 79–84.
37. Notes on an Orthocentric Triangle, Vol. xii., 1894, pp. 118, 119.
38. Parabolic Note: Co-normal Points, Vol. xiii., 1895, pp. 29–35.
39. Properties of some Groups of Wallace Lines, Vol. xiv., 1896, pp. 116–120.
40. Geometrical Note, Vol. xv., 1897, pp. 98, 99.

VI.) From Vol. xliv., 1886, of the *Mathematical Questions from the Educational Times*:—

41. Some Properties of a Quadrilateral in a Circle the Rectangles under whose Opposite Sides are Equal.†

In addition to the above Mr. Tucker contributed a large number of questions to the *Educational Times*.

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* The Tucker circles.

† The harmonic quadrilateral.

# NOTES AND CORRECTIONS.

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Dr. W. H. Bussey sends the following correction of his paper printed in this volume, pp. 296–315 :—

P. 312, line 8, *for* "$[A^{29}BA^{24}B\,(A^{16}B\ldots$" *read* "$[A^{29}BA^{24}BA^{16}(B\ldots.$"

Mr. P. W. Wood sends the following corrections of his paper printed in this volume, pp. 316–333 :—

P. 318. In the expansion of $P_4$ the coefficient of $s_1^3s_2^4s_3s_4^2$ should be "$-1$," instead of "1"; and the coefficient of $s_1s_2^5s_3^5s_4^4$ should be "$-14$," instead of "14."

The Rev. F. H. Jackson sends the following abstract of his paper "Note on a case of $F(\alpha, \beta, \gamma, \delta, \epsilon, 1)$," communicated at the April meeting :—

In connection with the product expressions for the sum of the series of cubes of the binomial coefficients, and the sum of the series $F(a, a-\delta+1, a-\epsilon+1, \delta, \epsilon, 1)$, investigated by Prof. F. Morley and Prof. A. C. Dixon in the *Proceedings*, Vol. xxxiv., p. 397, and Vol. xxxv., p. 284, I think that the following binomial form analogous to Vandermonde's may be of some interest.

If $\qquad [x] = (q^x-1)/(q-1)$ and $[x]_r = [x][x-1]\ldots[x-r+1]$,

it is easy to establish

$$q\,[x]_2[y]_2-[2][x][y][x][y]+q^3[x]_2[y]_2 = -[2][x][y][x+y-1],$$

and similar identities corresponding to binomial forms for $n = 4, 6, 8$. I have verified for $n = 2, 4, 6, 8$

$$\sum_{r=0}^{n}(-1)^r\frac{[n]!}{[n-r]!\,[r]!}\,q^{(r)}\,[x]_{n-r}[y]_{n-r}[x]_r[y]_r = (-1)^{\frac{1}{2}n}\frac{[n]!}{[\frac{1}{2}n]!}[x]_{\frac{1}{2}n}[y]_{\frac{1}{2}n}[x+y-\tfrac{1}{2}n]_{\frac{1}{2}n}, \quad \text{(A)}$$

in which the index $(r)$ of the $q$-factor of the general term on the left is $\frac{1}{8}\{3(n-2r)^2-2(n-2r)\}$, and shown that for $n$ an even integer in general the $q^{(r)}$-factor of the general term must necessarily be of this form if the equation (A) is to be valid. Although such forms are easily obtained, it is not easy to establish them in their generality. Form (A) may be of interest because, if $n$ be unrestricted except by conditions for convergence, it contains

$$1+[n]^3q^{1-\frac{3}{2}n}+\left(\frac{[n][n+1]}{[2]!}\right)^3q^{2-3n}+\ldots = \frac{1}{(q^{\frac{1}{2}n}+1)}\,\frac{S_q(n)}{S_q(\frac{1}{2}n)}\,\frac{\Gamma_q(1-\frac{3}{2}n)}{\Gamma_q^3(1-\frac{1}{2}n)}\,q^{\frac{1}{8}[-(3n^2+2n)]}$$

(cf. Prof. Morley's paper, Vol. xxxiv.), $S_q(n)$ being a certain sigma-function reducing, in case $q = 1$, to $\sin n\pi$.

Mr. Western sends the following statement concerning his "Note on Fermat's Numbers and the Converse of Fermat's Theorem," communicated at the April meeting :—

The note contained a statement that the author had calculated the residue of $3^{2^{127}}$ (mod $2^{128}+1$), and had thence proved that $2^{128}+1$ is not a prime number. Reference should be made to Lucas,

**Théorie des Nombres, t. I., p. 441. The calculation has been carried out simultaneously and independently by Dr. J. C. Morehead (Bulletin Amer. Math. Soc., for July, 1905, Vol. VII., pp. 543–545), and the two calculations have been found to agree exactly. The composition of $2^{128}+1$ is therefore definitely established.**

Lt.-Col. Cunningham sends the following abstract of his "Note on High Pellian Factorisations," communicated at the May meeting;—

**A method was explained for constructing very large factorisable numbers of the form $N = y^2+1$ (with complete resolution into prime factors) from the Pellian equation $y^2 - Dx^2 = -1$. Examples were given, ending with a number of 78 figures,**

$$N = (2^{128} + 3 \cdot 2^{42})^2 + 1 = (2^{84}+1)(2^{86}+1)^2.$$

**The resolution of the factors on the right-hand side has been given by Lucas.**

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

# THE APPLICATION OF BASIC NUMBERS TO BESSEL'S AND LEGENDRE'S FUNCTIONS

## (SECOND PAPER)

*By* Rev. F. H. JACKSON.

[Received December 1st, 1904—Read December 8th, 1904.]

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Introduction.—§ 1. Lemmas I.-V.—§ 2. Theorems corresponding to

$$e^{r\cos\theta}J_0(r\sin\theta) = \Sigma\frac{r^n}{[n]!}P_n(\cos\theta),$$

and Summation of a Series analogous to $\Sigma(-1)^r\frac{n!}{r!\,\overline{n-r}!}x^rP_{n-r}$.—§ 3. Relations analogous to $e^{ir\cos\theta} = 2^{\frac{1}{2}q-1}\Gamma(\frac{1}{2}q-1)\,\Sigma i^n(n+\frac{1}{2}q-1)\,P_n(q,\cos\theta)\,J_{n+\frac{1}{2}q-1}(r)/r^{\frac{1}{2}q-1}$.—§ 4. Function analogous to $J(2x\cos\theta)$, and certain Theta Function Products.—§ 5. Expansions of Arbitrary Functions in Series of $J_{[n]}$ and $P_{[n]}$ Functions.—§ 6. Relations between $J_{[n]}$ and $\mathfrak{J}_{[n]}$ is

$$J(x) = E_p(ix)\,E_p(-ix)\,\mathfrak{J}(x).$$

Results derived from this Relation. A Multiple Function $J_{m,n}$; its relation to $J_m$.

### *Introduction.*

In a previous paper the basic exponential function $E_p(x)$ was used to obtain a generalized form of Neumann's addition theorem for Bessel's function $J_0$. The generalized functions $J_{[n]}$ belong to the class commonly known as $q$ functions. In this paper the function $E_p(x)$ will be used to obtain various relations between the generalized forms of Bessel's and Legendre's functions. An example of such a relation is

$$E_p(ir\cos\theta) = \{q-4\}!\sum_{n=0}^{\infty} i^n[2n+q-2]\,\mathfrak{P}_n(q,\cos\theta)\,\frac{J_{[n+\frac{1}{2}q-1]}(r)}{r^{\frac{1}{2}q-1}},$$

which reduces when $p = 1$ to

$$e^{ir\cos\theta} = 2^{\frac{1}{2}q-1}\Gamma(\tfrac{1}{2}q-1)\sum_{n=0}^{\infty} i^n(n+\tfrac{1}{2}q-1)P_n(q, \cos\theta)\frac{J_{n+\frac{1}{2}q-1}(r)}{r^{\frac{1}{2}q-1}},$$

a result due to Dr. Hobson ("Bessel Functions," *Proc. London Math. Soc.*, Vol. xxv., p. 59).

A variety of such forms will be obtained, and by means of these theorems the following series of generalized Legendre functions will be formed and its summation effected for all values of $n$:—

$$\sum_{r=0}^{\infty}(-1)^r\frac{[n]!}{[r]!\,[n-r]!}x^r P_{[n-r]}(x).$$

This series and another of similar type afford remarkable parallels to the binomial form. As in previous papers, $[n]!$ is a convenient expression for $\Gamma_p(n+1)$ and $\{2n\}! = [2]^n\Gamma_{p^2}(n+1)$ while $[n] = (p^n-1)/(p-1)$.

The function

$$E_p(x) = \prod_{m=1}^{\infty}\{1+x(p-1)p^{-2m}\} = 1+\frac{x}{[1]!}+\frac{x^2}{[2]!}+\dots \quad (|p|>1);$$

so

$$E_p\left(\frac{x}{1-p}\right) = \prod_{m=1}^{\infty}(1-xp^{-2m}).$$

This infinite product expression for the basic exponential enables us to connect the theory of the functions $J_{[n]}$ and $P_{[n]}$ with the $q$ theory of theta function products. In previous papers it was found that on inverting the base $p$ in the function $J_{[n]}(x)$ another form of function $p^{n^2}\mathfrak{J}_{[n]}(x)$ was obtained, and that $J_{[n]}\mathfrak{J}_{[m]} = \mathfrak{J}_{[n]}J_{[m]}$ (p. 195, Ser. 2, Vol. 2). In this paper a very simple equation is found to connect the functions, viz.,

$$J_{[n]}(x) = E_p(ix)\,E_p(-ix)\,\mathfrak{J}_{[n]}(x),$$

which leads to many remarkable identities, and opens up a theory for a double $J$ function, denoted $J^p_{m,n}(x)$. The function $E_p$ will, moreover, be used to form a function

$$\mathfrak{J}_{[0]}(2x, \cos\theta) = \sum_{r=0}^{\infty}(-1)^r\frac{(2x)^{2r}}{\{2r\}!\,\{2r\}!}p^{2r^2}\cos^2\theta\cos^2(\theta+\omega)\dots\cos^2(\theta+r\omega-\omega)$$

such that $$\mathfrak{J}_{[0]}(2xy, \cos\theta) = J_{[0]}\mathfrak{J}_{[0]}+2\sum_{n=1}^{\infty}(-1)^n J_{[n]}\mathfrak{J}_{[n]},$$

in which the arguments of the functions $J$ and $\mathfrak{J}$ on the right side of the equation are $x^2e^{i\theta}$ and $y^2e^{-i\theta}/p$ respectively.

In the section of the paper which deals with theta function products reference is made to the memoirs on $q$ products and series by Prof. L. J, Rogers ("Certain Infinite Products," *Proc. London Math. Soc.*, Vols. xxiv., xxv.). Various relations are obtained between the coefficients denoted by Prof. Rogers $A_r(\theta)$, $B_r(\theta)$ and the functions denoted $J_{[n]}$ in this and previous papers. As an example,

$$I_{[0]}\left(\frac{x}{p-p^2}\right) = J_{[0]} - p\frac{x}{(p-1)}\mathfrak{J}_{[1]}A_1 + p^4\frac{x^2}{(p-1)(p^2-1)}\mathfrak{J}_{[2]}A_2 - \dots,$$

in which the argument of $J$ is $xe^{-i\theta}/(p-1)$, and the argument of $\mathfrak{J}$ is $xe^{-i\theta}/(p^2-p)$. $I$ is used to denote the generalized Bessel series in which all the terms are positive. A coefficient analogous to Legendre's is expressed in terms of Rogers' coefficients as

$$(1-p)^n P_{[n]}(q, \cos\theta) = \sum_{r=0}^{r=n}(-1)^r \frac{A_{n-r}(\theta)\,B_r(\theta)}{\{2n-2r\}!\,\{2r\}!}\,p^{(4-q)r}.$$

A series of novel form is found to express the theta function product

$$\prod_{m=1}^{\infty}\frac{(1-2xp^{2m}\cos\theta+x^2p^{4m})}{(1+2xp^{2m}\cos\theta+x^2p^{4m})} = \Sigma b_n x^n,$$

$$b_n = (-1)^n\frac{\cos\theta\dots\cos(\theta+n\omega-\omega)}{(p^2-1)(p^4-1)\dots(p^{2n}-1)}$$

$$\times\left\{\sum_{r=0}^{r=n} p^{r^2-rn}\frac{\{2n\}!}{\{2n-2r\}!\,\{2r\}!}\,\frac{\cos\theta\dots\cos(\theta+r\omega-\omega)}{\cos(\theta+n\omega-\omega)\dots\cos(\theta+n\omega-r\omega)}\right\}.$$

The properties of the functions $A_r(\theta)$, $B_r(\theta)$ and their use in the investigation of theta function products is shown in the papers of Prof. L. J. Rogers (*loc. cit.*). Finally, two theorems analogous to Neumann's theorems for the expansion of arbitrary functions in series (1) of Bessel's functions, (2) of Legendre's functions, are given.

$$f(x) = \sum_{0}^{\infty} a_n J_{[n]}(x),$$

where
$$a_n = \frac{1}{2\pi i}\int_c O_n(t)\,f(t)\,dt,$$

and
$$O_n(t) = \frac{\{2n\}!}{t^{n+1}}\left\{1 - \frac{t^2}{[2][2n-2]} + \dots\right\}.$$

The expression for $f(x)$ in a series of $P_{[n]}(x)$ functions gives

$$a_n = \frac{[2n+1]}{2\pi i}\int_c Q_{[n]}(t)\,f(t)\,dt.$$

I hope that the results given in the paper may be of interest not merely because of their remarkable likeness to well known results in the case of ordinary functions, but because they link the functions $J_n$, $P_n$, ...

by means of their $q$ generalizations with the $q$ theory of elliptic functions. With regard to the expansions of arbitrary functions in basic (or $q$) series, it seems noteworthy that functions not involving $p$ should be capable of expression in terms of functions involving this arbitrary parameter.

## § 1.

It will be well to collect together in one section certain subsidiary results required in subsequent work. The analysis required to establish these results has no novelty of method, and if given at length would be somewhat tedious, so the main points only of the analysis will be given. Some of these results, such as Lemma II., Lemmas IV., V., have a certain interest in themselves apart from the work which is based upon them.

LEMMA I.—

$$1-p^{2-2r}\frac{[2r][2n+q-6]}{[2][2n+q-2r-4]}$$
$$+p^{6-4r}\frac{[2r][2r-2]}{[2][4]}\,\frac{[2n+q-4][2n+q-10]}{[2n+q-2r-4][2n+q-2r-6]}-\dots,$$

the $p$ prefix of the general term being $p^{s(s+1)-2sr}$ $(s = 1, 2, 3, \dots)$. The sum of this series to $(r+1)$ terms is zero. In general the series can easily be summed term by term; the sum of $s$ terms is always a factor of the $(s+1)$-th term: for example, the sum of the first three terms is

$$p^{6-4r}\frac{[2r-2][2r-4][2n+q-4][2n+q-6]}{[2][4][2n+q-2r-4][2n+q-2r-6]}.$$

In case $r$ is a positive integer, the sum to $(r+1)$ terms is zero, owing to the appearance of a zero factor in the numerator of the product which represents the sum of the series.

LEMMA II.—

$$\frac{x^n}{\{2n\}!}=J_{[n]}+\frac{[2n+4]}{[2]}J_{[n+2]}+\dots$$
$$+p^{s(s-1)}[2n+4s]\frac{[2n+2s-2][2n+2s-4]\dots[2n+2]}{[2][4]\dots[2s]}J_{[n+2s]}+\dots.$$

There is no difficulty, on taking out the terms involving $x^{n+2r}$, in showing that the sum of the coefficients of $x^{n+2r}$ vanishes, $x^n$ arises only from $J_{[n]}$, and its coefficient is $1/\{2n\}!$ The convergence, &c., of this series is discussed in a note at the end of the paper. A similar series for $\mathfrak{J}$ functions is there given.

LEMMA III.—

$$P_{[n]}(x) = p^{\frac{1}{2}(n^2-n)}\Sigma x^n \frac{[n][n-1]\dots[n-2r+1]}{[2]^2[4]^2\dots[2r]^2}$$
$$\times\left(1-\frac{p^3}{x^2}\right)\left(1-\frac{p^5}{x^2}\right)\dots\left(1-\frac{p^{2r+1}}{x^2}\right)p^{2r^2+r-2rn}. \qquad (\alpha)$$

It has been shown in a preceding paper, Ser. 2, Vol. 2, p. 215, that

$$P_{[n]}(x) = \Sigma x^n \frac{[n][n-1]\dots[n-2r+1]}{[2]^2[4]^2\dots[2r]^2}\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)\dots\left(p^{4r-2}-\frac{p^{2r+1}}{x^2}\right). \qquad (\beta)$$

Invert the base $p$ in this theorem ; then

$$P_{[n]}(x) \quad \text{becomes} \quad p^{-\frac{1}{2}[n(n-1)]-2n}P_{[n]}(xp^2)$$

and the expression on the right side of $(\beta)$ becomes

$$x^n\frac{[n]\dots[n-2r+1]}{[2]^2\dots[2r]^2}p^{2r^2+r-2nr}\left(1-\frac{p^{-1}}{x^2}\right)\dots\left(1-\frac{p^{2r-3}}{x^2}\right).$$

On replacing $x$ by $xp^{-2}$, we have the theorem as stated in $(\alpha)$.

LEMMA IV.—

$$\frac{\mu^n}{[n]!} = \frac{\{q-4\}!}{\{2n+q-4\}!}\left[P_{[n]}(q, \mu)+p^3\frac{[2n+q-6]}{[2]}P_{[n-2]}(q, \mu)\right.$$
$$\left.+p^6\frac{[2n+q-4][2n+q-10]}{[2][4]}P_{[n-4]}+\dots\right]. \qquad (\gamma)$$

The $p$ prefix of the general term of the series within the large brackets is $p^{3r}$, and

$$P_{[n]}(q, \mu) = \sum_{r=0}^{\infty}(-1)^r p^{r(r+2)}\frac{\{2n+q-2r-4\}!}{\{q-4\}!\,\{2r\}!\,\{n-2r\}!}\mu^{n-2r}.$$

Comparing $P_{[n]}(q, \mu)$ with the function denoted $P_n(p, \mu)$ by Dr. Hobson (see *Proc. London Math. Soc.*, Vol. xxv., p. 57), it is seen that when the base $p$ is unity the function $P_{[n]}(q, \mu)$ reduces to $P_n(p, \mu)$, allowing for the use of $q$ in place of Hobson's parameter $p$, since $p$ is already appropriated to denote the base of the functions. In series $(\gamma)$, if we expand the $P$ functions and arrange according to powers of $\mu$, it is plain that $\mu^n$ arises only from $P_{[n]}$ and that the term involving it is $\mu^n/[n]!$, and the coefficients of the terms involving $\mu^{n-2r}$ form the series

$$(-1)^r p^{r^2-r}\frac{\{2n+q-2r-4\}!}{\{2n+q-4\}!\,\{2r\}!\,[n-2r]!}\left\{1-p^{2-2r}\frac{[r][2n+q-6]}{[2][2n+q-2r-4]}+\dots\right\},$$

which is identically zero by Lemma I., and so the required result $(\gamma)$ is established.

LEMMA V.—

$$\frac{\mu^n}{[n]!} = \frac{\{q-4\}!}{\{2n+q-4\}!}\Big\{\mathfrak{P}_n(q,\mu)+\frac{[2n+q-6]}{[2]}\mathfrak{P}_{n-2}(q,\mu)$$

$$+\frac{[2n+q-4][2n+q-10]}{[2][4]}\mathfrak{P}_{n-4}(q,\mu)+\ldots\Big\}, \quad (\delta)$$

$$\mathfrak{P}_n(q,\mu) = \sum_{r=0}^{\infty}(-1)^r p^{r(r-1)}\frac{\{2n+q-2r-4\}!}{\{q-4\}!\,\{2r\}!\,[n-2r]!}\mu^{n-2r}.$$

Series ($\delta$) differs from the analogous series in Lemma IV., in that there are no $p$ prefixes to the terms of the series. This lemma may be established on just the same lines as Lemma IV., and depends on the identity given in Lemma I. We notice that

$$\mathfrak{P}_n(q,\mu) = p^{-\frac{1}{2}n}P_{[n]}(q,\mu p^{\frac{1}{2}}).$$

In case $q = 3$, Lemma IV. reduces to a result given in a previous paper [(74), page 217, Ser. 2, Vol. 2]. The double forms of these theorems afford examples of the double forms of all ($q$) function series and products. This duality of form is necessary to give convergent series, as the base $|p| > 1$ or $< 1$.

## § 2.

Form the product

$$E_p(r\cos\theta)\{A_0+A_1r^2+A_2r^4+\ldots\}$$

in which $A_0$, $A_1$, ... are $p$ functions of $\theta$, viz.,

$$A_n = (-1)^n\frac{(p^3-p^2\cos^2\theta)(p^5-p^6\cos^2\theta)\ldots(p^{2n+1}-p^{4n-2}\cos^2\theta)}{[2]^2[4]^2\ldots[2n]^2}.$$

The series $\Sigma A_n r^{2n}$ and $E_p(r\cos\theta)$ are absolutely convergent if $|p| > 1$. The coefficient of $r^n$ in the product is therefore

$$\frac{1}{[n]!}\Sigma\cos^n\theta\frac{[n][n-1]\ldots[n-2r+1]}{[2]^2[4]^2\ldots[2r]^2}\frac{(p^3-p^2\cos^2\theta)\ldots(p^{2r+1}-p^{4r-2}\cos^2\theta)}{\cos^{2r}\theta},$$

and this series, by Lemma III., is $\dfrac{1}{[n]!}P_{[n]}(\cos\theta)$; so that we write

$$E_p(r\cos\theta)\,B(r,\theta) = \sum_{n=0}^{\infty}\frac{r^n}{[n]!}P_{[n]}(\cos\theta),$$

in which

$$B(r,\theta) = \sum_{n=0}^{\infty}(-1)^n\frac{(p^3-p^2\cos^2\theta)\ldots(p^{2n+1}-p^{4n-2}\cos^2\theta)}{[2]^2[4]^2\ldots[2n]^2}r^{2n},$$

reducing when $p = 1$ to

$$e^{r\cos\theta} J_0(r \sin\theta) = \Sigma \frac{r^n}{n!} P_n(\cos\theta).$$

In a similar way, forming the product $E_{p^{-1}}(r\cos\theta)\,\mathfrak{B}(r, \theta)$, viz.,

$$E_{p^{-1}}(r\cos\theta)\,\{\mathfrak{A}_0 + \mathfrak{A}_1 r^2 + \mathfrak{A}_2 r^4 + \dots\},$$

in which $$E_{p^{-1}}(r\cos\theta) = 1 + \frac{r\cos\theta}{[1]!} + p\,\frac{r^2\cos^2\theta}{[2]!} + \dots,$$

$$\mathfrak{A}_n = (-1)^n \frac{(p^3-\cos^2\theta)(p^5-\cos^2\theta)\dots(p^{2n+1}-\cos^2\theta)}{[2]^2[4]^2\dots[2n]^2}\, p^{-2n},$$

we find that the coefficient of $r^n$ in the resulting series is

$$\frac{1}{[n]!}\, p^{\frac{1}{2}[n(n-1)]} \Sigma \cos^n\theta\, \frac{[n][n-1]\dots[n-2r+1]}{[2]^2[4]^2\dots[2r]^2}$$
$$\times \frac{(p^3-\cos^2\theta)\dots(p^{2r+1}-\cos^2\theta)}{\cos^{2r}\theta}\, p^{2r^2+r-2rn}.$$

This, by ($a$), Lemma III., is $\dfrac{1}{[n]!} P_{[n]}(\cos\theta)$.

We write the theorems

$$\left.\begin{aligned} E_p(r\cos\theta)\, B(r, \theta) &= \Sigma \frac{r^n}{[n]!} P_{[n]}(\cos\theta), \\ E_{p^{-1}}(r\cos\theta)\, \mathfrak{B}(r, \theta) &= \Sigma \frac{r^n}{[n]!} P_{[n]}(\cos\theta). \end{aligned}\right\} \tag{1}$$

Since $E_p(r\cos\theta)\, E_{1\,p}(-r\cos\theta) = 1$ and the $B$ functions involve only even powers of $r$, it follows that

$$B(r, \theta)\,\mathfrak{B}(r, \theta)$$
$$= \left\{1 + \frac{r^2}{[2]!} P_{[2]} + \frac{r^4}{[4]!} P_{[4]} + \dots\right\}^2 - \left\{r P_{[1]} + \frac{r^3}{[3]!} P_{[3]} + \dots\right\}^2$$

(cf. *Proc. London Math. Soc.*, Vol. xxv., p. 66).

Again, since for the $E$ function $E_p(a)\, E_{p^{-1}}(-a) = 1$, we can write

$$E_{p^{-1}}(-r\cos\theta) \sum_{n=0}^{\infty} \frac{r^n}{[n]!} P_{[n]}(\cos\theta) = B(r, \theta). \tag{2}$$

This gives us a new form of $B(r, \theta)$ expressed in a series of ascending powers of $r$. The coefficient of $r^n$ in the expression on the left side of (2) is

$$\frac{1}{[n]!}\Big[P_{[n]}-\frac{[n]}{[1]}\cos\theta\, P_{[n-1]}+p\,\frac{[n][n-1]}{[2]!}\cos^2\theta\, P_{[n-2]}$$
$$-\dots(-1)^n p^{\frac{1}{2}[n(n-1)]}\cos^n\theta\, P_{[0]}\Big]. \quad (3)$$

The coefficients of the series within the large brackets follow the binomial form

$$(1-x)_n = 1-\frac{[n]}{[1]}x+p\,\frac{[n][n-1]}{[2]!}x^2-\dots$$

(*Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 193).

Now $B(r, \theta)$ is an even function of $r$; so that when $n$ is an odd integer it necessarily follows that the expression (3) is identically zero. In case $n$ is an even integer we obtain, by equating coefficients of $r^{2n}$ and replacing $\cos\theta$ by $x$,

$$\frac{1}{[2n]!}\Big[P_{[2n]}-\frac{[2n]}{[1]}x\,P_{[2n-1]}+p\,\frac{[2n][2n-1]}{[2]!}x^2 P_{[2n-2]}-\dots\Big]$$
$$=\frac{(p^2x^2-p^3)(p^6x^2-p^5)\dots(p^{4n-2}x^2-p^{2n+1})}{[2]^2[4]^2\dots[2n]^2}. \quad (4)$$

Similarly, from the second expression of (1) we obtain that, if $n$ be odd,

$$P_{[n]}-\frac{[n]}{[1]}x\,P_{[n-1]}+\frac{[n][n-1]}{[2]!}x^2P_{[n-2]}-\dots = 0,$$

and for $2n$ (even integer)

$$P_{[2n]}-\frac{[2n]}{[1]}xP_{[2n-1]}+\frac{[2n][2n-1]}{[2]!}x^2P_{[2n-2]}-\dots$$
$$=[2n]!\,\frac{(x^2-p^3)(x^2-p^5)\dots(x^2-p^{2n+1})}{[2]^2[4^2]\dots[2n]^2}. \quad (5)$$

These theorems are easily verified for integral values of $n$, and suggest interesting extensions in case $n$ be not integral.

In (5), if we take out the terms involving $x^{2n-2r}$ from the left side, the coefficient of $x^{2n-2r}$ is found to be

$$p^{r(r+2)}\,\frac{[4n-2r]!}{\{2r\}!\,\{4n-2r\}!\,[2n-2r]!}$$
$$\times\Big\{1-\frac{[2n]}{[1]}\,\frac{[2n-2r]}{[4n-2r-1]}+\frac{[2n][2n-1]}{[2]!}\,\frac{[2n-2r][2n-2r-1]}{[4n-2r-1][4n-2r-3]}-\dots\Big\}.$$

The sum of the Heinean series within the large brackets is easily found to be for all values of $n$

$$\frac{[2n]!\,\{2n-2r-1\}!}{\{2n\}!\,\{4n-2r-1\}!},$$

and the coefficient of $x^{2n-2r}$ may be written

$$p^{r(r+2)}\frac{[2n]!}{\{2n\}!^2}\,\frac{\{2n\}!}{\{2n-2r\}!\,\{2r\}!}. \tag{6}$$

Now

$$x^{2n}\mathop{\mathrm{L}}_{\kappa=\infty}\frac{(x^2-p^3)(x^2-p^5)\,\dots\,(x^2-p^{2\kappa+1})}{(x^2-p^{2n+3})(x^2-p^{2n+5})\,\dots\,(x^2-p^{2n+2\kappa+1})}.$$

may be expanded without difficulty (cf. *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 73) as

$$\sum_{r=0}^{r=\infty}\frac{[2n][2n-2]\,\dots\,[2n-2r+2]}{[2][4]\,\dots\,[2r]}\,p^{r(r+2)}x^{2n-2r}.$$

On comparing this with (6), viz., the coefficient of $x^{2n-2r}$ obtained from $P_{[2n]}-\frac{[2n]}{[1]}xP_{[2n-1]}+\dots$, we are justified in writing generally for all values of $2n$, except positive odd integral values,

$$\sum_0^\infty(-1)^r\frac{[2n]!}{[r]!\,[2n-r]!}\,x^rP_{[2n-r]}(x)$$

$$=x^{2n}\frac{[2n]!}{\{2n\}!^2}\mathop{\mathrm{L}}_{\kappa=\infty}\frac{(x^2-p^3)(x^2-p^5)\,\dots\,(x^2-p^{2\kappa+1})}{(x^2-p^{2n+3})(x^2-p^{2n+5})\,\dots\,(x^2-p^{2n+2\kappa+1})}\quad(|p|<1). \tag{7}$$

A similar theorem may be obtained as a general expression of

$$\sum_{r=0}^{\infty}(-1)^rp^{r(r-1)}\frac{[2n]!}{[r]!\,[2n-r]!}\,x^rP_{[2n-r]}(x)$$

$$=\frac{[2n]!}{\{2n\}!}(p^2x^2-p^3)\,\dots\,(p^{4n-2}x^2-p^{2n+1}). \tag{8}$$

A result which is easily derived from (1) is

$$\left\{1-\frac{2(1+p)}{[2]!}r^2\cos^2\theta+\frac{2(1+p)(1+p^2)(1+p^3)}{[4]!}r^4\cos^4\theta-\dots\right\}B(ir,\theta)\,\mathfrak{B}(ir,\theta)$$

$$=1+\sum_{n=1}^{\infty}(-1)^n\left\{P_{[2n]}+\frac{[2n]}{[1]}P_{[2n-1]}\,P_{[1]}+\frac{[2n][2n-1]}{[2]!}P_{[2n-2]}\,P_{[2]}+\dots\right.$$

$$\left.+P_{[2n]}\right\}\frac{r^{2n}}{[2n]!}, \tag{9}$$

reducing, in case $p=1$, to

$$\cos(2r\cos\theta)\,J_0^2(r\sin\theta)=1+\sum_{n=1}^{\infty}(-1)^n\left\{2P_{2n}+2\,\frac{2n}{1}\,P_{2n-1}P_1+\dots\right\}\frac{r^{2n}}{2n!}.$$

## § 3.

Writing (Lemma V.)

$$I_{[n+\frac{1}{2}q-1]}(r) = \frac{r^{n+\frac{1}{2}q-1}}{\{2n+q-2\}!}\left\{1+\frac{r^2}{[2]\,[2n+q]}+\dots\right\},$$

$$\mathfrak{P}_n(q,\mu) = \frac{\{2n+q-4\}!}{[n]!\,\{q-4\}!}\left\{\mu^n-\frac{[n][n-1]}{[2][2n+q-4]}\mu^{n-2}\right.$$

$$\left.+p^2\frac{[n][n-1][n-2][n-3]}{[2][4][2n+q-4][2n+q-6]}\mu^{n-4}+\dots\right\},$$

consider the series

$$\{q-4\}!\sum_{n=0}^{\infty}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}I_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}_n(q,\mu).$$

The coefficient of $r^n$ in this series is

$$\frac{\{q-4\}!}{\{2n+q-4\}!}\left\{\mathfrak{P}_n(q,\mu)+\frac{[2n+q-6]}{[2]}\mathfrak{P}_{n-2}(q,\mu),+\dots\right\},$$

and this, by Lemma V., is equal to $\mu^n/[n]!$; so that we are able to write, subject to the absolute convergence of the series,

$$\{q-4\}!\sum_{n=0}^{\infty}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}I_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}_n(q,\cos\theta)$$

$$= 1+\frac{r\cos\theta}{[1]!}+\frac{r^2\cos^2\theta}{[2]!}+\dots = E_p(r\cos\theta). \qquad (10)$$

Replacing $r$ by $ir$,

$$\{q-4\}!\sum_{n=0}^{\infty}i^n\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}J_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}_n(q,\cos\theta) = E_p(ir\cos\theta). \qquad (11)$$

By means of Lemma IV. we may show in the same way that

$$\{q-4\}!\sum_{n=0}^{\infty}i^n p^{-\frac{1}{2}n}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}J_{[n+\frac{1}{2}q-1]}(r)\,P_{[n]}(q,\cos\theta) = E_p(irp^{-\frac{1}{2}}\cos\theta) \qquad (12)$$

and again from these theorems, by inversion of the base $p$,

$$\{q-4\}!\sum_{n=0}^{\infty}i^n\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}p^{2q+\frac{1}{2}[n(n-1)]}\mathfrak{J}_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}'_n(q,\cos\theta) = E_{p^{-1}}(ir\cos\theta),$$

$$(13)$$

$$\{q-4\}!\sum_{n=0}^{\infty}i^n p^{\frac{1}{2}n}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}p^{\frac{1}{2}[n(n-1)]}\mathfrak{J}_{[n+\frac{1}{2}q-1]}(r)\,P'_{[n]}(q,\cos\theta)$$

$$= E_{p^{-1}}(irp^{\frac{1}{2}}\cos\theta), \qquad (14)$$

in which $P_{[n]}$ and $\mathfrak{P}_n$ are defined as in Lemmas IV. and V., while we define

$$\mathfrak{P}'_n(q,\mu) = \sum_{r=0}^{\infty}(-1)^r p^{r(r+q-2)} \frac{\{2n+q-2r-4\}!}{\{q-4\}!\,\{2r\}!\,[n-2r]!}\,\mu^{n-2r},$$

$$P'_{[n]}(q,\mu) = \sum_{r=0}^{\infty}(-1)^r p^{r(r+q-5)} \frac{\{2n+q-2r-4\}!}{\{q-4\}!\,\{2r\}!\,[n-2r]!}\,\mu^{n-2r}.$$

The four forms (11), (12), (13), (14) reduce to Dr. Hobson's result if the base $p$ be made unity. When the parameter $q=3$ the functions $P$ reduce to the basic generalizations of Legendre's function.

## § 4.

It can be shown without difficulty that

$$\prod_{m=1}^{\infty}\{1+x(p-1)p^{-2m}\} = 1+\frac{x}{[2]}+\frac{x^2}{[2][4]}+\ldots = E_{p^2}\left(\frac{x}{[2]}\right).$$

In a previous paper it has been shown (p. 198, Ser. 2, Vol. 2) that

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right)E_{1\,p^2}\left(-\frac{\lambda t^{-1}}{[2]}\right) = 1+\frac{\lambda}{[2]}(t-t^{-1})+\frac{\lambda^2}{[2][4]}(t-t^{-1})(t-p^2t^{-1})+\ldots.$$

Replacing $t$ by $e^{i\theta}$, $p$ by $e^{i\omega}$, we can write

$$\frac{E_{p^2}\left(\frac{\lambda t}{[2]}\right)}{E_{p^2}\left(\frac{\lambda}{[2]\,t}\right)} = 1+\frac{i\lambda}{[2]}\,2\sin\theta+\frac{i^2\lambda^2}{[2][4]}\,4p\sin\theta\sin(\theta+\omega)+\ldots$$
$$+\frac{i^n\lambda^n}{\{2n\}!}\,p^{\frac{1}{2}[n(n-1)]}\,2^n\sin\theta\sin(\theta+\omega)\ldots\sin(\theta+n\omega-\omega). \qquad (15\alpha)$$

Similarly

$$\frac{E_{p^2}\left(-\frac{\lambda t^{-1}}{[2]}\right)}{E_{p^2}\left(-\frac{\lambda t}{[2]}\right)} = 1+\frac{i\lambda}{[2]}\,2\sin\theta+\frac{i^2\lambda^2}{[2][4]}\,4p\sin\theta\sin(\theta-\omega)+\ldots$$
$$+\frac{i^n\lambda^n}{\{2n\}!}\,p^{\frac{1}{2}[n(n-1)]}\,2^n\sin\theta\sin(\theta-\omega)\ldots\sin(\theta-n\omega+\omega). \qquad (15\beta)$$

Take the product of these series; then, by result (8), p. 196, Ser. 2, Vol. 2, we obtain

$$\frac{\sum_{-\infty}^{+\infty} t^n J_{[n]}(\lambda)}{\sum_{-\infty}^{+\infty} t^{-n} J_{[n]}(\lambda)} = 1+c_1\lambda+c_2\lambda^2+\ldots,$$

in which

$$c_n = \frac{2^n i^n}{\{2n\}!} p^{\frac{1}{2}[n(n-1)]} \sin\theta \sin(\theta+\omega) \dots \sin(\theta+n\omega-\omega)$$

$$\times \left\{ \sum_{r=0}^{n} p^{r^2-rn} \frac{\{2n\}!}{\{2r\}!\,\{2n-2r\}!} \, \frac{\sin\theta \dots \sin(\theta-r\omega+\omega)}{\sin(\theta+n\omega-\omega)\dots\sin(\theta+n\omega-r\omega)} \right\}. \quad (15\gamma)$$

The product of the $E$ functions, besides being expressed in series of $J_{[n]}$ functions, may be expressed in theta function product form: for

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = \prod_{m=1}^{\infty} \{1+\lambda(p-1)tp^{-2m}\}.$$

On replacing $\lambda$ by $ix/(1-p)$ we obtain

$$E_{p^2}\left(\frac{ixt}{1-p^2}\right) = \prod_{m=1}^{\infty} \{1-ixtp^{-2m}\};$$

so that the product of ($15\alpha$) and ($15\beta$) gives us, if $|p| > 1$,

$$\left.\begin{aligned} &\prod_{m=1}^{\infty} \frac{(1-2xp^{-2m}\sin\theta+x^2p^{-4m})}{(1+2xp^{-2m}\sin\theta+x^2p^{-4m})} = \sum_{n=0}^{\infty} a_n x^n, \\ &a_n = (-1)^n \frac{\sin\theta \dots \sin(\theta+n\omega-\omega)}{(p^2-1)(p^4-1)\dots(p^{2n}-1)} \left\{ \sum_{r=0}^{n} p^{r^2-rn} \frac{\{2n\}!}{\{2r\}!\,\{2n-2r\}!} \right. \\ &\qquad\qquad \left. \times \frac{\sin\theta \dots \sin(\theta+\omega-r\omega)}{\sin(\theta+n\omega-\omega)\dots\sin(\theta+n\omega-r\omega)} \right\} p^{\frac{1}{2}(n^2-n)}. \end{aligned}\right\} \quad (16)$$

Similarly,

$$\left.\begin{aligned} &\prod_{m=1}^{\infty} \frac{(1-2xp^{-2m}\cos\theta+p^{-4m}x^2)}{(1+2xp^{-2m}\cos\theta+p^{-4m}x^2)} = \sum_{n=0}^{\infty} b_n x^n, \\ &b_n = (-1)^n \frac{\cos\theta \dots \cos(\theta+n\omega-\omega)}{(p^2-1)\dots(p^{2n}-1)} \left\{ \sum p^{r^2-rn} \frac{\{2n\}!}{\{2r\}!\,\{2n-2r\}!} \right. \\ &\qquad\qquad \left. \times \frac{\cos\theta \dots \cos(\theta+\omega-r\omega)}{\cos(\theta+n\omega-\omega)\dots\cos(\theta+n\omega-r\omega)} \right\} p^{\frac{1}{2}(n^2-n)}. \end{aligned}\right\} \quad (17)$$

Since

$$E_{p^2}\left(\frac{\lambda\kappa t}{[2]}\right) E_{1/p^2}\left(\frac{\lambda\kappa}{[2]t}\right) = 1 + \frac{\lambda\kappa}{[2]} 2\cos\theta + p\frac{\lambda^2\kappa^2}{[2][4]} 2^2 \cos\theta\cos(\theta+\omega) + \dots$$

$$+ p^{\frac{1}{2}[n(n-1)]} \frac{\lambda^n\kappa^n}{\{2n\}!} 2^n \cos\theta \dots \cos[\theta+(n-1)\omega] + \dots \quad (18)$$

and

$$E_{p^2}\left(-\frac{\lambda\kappa^{-1}t}{[2]}\right) E_{1/p^2}\left(-\frac{\lambda\kappa^{-1}}{[2]\,t}\right)$$

$$= 1-\frac{\lambda\kappa^{-1}}{[2]}\,2\cos\theta+p\,\frac{\lambda^2\kappa^{-2}}{[2][4]}\,2^2\cos\theta\cos(\theta+\omega)+\ldots, \tag{19}$$

also we know that

$$E_{p^2}\left(\frac{\lambda\kappa t}{[2]}\right) E_{p^2}\left(-\frac{\lambda\kappa^{-1}t}{[2]}\right) = J_{[0]}(\lambda t)+(\kappa-\kappa^{-1})J_{[1]}(\lambda t)+(\kappa^2+\kappa^{-2})J_{[2]}(\lambda t)+\ldots, \tag{20}$$

$$E_{1/p^2}\left(\frac{\lambda\kappa}{[2]\,t}\right) E_{1/p^2}\left(-\frac{\lambda\kappa^{-1}}{[2]\,t}\right)$$

$$= \mathfrak{J}_{[0]}\left(\frac{\lambda}{pt}\right)+p(\kappa-\kappa^{-1})\mathfrak{J}_{[1]}\left(\frac{\lambda}{pt}\right)+p^4(\kappa^2+\kappa^{-2})\mathfrak{J}_{[2]}\left(\frac{\lambda}{pt}\right)+\ldots. \tag{21}$$

On taking the product of (20) and (21) we find

$$\left\{J_{[0]}(\lambda e^{i\theta})+\sum_{n=1}^{\infty}[\kappa^n+(-\kappa)^n]J_{[n]}(\lambda e^{i\theta})\right\}$$

$$\times\left\{\mathfrak{J}_{[0]}\left(\frac{\lambda}{p}e^{-i\theta}\right)+\sum_{n=1}^{\infty}p^{n^2}[\kappa^n+(-\kappa)^n]\mathfrak{J}_{[n]}\left(\frac{\lambda}{p}e^{-i\theta}\right)\right\}$$

$$= \mathfrak{J}_{[0]}(2\lambda, \cos\theta)+\Sigma p^{n^2}[\kappa^n+(-\kappa)^{-n}]\mathfrak{J}_{[n]}(2\lambda, \cos\theta), \tag{22}$$

in which $J_{[n]}(\lambda e^{i\theta})$ and $\mathfrak{J}_{[n]}(\lambda e^{-i\theta}/p)$ are the generalizations of Bessel's function defined p. 195, Ser. 2, Vol. 2, but

$$\mathfrak{J}_{[n]}(2\lambda, \cos\theta) = \Sigma(-1)^n\frac{(2\lambda)^{n+2r}}{\{2r\}!\,\{2n+2r\}!}\,p^{2r(n+r)}\cos\theta\cos(\theta+\omega)\ldots$$

$$\times\cos[\theta+(n+r-1)\omega]\cos\theta\cos(\theta+\omega)\ldots\cos[\theta+(r-1)\omega]. \tag{23}$$

In case $p = 1$, this function reduces immediately to $J_n(2\lambda\cos\theta)$.

By equating coefficients in (22) we obtain interesting forms of quasi-addition theorems. For example, from the terms independent of $\kappa$ we find

$$J_{[0]}\mathfrak{J}_{[0]}+2\Sigma(-1)^n p^{n^2}J_{[n]}\mathfrak{J}_{[n]} = \mathfrak{J}_{[0]}(2\lambda, \cos\theta),$$

in which

$$\mathfrak{J}_{[0]}(2\lambda, \cos\theta) = \sum_{r=1}^{\infty}(-1)^r\frac{(2\lambda)^{2r}}{\{2r\}!\,\{2r\}!}\,p^{2r^2}\cos^2\theta\cos^2(\theta+\omega)\ldots\cos^2[\theta+(r-1)\omega], \tag{24}$$

the arguments of the functions $J$ and $\mathfrak{J}$ on the left side of (24) being $\lambda e^{i\theta}$ and $\lambda e^{-i\theta}/p$ respectively. By similar reasoning we may show that more generally

$$\mathfrak{J}_{[0]}(2xy, \cos\theta) = J_{[0]}(x^2e^{i\theta})\mathfrak{J}_{[0]}(y^2p^{-1}e^{-i\theta})+2\Sigma(-1)^n p^{n^2}J_{[n]}\mathfrak{J}_{[n]}, \tag{24a}$$

and other theorems of similar kind may be obtained by equating coefficients of $\kappa^n$ in (22).

## § 5.

The expansion $\dfrac{1}{(1-2x\cos\theta+x^2)^{\frac{1}{2}q-1}} = \sum\limits_{n=0}^{\infty} x^n P_n(q,\cos\theta)$ (25)

is well known (cf. *Proc. London Math. Soc.*, Vol. xxv., p. 57), and the properties of the coefficients $P_n(q,\cos\theta)$ have been investigated by Dr. Hobson. It may be of interest to form the basic coefficient analogous to $P_n(q,\cos\theta)$ and connect it by means of the $E$ function with other generalizations of $P_n$ and $J_n$. Take the basic analogue of the binomial theorem

$$\prod_{r=1}^{\infty}\frac{(1-p^r x)}{(1-p^{n+r}x)} = 1-p\frac{[n]}{[1]}x+p^3\frac{[n][n-1]}{[2]!}x^2-\dots,$$

and let us apply this to obtain the analogue of (25). Changing $p$ into $p^2$, and making $n = 1-\frac{1}{2}q$, we find

$$\prod_{r=1}^{\infty}\frac{(1-p^{2r}xe^{i\theta})}{(1-p^{2-q+2r}xe^{i\theta})} = 1-p^2\frac{[2-q]}{[2]}xe^{i\theta}+p^6\frac{[2-q][-q]}{[2][4]}x^2e^{2i\theta}-\dots. \quad (26)$$

In the usual way, by taking the product of two such series, we find

$$\prod_{r=1}^{\infty}\frac{(1-2p^{2r}x\cos\theta+x^2p^{4r})}{(1-2p^{2-q+2r}x\cos\theta+x^2p^{4-2q+4r})} = 1+\Sigma(-1)^n x^n p^{(4-q)n}P_{[n]}(q,\cos\theta), \quad (27)$$

in which

$$P_{[n]}(q,\cos\theta) = \frac{[q-2][q]\dots[q+2n-4]}{\{2n\}!}$$

$$\times\left\{2\cos n\theta+\frac{[q-2][2n]}{[2][q+2n-4]}2\cos(n-2)\theta+\dots\right\}. \quad (28)$$

The infinite products

$$\prod_{r=0}^{\infty}\frac{1}{(1-2xp^m\cos\theta+x^2p^{2m})} = 1+\Sigma\frac{A_r(\theta)}{(1-p^r)!}x^r,$$

$$\prod_{r=0}^{\infty}(1+2xp^{m+1}\cos\theta+x^2p^{2m+2}) = 1+\Sigma\frac{B_r(\theta)}{(1-p^r)!}x^r$$

have been discussed by Prof. L. J. Rogers (*Proc. London Math. Soc.*, Vols. xxiv., xxv.). In the notation of Rogers' papers we have

$$\left\{1+\Sigma\frac{A_r(\theta,p^2)}{\{2r\}!}\left(\frac{xp^{4-q}}{1-p}\right)^r\right\}\left\{1+\Sigma\frac{B_r(\theta,p^2)}{\{2r\}!}\left(\frac{x}{1-p}\right)^r\right\}$$

$$= 1+\Sigma(-1)^n x^n p^{(4-q)n}P_{[n]}(q,\cos\theta). \quad (29)$$

Equating coefficients of $x^n$, we see that

$$p^{(4-n)r}P_{[n]}(q,\cos\theta) = \frac{1}{(1-p)^n}\sum_{r=0}^{n}(-1)^r\frac{A_rB_{n-r}}{\{2r\}!\{2n-2r\}!}p^{(4-q)r}. \quad (30)$$

The discussion of this for general values of $n$ would be interesting, but is not attempted here.

The properties of the coefficients $A_r$, $B_r$ are discussed by Prof. L. J. Rogers (*loc. cit.*). Some interesting relations may be found between the coefficients $J_{[n]}$ and $A_n$. We have

$$\prod_{m=1}^{\infty}\{1+2\kappa x p^{-2m}\cos\theta+\kappa^2x^2p^{-4m}\}^{-1}$$

$$=\left\{1+\frac{\kappa x e^{i\theta}}{p-1}+\frac{\kappa^2x^2e^{2i\theta}}{(p-1)(p^2-1)}+\ldots\right\}\left\{1+\frac{\kappa x e^{-i\theta}}{p-1}+\ldots\right\}$$

$$=1+\frac{\kappa x}{(p-1)}A_1(\theta)+\frac{\kappa^2x^2}{(p-1)(p^2-1)}A_2(\theta)+\ldots, \tag{31}$$

$$A_r(\theta)=2\cos r\theta+2\frac{[r]}{[1]}\cos(r-2)\theta+2\frac{[r][r-1]}{[2]!}\cos(r-4)\theta+\ldots.$$

Consider the product $\qquad E_{p^2}\left(\frac{\kappa\lambda}{[2]t}\right)E_{1\,p^2}\left(-\frac{\lambda}{[2]\kappa t}\right).$ (32)

Since $E_p(a)E_{p^{-1}}(-a)=1$, we can write (32) as

$$\frac{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right)E_{p^2}\left(\frac{\kappa\lambda}{[2]t}\right)}{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right)E_{p^2}\left(\frac{\lambda}{[2]\kappa t}\right)}=\prod_{m=1}^{\infty}\frac{\{1+\kappa\lambda t(p-1)p^{-2m}\}\{1+\kappa\lambda t^{-1}(p-1)p^{-2m}\}}{\{1+\kappa\lambda t(p-1)p^{-2m}\}\{1+\kappa^{-1}\lambda t^{-1}(p-1)p^{-2m}\}}. \tag{33}$$

Replace $\lambda$ by $x/(p-1)$, $\kappa$ by $e^{i\phi}$. $t$ by $e^{i\theta}$; then the infinite product (33) is

$$\prod_{m=1}^{\infty}\frac{(1+2\kappa xp^{-2m}\cos\theta+\kappa^2x^2p^{-4m})}{(1+2xp^{-2m}\cos(\theta+\phi)+x^2p^{-4m})}, \tag{34}$$

while (32) becomes $\qquad \sum_{r=0}^{\infty}\kappa^rJ_{[r]}+\sum_{r=1}^{\infty}(-1)^rp^{r^2}\kappa^{-r}\mathfrak{J}_{[r]},$

the arguments of $J$ and $\mathfrak{J}$ being respectively $xe^{-i\theta}/(p-1)$ and $xe^{-i\theta}/(p^2-p)$ [cf. (22), p. 201, Ser. 2, Vol. 2]. Hence

$$\prod_{m=1}^{\infty}\frac{1}{(1+2xp^{-2m}\cos(\theta+\phi)+x^2p^{-4m})}$$

$$=\left\{1+\sum_{r=1}^{\infty}(-1)^r\frac{\kappa^rx^r}{(1-p^r)!}A_r(\theta)\right\}\left\{\sum_0^{\infty}\kappa^rJ_{[r]}+\sum_1^{\infty}(-1)^r\kappa^{-r}p^{r^2}\mathfrak{J}_{[r]}\right\}. \tag{35}$$

The infinite product on the left may be expressed in a series of generalized Bessel functions as

$$I_{[0]}\left(\frac{x}{p-p^2}\right)+\sum_{r=1}^{\infty}2p^{r^2}\cos(r\theta+r\phi)\,I_{[n]}\left(\frac{x}{p-p^2}\right), \tag{36}$$

in which $$I_{[n]}(a) = \sum_{r=0}^{\infty} \frac{a^{n+2r}}{\{2r\}!\,\{2n+2r\}!}\, p^{2r(n+r)},$$

because in $E$ function form the product is

$$\frac{1}{E_{p^2}\left(\frac{\kappa x t}{(p-1)[2]}\right) E_{p^2}\left(\frac{x\kappa^{-1}t^{-1}}{(p-1)[2]}\right)} = E_{p^{-2}}\left(-\frac{\kappa x t}{(p-1)[2]}\right) E_{p^{-2}}\left(-\frac{x\kappa^{-1}t^{-1}}{(p-1)[2]}\right)$$
$$= I_{[0]} + \sum_{n=1}^{\infty} (-1)^n p^{n^2} (\kappa^n t^n + \kappa^{-n} t^{-n})\, I_{[n]}.$$

Equating coefficients in (35) and (36), we find

$$I_{[0]}\left(\frac{x}{p-p^2}\right) = J_{[0]} - p\,\frac{x}{(p-1)}\,\mathfrak{J}_{[1]} A_1 + p^4\,\frac{x^2}{(p-1)(p^2-1)}\,\mathfrak{J}_{[2]} A_2 - \dots, \tag{37}$$

$$p^{n^2} e^{-ni\theta} I_{[n]} = p^n \mathfrak{J}_{[n]} - p^{(n+1)^2}\,\frac{x}{(p-1)}\,\mathfrak{J}_{[1]} A_1 + \dots, \tag{38}$$

in which the argument of $I$ is $x/(p-p^2)$, of $\mathfrak{J}$ is $xe^{-i\theta}/(p^2-p)$, of $J$ is $xe^{-i\theta}/(p-1)$.

## § 6.

### *Expansion of an Arbitrary Function in Series of Functions* $J_{[n]}$ or $\mathfrak{J}_{[n]}$.

If $|t| > |x|$,

$$\frac{1}{t-x} = \frac{1}{t} + \frac{x}{t^2} + \frac{x^2}{t^3} + \dots .$$

Now express each power of $x$ in a series of $J_{[n]}$ functions by Lemma II. as

$$\frac{x^n}{t^{n+1}} = \frac{\{2n\}!}{t^{n+1}} \left\{ J_{[n]} + \frac{[2n+4]}{[2]} J_{[n+2]} + p^2 \frac{[2n+8][2n+2]}{[2][4]} J_{[n+4]} + \dots \right\}.$$

Arrange the series according to the order of the $J$ functions. Terms involving $J_{[n]}$ will arise from the series which represent

$$\frac{x^n}{t^{n+1}},\quad \frac{x^{n-2}}{t^{n-1}},\quad \frac{x^{n-4}}{t^{n-3}},\quad \dots,$$

and these terms will give

$$J_{[n]}(x)\,\frac{\{2n\}!}{t^{n+1}} \left\{ 1 - \frac{t^2}{[2n-2][2]} + p^2 \frac{t^4}{[2n-2][2n-4][2][4]} - \dots \right\}. \tag{39}$$

If $O_{[n]}(t) = \dfrac{\{2n\}!}{t^{n+1}} \left\{ 1 + \dfrac{t^2}{[2][2n-2]} + p^2 \dfrac{t^4}{[2][4][2n-2][2n-4]} + \dots \right\}$

in which the $p$ prefix of the general term is $p^{r(r-1)}$, we may write

$$\frac{1}{t-x} = O_{[0]}(t)\,J_{[0]}(x) + \sum_{n=1}^{\infty} O_{[n]}(t)\,J_{[n]}(x) \tag{40}$$

(cf. Heine, *Kugelfunctionen*, Vol. I., p. 249). Thus

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-x}\,dt \\ &= \frac{1}{2\pi i}\int_C f(t)\left\{O_{[0]}(t)\,J_{[0]}(x) + \sum_{n=1}^{\infty} O_{[n]}(t)\,J_{[n]}(x)\right\}dt \\ &= a_0 J_{[0]}(x) + a_1 J_{[1]}(x) + a_2 J_{[2]}(x) + \dots, \end{aligned}$$

$$a_n = \frac{1}{2\pi i}\int_C O_{[n]}(t)\,f(t)\,dt. \tag{41}$$

In a similar way by means of (77), p. 218, Ser. 2, Vol. 2,

$$f(x) = \sum_{n=0}^{\infty} a_n P_{[n]}(x),$$

$$a_n = \frac{[2n+1]}{2\pi i}\int_C f(t)\,Q_{[n]}(t)\,dt \qquad (|\,t\,| > |\,x\,| > 1). \tag{42}$$

## § 7.

In connection with the theory of the function $E_p(x)$, we may show that

$$J_{[n]}(x) = E_p(ix)\,E_p(-ix)\,\mathfrak{J}_{[n]}(x),$$

and derive thence many interesting results. It is easily shown that

$$\begin{aligned} E_{p^{-1}}(-x)\,I_{[n]}(x) = \frac{x^n}{\{2n\}!}\Big\{1 - x + \frac{[2n+3]}{[2][2n+2]}x^2 - \frac{[2n+5]}{[3]!\,[2n+2]}x^3 \\ + \frac{[2n+5][2n+7]}{[4]!\,[2n+2][2n+4]}x^4 - \dots\Big\}, \end{aligned} \tag{43}$$

$$I_{[n]}(x) = i^{-n} J_{[n]}(ix).$$

Inverting the base $p$, we obtain

$$E_p(-x)\,\mathrm{I}_{[n]}(x) = \frac{x^n}{\{2n\}!}\left\{1 - x + \frac{[2n+3]}{[2][2n+2]}x^2 - \dots\right\}, \tag{43a}$$

in which $\mathrm{I}_{[n]}(x) = i^{-n}\,\mathfrak{J}_{[n]}(ix),$

because $\dfrac{x^n}{\{2n\}!}$ becomes $p^{n^2}\dfrac{x^n}{\{2n\}}$, $I_{[n]}(x)$ becomes $p^{n^2}\mathrm{I}_{[n]}(x)$, and the series within the large brackets of (43) is unchanged by the inversion of $p$.

Hence
$$E_{p^{-1}}(-x)\, I_{[n]}(x) = E_p(-x)\, \mathrm{I}_{[n]}(x), \tag{44}$$

and it easily follows that
$$E_{p^{-1}}(-ix)\, J_{[n]}(x) = E_p(-ix)\, \mathfrak{J}_{[n]}(x), \tag{45}$$

also
$$J_{[n]}(x) = E_p(ix)\, E_p(-ix)\, \mathfrak{J}_{[n]}(x); \tag{46}$$

so that the extension of the series for Lommel's product $J_m(x)\, J_{[n]}(x)$ (*Proc. R.S.*, Vol. LXXIV., p. 67) may be written

$$\begin{aligned} &E_p(ix)\, E_p(-ix)\, \mathfrak{J}_{[m]}\, \mathfrak{J}_{[n]} \\ &\quad = E_{p^{-1}}(ix)\, E_{p^{-1}}(-ix)\, J_{[m]}\, J_{[n]} \\ &\quad = \sum_{r=0}^{\infty} (-1)^r \frac{\{2m+2n+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!}\, x^{m+n+2r}. \end{aligned} \tag{47}$$

It follows also that
$$J_{[m]}\, J_{[n]} = E_p^2(ix)\, E_p^2(-ix)\, \mathfrak{J}_{[m]}\, \mathfrak{J}_{[n]}.$$

At this point we see that interesting equations may be formed giving relations between the functions $J_{[n]}$ and analogous functions which may be conveniently termed multiple Bessel functions. The notation for such a function will be

$$J^p_{m,n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2n+2m+4r}}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!}. \tag{48}$$

In the denominator of the general term we have the same expression as appears in the denominator of the general term of (47), which is a generalization of a series due to Lommel.

When $m$ and $n$ are integers the function may be derived from the function $J_{[n]}$ in the same way that $J_{[n]}$ was derived from $E$ (cf. p. 196, Ser. 2, Vol. 2).

Consider the product $J_{[n]}(xt)\, J_{[n]}(xt^{-1})$.

We may arrange in a series of ascending and descending powers of $t$ as

$$J^p_{n,0}(x) + \sum_{m=1}^{\infty} (-1)^m (t^{2m} + t^{-2m})\, J^p_{n,m}(x). \tag{49}$$

Consider the equations

$$E_{p^2}\left(\frac{xt}{p+1}\right) E_{p^2}\left(-\frac{xt^{-1}}{p+1}\right) = J_{[0]}(x) + \sum_{n=1} (t^n + t^{-n})\, J_{[n]}(x), \tag{50}$$

$$E_{p^{-2}}\left(\frac{xt}{p+1}\right) E_{p^{-2}}\left(-\frac{xt^{-1}}{p+1}\right) = \mathfrak{J}_{[0]}\left(\frac{x}{p}\right) + \Sigma p^{n^2} (t^n + t^{-n})\, \mathfrak{J}_{[n]}\left(\frac{x}{p}\right). \tag{50a}$$

Since
$$E_p\mathfrak{J}_{[n]} = E_{p\ 1}J_{[n]} \tag{51}$$

[cf. (45)], we can write the quotient of the two expressions on the left sides of (50) and (50*a*) as

$$\frac{J^{p^2}_{[n]}\left(\frac{ixt}{p+1}\right)J^{p^2}_{[n]}\left(-\frac{ixt^{-1}}{p+1}\right)}{\mathfrak{J}^{p^2}_{[n]}\left(\frac{ixt}{p+1}\right)\mathfrak{J}^{p^2}_{[n]}\left(-\frac{ixt^{-1}}{p+1}\right)}, \tag{52}$$

the index $p^2$ being necessary to indicate that the base of the functions is $p^2$. This quotient may be expressed in series of $J_{m,n}$ functions by means of result (49), viz., as

$$\frac{J^{p^2}_{n,0}\left(\frac{ix}{[2]}\right)+\sum_{m=1}^{\infty}(t^{2m}+t^{-2m})(-1)^m J_{n,m}\left(\frac{ix}{[2]}\right)}{\mathfrak{J}^{p^2}_{n,0}\left(\frac{ix}{[2]}\right)+\sum_{m=1}^{\infty}(t^{2m}+t^{-2m})(-1)^m \mathfrak{J}_{n,m}\left(\frac{ix}{[2]}\right)}. \tag{53}$$

$\mathfrak{J}_{n,m}$ denotes the function derived from the product $\mathfrak{J}_{[n]}(xt)\,\mathfrak{J}_{[n]}(xt^{-1})$ in the same way that $J_{n,m}$ was derived from $J_{[n]}$.

Equating the above expression to the quotient of the two series which form the right sides of equations (50) and (50*a*), we obtain from cross multiplication, equating constant terms (independent of *t*),

$$\begin{aligned}J_{[0]}(x)\,\mathfrak{J}^{p^2}_{n,0}\left(\frac{ix}{[2]}\right)+2\sum_{m=1}^{\infty}(-1)^m J_{[2m]}\,\mathfrak{J}^{p^2}_{n,m}\qquad&\\ =\mathfrak{J}_{[0]}J^{p^2}_{n,}\ +2\sum_{m=1}^{\infty}(-)^m p^{2m^2}\,\mathfrak{J}_{[2m]}J^{p^2}_{n,m}.&\end{aligned} \tag{54}$$

I do not propose here to enter any further into the theory of multiple *J* functions; but I consider that the simple relations

$$\left.\begin{aligned}J_{[n]}(x)\,\mathfrak{J}_{[m]}(x) &= \mathfrak{J}_{[m]}(x)\,J_{[n]}(x)\\ \frac{J_{[n]}(x)}{\mathfrak{J}_{[n]}(x)} &= \frac{E_p(ix)}{E_{p^{-1}}(ix)}\\ E_p(x)E_{p^{-1}}(y) &= 1+\frac{(x+y)}{[1]}+\frac{(x+y)(x+py)}{[2]!}+\dots\end{aligned}\right\} \tag{55}$$

will afford a considerable field for investigation, in forms connecting theta function products with the basic or *q* forms of Bessel's and Legendre's functions.

Taking the product of (43) and (43*a*),

$$E_p(-x)\,E_{p^{-1}}(-x)\,\mathrm{I}_{[n]}(x)\,I_{[m]}(x)$$

$$=\frac{x^{m+n}}{\{2m\}!\,\{2n\}!}\left\{1-x+\frac{[2m+3]}{[2][2m+2]}x^2-\ldots\right\}\left\{1-x+\frac{[2n+3]}{[2][2n+2]}x^2-\ldots\right\}. \tag{56}$$

The left side of this may be written

$$\left\{1-\frac{2x}{[1]}+\frac{2(1+p)}{[2]!}x^2-\frac{2(1+p)(1+p^2)}{[3]!}x^3+\ldots\right\}$$

$$\times\sum_{r=0}^{\infty}\frac{\{2n+2m+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!}\,x^{m+n+2r}.$$

Also, since $\qquad E_p(x)E_{p^{-1}}(x)=\dfrac{1}{E_{p^{-1}}(-x)\,E_p(-x)}$,

$\displaystyle\sum_{r=0}^{\infty}\frac{\{2n+2m+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!}\,x^{m+n+2r}$ may be expressed as the product of three series, viz.,

$$\frac{x^{m+n}}{\{2m\}!\,\{2n\}!}\left\{1-x+\frac{[2m+3]}{[2][2m+2]}x^2-\ldots\right\}\left\{1-x+\frac{[2n+3]}{[2][2n+2]}x^2-\ldots\right\}$$

$$\times\left\{1+\frac{2x}{[1]}+\frac{2(1+p)}{[2]!}x^2+\frac{2(1+p)(1+p^2)}{[3]!}x^3+\ldots\right\}. \tag{57}$$

By equating coefficients and by substituting *ix* for *x*, various identities may be deduced from these series: for example, taking

$$2C(x)=E_p(ix)+E_p(-ix)=1-\frac{x^2}{[2]!}+\frac{x^4}{[4]!}-\ldots,$$

$$2iS(x)=E_p(ix)-E_p(-ix)=x-\frac{x^3}{[3]!}+\ldots,$$

$$J_{[n]}(x)\doteq\frac{x^nC(x)}{(2)_n\Gamma_p(n+1)}\left\{1-\frac{[2n+3]}{[2n+2]}\frac{x^2}{[2]!}+\ldots\right\}$$

$$+\frac{x^nS(x)}{(2)_n\Gamma_p(n+1)}\left\{x-\frac{[2n+5]}{[2n+2]}\frac{x^3}{[3]!}+\ldots\right\}. \tag{58}$$

This theorem reduces, when $p=1$, to a well known theorem in Bessel's function theory. I hope that the results given may be interesting not merely because of their remarkable likeness to well known results in the case of ordinary functions $J_n$, $P_n$, …, but because the theory links the functions $J_n$, $P_n$, …, by means of their basic or $q$ generalizations, with the $q$ theory of elliptic functions.

*Addition to the preceding paper.*

[Received January 18th, 1905.]

# EXPANSIONS OF FUNCTIONS IN SERIES OF BASIC BESSEL COEFFICIENTS

WE may show that, if $f(x)$ be a function capable of expansion in an absolutely convergent series

$$\sum_0^\infty a_s x^s,$$

then, on replacing each power of $x$ by its expression in an infinite series of $J_{[n]}$ functions, we obtain an infinite series $\sum_0^x b_s J_{[s]}$, and that the absolute convergence of this series depends only on the absolute convergence of $\Sigma\, a_s x^s$ (cf. Gray and Mathews, *Treatise on Bessel Functions*, pp. 19–20, 29–30; Neumann, *Leipzig Berichte*, 1869). A similar generalization of Neumann's result

$$\sum_0^\infty a_s x^s = \sum_0^\infty (b_{2s} J_{[s]}^2 + b_{2s+1} J_s J_{s+1})$$

will be given.

Firstly, we may establish without difficulty the three following identities:—

$$\begin{aligned}\frac{x^n}{\{2n\}!} &= J_{[n]} + \frac{[2n+4]}{[2]} J_{[n+2]} + \dots \\ &\quad + p^{s(s-1)}[2n+4s]\frac{[2n+2s-2][2n+2s-4]\dots[2n+2]}{[2][4]\dots[2s]} J_{[n+2s]} + \dots \\ &= \Sigma\, c_s J_{[n+2s]} \qquad (a)\end{aligned}$$

[cf. Gray and Mathews, *Bessel Functions*, (47), p. 19],

$$\begin{aligned}\frac{x^{2n}}{\{2n\}!\,\{2n\}!} &= J_{[n]}\mathfrak{J}_{[n]} + \frac{[4n+4]}{[2]} J_{[n+1]}\mathfrak{J}_{[n+1]} + \dots \\ &\quad + p^{s(s-1)}[4n+4s]\frac{[4n+2s-2][4n+2s-4]\dots[4n+2]}{[2][4]\dots[2s]} J_{[n+s]}\mathfrak{J}_{[n+s]} + \dots, \qquad (\beta)\end{aligned}$$

$$\begin{aligned}\frac{x^{2n-1}}{\{2n\}!\,\{2n-2\}!} &= J_{[n-1]}\mathfrak{J}_{[n]} + \frac{[4n+2]}{[2]} J_{[n]}\mathfrak{J}_{[n+1]} + \dots \\ &\quad + p^{s(s-1)}[4n][4n+4s-2]\frac{[4n+2][4n+4]\dots[4n+2s-4]}{[2][4]\dots[2s]} \\ &\qquad \times J_{[n+s-1]}\mathfrak{J}_{[n+s]} + \dots \qquad (\gamma)\end{aligned}$$

[cf. Gray and Mathews, *Bessel Functions*, (75), (76), p. 29].

In series ($a$), if we take out from the $J$ series in each term the part involving $x^{n+2r}$, we find that the coefficient of $x^{n+2r}$ is a series of $r+1$ terms, which reduces identically to zero by a simple property of Heine's series. Moreover, it may be shown that, if $S_h$ denote the sum of the first $h$ terms of series ($a$),

$$\{2n\}!\,S_h = x^n + \frac{\{2n+2h\}!}{\{2n+4h+2\}!\,\{2h+2\}!}\,x^{n+2h+2}$$
$$\times\left[1-\frac{x^2}{[2h+4][2n+2h+4][2]}+\dots\right].$$

The series within the large brackets on the right is absolutely convergent for all values of $x$ if $p>1$, and for limited values of $x$ if $p=1$ or $p<1$. In case $p>1$ or $p=1$ the right side may be made as near to $x^n$ as we please by taking $h$ large enough. The series $\sum\limits_{s=h+1}^{\infty} c_s J_{[n+2s]}$ becomes infinitesimal by taking $h$ infinitely large. The relation

$$x^n = \{2n\}!\sum_{0}^{\infty} c_s J_{[n+2s]}$$

holds then for all finite values of $x$ if $p \geqslant 1$.

In series ($a$), if we invert the base $p$, we find (since $\{2n\}!$ becomes $p^{-n^2}\{2n\}!$ and $J_{[n]}$ becomes $p^{n^2}\mathfrak{J}_{[n]}$) that

$$\frac{x^n}{\{2n\}!} = \mathfrak{J}_{[n]} + p^{2n+2}\frac{[2n+4]}{[2]}\mathfrak{J}_{[n+2]}+\dots$$
$$+p^{s(3s-1)+n(n+2s)}[2n+4s]\frac{[2n+2s-2]\dots[2n+2]}{\{2s\}!}\mathfrak{J}_{[n+2s]}+\dots$$
$$=\Sigma\, c_s\,\mathfrak{J}_{[n+2s]}.$$

In case $p<1$ this series is absolutely convergent.

Now suppose $\quad f(x) = a_0+a_1x+a_2x^2+\dots$

an absolutely convergent series; then, on substituting for each power of $x$ its expression in $J^p$ functions ($p>1$) or $\mathfrak{J}^p$ functions ($p<1$), we obtain in the case of the $J$ functions a series $\sum\limits_{0}^{\infty} b_s J_{[s]}$, where

$$b_s = \{2s\}!\left[a_s+\frac{a_{s-2}}{[2s-2][2]}+\frac{a_{s-4}}{[2s-2][2s-4][2][4]}+\dots\right].$$

If the series are both absolutely convergent, we may put

$$\Sigma a_s x^s = \Sigma b_s J_{[s]}.$$

An important special case is when the series $\Sigma a_s x^s$ satisfies Cauchy's first test of convergence,

$$a_s x / a_{s-1} < k,$$

where $k$ is a definitely assigned proper fraction.

In this case

$$L \frac{b_s J_{[s]}}{b_{s-1} J_{[s-1]}}$$

$$= L \frac{\{2s\}!\left[a_s + \frac{1}{[2s-2]} a_{s-2} + \dots\right]\left\{\frac{x^s}{\{2s\}!} - \frac{x^{s+2}}{\{2s+2\}!\,\{2\}!} + \dots\right\}}{\{2s-2\}!\left[a_{s-1} + \frac{1}{[2s-4]} a_{s-3} + \dots\right]\left\{\frac{x^{s-1}}{\{2s-2\}!} - \frac{x^{s+2}}{\{2s\}!\,\{2\}!} + \dots\right\}}$$

$$= \frac{a_s}{a_{s-1}} x,$$

and the series $\Sigma b_s J_{[s]}$ is absolutely convergent.

Similar arguments will suffice to show that we may expand $f(x)$ in a series

$$\Sigma b_s \mathfrak{J}_{[s]} \quad (p < 1).$$

In the same manner, if in series ($\beta$) and ($\gamma$) we replace each term on the right sides of these equations by a series of powers of $x$ of the form

$$J_{[n]} \mathfrak{J}_{[m]} = \sum_{r=0}^{\infty} (-1)^r \frac{\{2m+2n+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!} x^{m+n+2r},$$

the terms involving $x^{2n+2r}$ in ($\beta$) are $r+1$ in number, and the sum reduces identically to zero by a property of Heine's series, and similar arguments to those used above would suffice to give us a generalization of Neumann's expansion

$$\Sigma a_s x^s = \Sigma (b_{2s} J_s^2 + b_{2s+1} J_s J_{s+1}).$$

I do not, however, discuss this case, my aim being to justify in one case the expansion (41) of the previous section.

# ALTERNANTS AND CONTINUOUS GROUPS

*By* H. F. BAKER.

[Received January 8th, 1905.—Read January 12th, 1905.]

THE present paper is concerned with the proof of a fundamental proposition of non-commutative algebra, and with shewing that this leads at once to various results fundamental for the theory of continuous groups which are usually developed only with much more detailed considerations than are here employed.

One of the most noticeable peculiarities of the algebra of two non-commutative quantities $A$, $B$ is the occurrence of the alternant

$$(A, B) = AB-BA\ ;$$

from this we may form the alternants $(A, (A, B))$, $(B, (A, B))$, and so on, each new quantity being the alternant of a previous quantity with either $A$ or $B$; and, if these be called simple alternants, we may proceed to form alternants of simple alternants; it is, however, easy to see that these are expressible by simple alternants with numerical coefficients. Now it is an obvious suggestion of Lie's theory of groups that the product

$$e^A e^B = \left(1+A+\frac{A^2}{2!}+\ldots\right)\left(1+B+\frac{B^2}{2!}+\ldots\right)$$

is of the form $e^C$, where $C$ is a series of alternants of $A$ and $B$. For, if, in the ordinary notation,

$$X = e_1 X_1+\ldots+e_r X_r, \qquad X' = e'_1 X_1+\ldots+e'_r X_r, \qquad X'' = e''_1 X_1+\ldots+e''_r X_r,$$

we have

$$e^X e^{X'} x_i = e^X f_i(x, e') = f_i[f(x, e), e'] = f_i(x, e'') = e^{X''} x_i,$$

and the ground for this simplification can only be the alternant relations

$$(X_\rho, X_\sigma) = \Sigma c_{\rho\sigma\tau} X_\tau.$$

The fact that the product $e^A e^B$ is capable of the form $e^C$ has been insisted on by Mr. Campbell in his recent interesting book on groups, and he has given an elementary proof of it (*Proc. London Math. Soc.*, Vol. XXIX., 1898, p. 14), which cannot, however, I think, be regarded as final. It follows from this that the terms of any the same dimension in $A$ and $B$ in the expansion of

$$(e^A e^B - 1) - \tfrac{1}{2}(e^A e^B - 1)^2 + \tfrac{1}{3}(e^A e^B - 1)^3 - \dots$$

must be simple alternants or sums of such with numerical coefficients; and to the fourth dimension we, in fact, find for this expansion

$$A + B + \tfrac{1}{2}(A, B) + \tfrac{1}{12}(A - B, (A, B)) - \tfrac{1}{24}(A, (B, (A, B))).$$

On the other hand, starting with Schur's formulæ for the first parameter group, and using the notation of the theory of matrices, I have, in several papers in these *Proceedings*, deduced from the theory of groups various propositions primarily for the parameter group; and, in particular, if $e''_\sigma = \psi_\sigma(e, e')$, which we shall write $e'' = \psi(e, e')$, be the equations of that group, with canonical variables $e$, $e''$ and canonical parameters $e'$, if $E$ be the matrix whose general element is

$$E_{\rho\sigma} = \sum_\tau c_{\sigma\tau\rho} e_\tau \quad (\rho, \sigma, \tau = 1, 2, \dots, r),$$

and $\Delta = e^E$, while $E'$, $\Delta'$ and $E''$, $\Delta''$ depend respectively on $e'$ and $e''$, as $E$, $\Delta$ depend on $e$, it has been shown (*Proc. London Math. Soc.*, Vol. xxxiv., 1901, p. 98) that

$$\Delta'' = \Delta'\Delta,$$

leading (*ib.*, Vol. xxxv., p. 333) to

$$E'' = (\Delta'\Delta - 1) - \tfrac{1}{2}(\Delta'\Delta - 1)^2 + \dots.$$

The present paper professes to show that these investigations for the matrix $E$, so long as they refer to the general group of unassigned order, have a much wider bearing, and that, in fact, they furnish a proof of the general theorem $e^A e^B = e^C$, of a simple character, and so also of the fundamental group property which arises when, as in Mr. Campbell's work, $A$ and $B$ are linear differential operators. But there is more than this: we have proved (*ib.*, Vol. xxxiv., 1902, p. 352) that the $E$ matrix of the row $Ee'$ is the alternant $(E, E')$; hence, if $E_1, \dots, E_m$ be each $E$ matrices, either of $e$, or $e'$, or of some other row, the $E$ matrix of the row $E_1 E_2 \dots E_m E e'$ is the alternant $(E_1, (E_2, \dots, (E_m, (E, E'))\dots))$; thus every alternant of $E$ matrices is an $E$ matrix. From this it follows that the equations of the first parameter group are of the form

$$e'' = e + e' + (H_0 + H_1 + \dots) E e',$$

where $H_n$ is a homogeneous polynomial of dimension $n$ in the matrices $E$, $E'$—in itself a remarkable result, since this is then the parameter group of every possible continuous group when the appropriate matrix $E$

is employed—but it also follows that every term in this expression gives rise to one of the terms in the expansion of

$$e^A e^B - 1 - \tfrac{1}{2}(e^A e^B - 1)^2 + \tfrac{1}{3}(e^A e^B - 1)^3 - \dots,$$

and that there are no terms in this latter but such as arise in this way. The series for $e''$ is manifestly simpler to deal with than the series of alternants: in this paper it is obtained without the assumption of any of the propositions of the theory of groups, or the use of the theory of matrices; and a law of recurrence for its terms, whereby any term is deducible from the preceding, is obtained. The result can then conversely be used to establish the existence of general functions possessing the group property, and in particular it gives incidentally the origin of Schur's formulæ. The method employed is of the kind usually called symbolical, the quantities being defined by their laws of operation and not belonging to the ordered aggregate of natural number; if it be not conceded that sufficient explanation is given to shew that these laws are in this case self-consistent, the work retains its validity when interpreted in terms of the theory of matrices; but it is claimed that the paper is a contribution to the calculus of alternants of any non-commutative quantities.

## 1. *Alternants and their Bases.*

The capitals $A$, $B$, $C$, ... denote any quantities which can be added and subtracted associatively and commutatively, and can be multiplied associatively and distributively, but not commutatively. We do not recognise the existence of solutions of the equations $xA = 1$ or $Ax = 1$. With each capital, $A$, we associate a quantity, $a$, which we call its *base*, and we call $A$ the *derivative* of $a$, either determining the other uniquely. By the base of a sum or difference of derivatives we understand the sum or difference of their bases; thus the base of $-A$ is $-a$, and the base of zero is zero, and conversely; and bases can be added associatively and commutatively. Further, with every alternant of capitals we associate a definite base of which it is the derivative; and a derivative is either a capital or an alternant of capitals. As we have not as yet introduced products of capitals and bases, it is legitimate to denote the base of $(A, B)$ by $Ab$; the identity $(A, B)+(B, A) = 0$ will then require $Ab+Ba = 0$; in particular $Aa = 0$. The base of an alternant $(A, (B, C))$ will then naturally be denoted by $Ad$, where $d$ is $Bc$, the base of $(B, C)$, and so may be denoted by $A(Bc)$; we shall, however, denote this by $ABc$; it being observed that the notation $Ab$ has as yet been introduced only for the case when $A$ is a single capital, and $b$ a base, there is no confusion possible of $ABc$ with the symbol $AB.c$ regarded as the base of $(AB, C)$ or $ABC-CAB$. In general we assign similarly, to an alternant

$$(A_1, (A_2, (\dots, A_n, B) \dots))),$$

a base denoted by $A_1 A_2 \dots A_n b$; if $c = A_k A_{k+1} \dots A_n b$, the alternant takes the form $(A_1, (A_2, \dots (A_{k-1}, C) \dots))$, and its base has the notation $A_1 \dots A_{k-1} c$; so that in the product $A_1 \dots A_n b$ the symbols are associative.

We shew next that this notation for the base of a simple alternant may conveniently be extended to the base of a compound alternant; as a compound alternant is expressible by a sum of simple alternants, its base is a sum of bases of the form considered; we shew, however, that its expression in a succinct form is equivalent to allowing the symbols in the base of a compound alternant to obey a distributive law. First, the alternant $((A, B), C)$ is easily verified to be the sum of $(A, (B, C))$ and $-(B, (A, C))$, of which the bases are respectively $ABc$ and $-BAc$; if then the base $(A, B)c$ be assigned to $((A, B), C)$, we have

$$(A, B)c = ABc - BAc \quad \text{or} \quad (AB-BA)c = ABc - BAc,$$

as is most natural. In passing we notice that the base of

$$((A, B), C) = -(C, (A, B))$$

is also denoted by $-CAb$; so that we have

$$ABc + BCa + CAb = 0.$$

In general, if $A = (A_1, (A_2, \dots (A_{n-1}, A_n) \dots))$ be any alternant, and $B$ be a single capital, we denote the base of $(A, B)$ by $Ab$, and we examine now the implications of this notation. Suppose, in expanded form,

$$A = \Sigma \pm K_1 K_2 \dots K_n,$$

where $K_1 K_2 \dots K_n$ is a permutation of $A_1 A_2 \dots A_n$, so that

$$Ab = (\Sigma \pm K_1 K_2 \dots K_n)\, b.$$

Any term $K_1 K_2 \dots K_n b$ is the base of an alternant $(K_1, (K_2, \dots (K_n, B) \dots))$; we show that

$$(A, B) = \Sigma \pm (K_1, (K_2, \dots (K_n, B) \dots));$$

when this is proved we can write

$$(\Sigma \pm K_1 K_2 \dots K_n) b = \Sigma \pm (K_1 K_2 \dots K_n b).$$

Let $P_{n-1}$ denote $(A_2, (A_3, \dots (A_{n-1}, A_n) \dots))$; then $(A, B)$, $= ((A_1, P_{n-1}), B)$, is, as just remarked, equal to $(A_1, (P_{n-1}, B)) - (P_{n-1}, (A_1, B))$; denote

$(A_1, B)$ by $C$, and assume the law of expansion just stated, which we wish to prove true for an alternant $(A, B)$ of $(n+1)$ dimensions, to be true for alternants $(P_{n-1}, B)$, $(P_{n-1}, C)$, of $n$ dimensions; we have remarked that it holds for $n = 3$. Supposing

$$P_{n-1} = \Sigma \pm H_1 \dots H_{n-1},$$

we have thence $(P_{n-1}, B) = \Sigma \pm (H_1, (\dots (H_{n-1}, B) \dots)),$

$$(P_{n-1}, C) = \Sigma \pm (H_1, (\dots (H_{n-1}, C) \dots)),$$

and thus

$$\begin{aligned}(A, B) &= (A_1, (P_{n-1}, B)) - (P_{n-1}, C) \\ &= \Sigma \pm (A_1, (H_1, (\dots (H_{n-1}, B) \dots))) - \Sigma \pm (H_1, (\dots (H_{n-1}, (A_1, B)) \dots)),\end{aligned}$$

while $A = (A_1, P_{n-1}) = \Sigma \pm A_1 H_1 \dots H_{n-1} - \Sigma \pm H_1 \dots H_{n-1} A_1$;

so that the result is established.

More generally, let $B = (B_1, (B_2, \dots (B_{m-1}, B_m) \dots))$ be any alternant, of which the base is $B_1 B_2 \dots B_{m-1} b_m$. When $A$ was a single capital, we agreed to denote the base of $(A, B)$ by $AB_1 \dots B_{m-1} b_m$; it is natural to seek to extend this to the case when, as above,

$$A = (A_1, (A_2, \dots (A_{n-1}, A_n) \dots)),$$

and to denote the base of $(A, B)$ by $(\Sigma \pm K_1 K_2 \dots K_n) B_1 B_2 \dots B_{m-1} b_m$; in fact, the proof given above for the expansion of $(A, B)$ when $B$ is a single capital holds equally here, and we have

$$\Sigma \pm (K_1 K_2 \dots K_n B_1 \dots B_{m-1} b_m) = (\Sigma \pm K_1 K_2 \dots K_n) B_1 B_2 \dots B_{m-1} b_m.$$

Again, if $C = (C_1, (C_2 \dots (C_{h-1}, C_h) \dots))$, the equation

$$\begin{aligned}\Sigma \pm C_1 C_2 \dots C_h K_1 K_2 \dots K_n B_1 \dots B_{m-1} b_m \\ &= C_1 C_2 \dots C_h (\Sigma \pm K_1 \dots K_n) B_1 \dots B_{m-1} b_m \\ &= C_1 C_2 \dots C_h \Sigma \pm (K_1 \dots K_n B_1 \dots B_{m-1} b_m)\end{aligned}$$

requires the obvious identity

$$\begin{aligned}\Sigma \pm (C_1, (C_2 \dots (C_h, (K_1 \dots (K_n, B) \dots)) \dots)) \\ &= (C_1, (C_2 \dots [C_h, (\Sigma \pm (K_1, (\dots (K_n, B) \dots)))] \dots)).\end{aligned}$$

Thus in a product or sum of products of the form $A_1 \dots A_{n-1} a_n$, where $A_1, \dots, A_n$ are single capitals, when the sum is of the form

$$A_1 \dots A_h P A_k \dots A_{n-1} a_n,$$

where $P$ is an alternant, the symbols obey the associative and distributive laws. The same is true when $A_1, \dots, A_n$ are themselves alternants, as may be similarly shewn. For instance, returning to the theorem of expansion proved above, the base of $(A, B)$ or $((A_1, (A_2 \dots (A_{n-1}, A_n) \dots)), B)$ may equally be denoted by $-BA_1 \dots A_{n-1}a_n$; let $p_{n-1} = A_2A_3 \dots A_{n-1}a_n$, so that the derivative $P_{n-1}$ is as above; then

$$-BA_1 \dots A_{n-1}a_n = -BA_1p_{n-1};$$

the alternants shew, as in the work above, that this is the same as

$$A_1P_{n-1}b+P_{n-1}Ba_1 = A_1P_{n-1}b-P_{n-1}A_1b;$$

but the base of $(A, B)$, $= ((A_1, P_{n-1}), B)$, is $(A_1, P_{n-1})\,b$; thus we have $(A_1P_{n-1}-P_{n-1}A_1)\,b$ equal to $A_1P_{n-1}b-P_{n-1}A_1b$. And so on. And by the way we notice the identities such as

$$BA_1 \dots A_{n-1}a_n+\Sigma \pm K_1 \dots K_nb = 0,$$

the general form of which is $Ab+Ba = 0$, including $Aa = 0$, where $A$, $B$ are any two derivatives, that is, either single capitals or alternants of capitals.

It is manifest that the notation of bases, once established, as here, furnishes a compendious way of expressing the relations among alternants, and may be of great use in expressing a compound alternant by simple alternants.

Many identities arise from the relation $Ab+Ba = 0$, by supposing $a$ to be a base of the form $A_1A_2 \dots A_ma_{m+1}$, and $b$ a base of the form $B_1B_2 \dots B_nb_{n+1}$, where $A_1, \dots, A_{m+1}, B_1, \dots, B_{n+1}$ are any derivatives. Restricting ourselves, as sufficient for purposes of illustration and as giving results to be utilised below, to the cases when each of $A_1, \dots, A_{m+1}$, $B_1, \dots, B_{n+1}$ is either $A$ or $B$, we find on examination that the following are all the identities so obtainable up to those of the sixth dimension.

(1) From $Aa = 0$, putting $Ab$ for $a$, we have $(AB-BA)\,Ab = 0$. This corresponds to the alternant relation $((A, B), (A, B)) = 0$, which is the same as $(A, (B, (A, B))) = (B, (A, (A, B)))$.

(2) Also from $Aa = 0$, by putting $A^2b$ for $a$, we have

$$(A, (A, B))\,A^2b = 0,$$

that is $$(A^2B-2ABA+BA^2)\,A^2b = 0,$$

or $$(A^3B-2ABA^2+BA^3)\,Ab = 0.$$

This corresponds to the identity

$$((A, (A, B)), (A, (A, B))) = 0,$$

which is equivalent to

$$(A,(A,(B,(A,(A,B)))))-2(A,(B,(A,(A,(A,B)))))$$
$$+(B,(A,(A,(A,(A,B))))) = 0,$$

an identity which we may agree to denote by

$$[1^2 21^2 2]-2[121^3 2]+[21^4 2] = 0.$$

By interchange of $A$ and $B$ we derive

$$(AB^3-2BAB^2+B^3A)Ab = 0.$$

(3) From $Ab+Ba = 0$, putting $A^2b$ for $a$, $BAb$ for $b$, we derive

$$(A,(A,B))BAb+(B,(A,B))A^2b = 0,$$

which is the same as

$$(A^2B^2-B^3A^2+2BABA+BA^2B-2ABAB-AB^2A)Ab = 0,$$

or, in virtue of $(AB-BA)Ab = 0$, the same as

$$(A^2B^2-B^3A^2+3BA^2B-3AB^2A)Ab = 0.$$

This corresponds to the identity

$$((A,(A,B)),(B,(A,B)))+((B,(A,B)),(A,(A,B))) = 0,$$

which expanded is the same as

$$[1^2 2^2 12]-[2^2 1^3 2]+3[21^2 212]-3[12^2 1^2 2] = 0.$$

The identities

$$S(A,B)Ab = (A^3B+BA^3-2ABA^2)Ab = 0,$$
$$T(A,B)Ab = (A^2B^2-B^2A^2+3BA^2B-3AB^2A)Ab = 0$$

are not really independent; for we find

$$S(A+B,B)Ab = [S(A,B)-S(B,A)+T(A,B)]Ab,$$
$$T(A+B,B)Ab = [T(A,B)-2S(B,A)]Ab,$$

while $$(A+B)b = Ab.$$

In what has preceded we have considered only bases of the form $Ab$, wherein $A$ and $B$ are alternants (or simple capitals); and, if

$$A = \Sigma \pm K_1K_2\dots K_n,$$

it has been shown that $Ab = \Sigma \pm (K_1K_2\dots K_n b)$.

If now $P_1$, $P_2$, ... be simple capitals or alternants, it is natural to regard $(P_1P_2\dots P_h+P_{h+1}P_{h+2}\dots P_k+\dots)b$ as an appropriate notation for the base which is the sum of the bases $P_1P_2 \dots P_h b$, $P_{h+1}P_{h+2} \dots P_k b$, …; so that its derivative will be

$$(P_1, (P_2, \dots (P_h, B) \dots)) + (P_{h+1}, (P_{h+2}, \dots (P_k, B) \dots)) + \dots .$$

When $P_1P_2 \dots P_h + P_{h+1}P_{h+2} \dots P_k + \dots$ is an alternant we have shewn this to be legitimate; we extend it to the general case, for which there is no corresponding condition for alternants satisfied, the derivative not being equal to

$$((P_1 \dots P_h + P_{h+1} \dots P_k + \dots), B);$$

this expression, in fact, not being an alternant of alternants, is not under consideration.

A particular case of this definition is

$$c = (\lambda_0 + \lambda_1 A + \lambda_2 A^2 + \dots)\, b,$$

where $A$ is a derivative; this we regard as the base of

$$C = \lambda_0 B + \lambda_1 (A, B) + \lambda_2 (A, (A, B)) + \dots .$$

If $\mu_0, \mu_1, \dots$ be such functions of $\lambda_0, \lambda_1, \dots$ that

$$(\mu_0 + \mu_1 A + \mu_2 A^2 + \dots)(\lambda_0 + \lambda_1 A + \lambda_2 A^2 + \dots) = 1,$$

we have

$$b = (\mu_0 + \mu_1 A + \mu_2 A^2 + \dots)\, c,$$

and the substitution in the series above given for $C$ of the series

$$B = \mu_0 C + \mu_1 (A, C) + \mu_2 (A, (A, C)) + \dots$$

reduces that series to $C$.

## 2. *Of the Substitutional Operation.*

We introduce a symbol to express the operation of replacing a base $a$ by a base $b$, and write

$$\left(b \frac{\partial}{\partial a}\right) a = b;$$

the operation of replacing the capital $A$ by the capital $B$ is also considered, and may be denoted by the same symbol, so that

$$\left(b \frac{\partial}{\partial a}\right) A = B;$$

and, if $P$ be a product of capitals, some of which are $A$, by the effect of the operator upon $P$ is meant a sum of terms each differing from $P$ by the substitution of $B$ for $A$ in one factor without change of order; for instance,

$$\left(b \frac{\partial}{\partial a}\right) ABA = B^2A + AB^2, \quad \left(b \frac{\partial}{\partial a}\right) A^2Ca = BACa + ABCa + A^2Cb.$$

Denoting $\left(b\frac{\partial}{\partial a}\right)$ by $\delta$, and the operation of forming the derivative $A$ of $a$ by $\epsilon$, so that $\epsilon a = A$, we have

$$\delta\epsilon a = \delta A = B = \epsilon b = \epsilon\delta a.$$

Let the alternant $(A, (A, \ldots (A, C) \ldots))$, where $A$ occurs $m$ times, be, for brevity, denoted by $(A_m C)$; then

$$\delta\epsilon(Ac) = \delta(AC-CA) = BC-CB = \epsilon(Bc) = \epsilon\delta(Ac),$$

$$\delta\epsilon(A^m c) = \delta(A_m C) = (B, (A_{m-1}C)) + (A, (B, (A_{m-2}C))) + \ldots + (A_{m-1}(B, C))$$

$$= \epsilon(BA^{m-1}c + ABA^{m-2}c + \ldots + A^{m-1}Bc) = \epsilon\delta(A^m c),$$

$$\delta\epsilon(A^n DA^m c) = \delta(A_n(D(A_m C))) = \delta_1(A_n(D(A_m C))) + \delta_2(A_n(D(A_m C))),$$

where $\delta_1$ operates on $A$ occurring under $A_n$, and $\delta_2$ on $A$ occurring under $A_m$,

$$= \epsilon\delta_1(A^n DA^m c) + \epsilon\delta_2(A^n DA^m c) = \epsilon\delta(A^n DA^m c);$$

and so on. Thus, if $F(A)$ be a symbol for the sum of a finite or an infinite number of products of capitals, of which each product involves $A$, and the other capitals in the product are unaffected by $\delta$, we infer that

$$\delta\epsilon F(A)c = \epsilon\delta F(A)c.$$

We shall presently be concerned with a base $b$ which is a particular case of the form $F(A)c$, namely, is

$$b = (\lambda_0 + \lambda_1 A + \lambda_2 A^2 + \ldots)c;$$

then
$$B = \lambda_0 C + \lambda_1(A_1 C) + \lambda_2(A_2 C) + \ldots,$$

and
$$\delta^2 a = \left(b\frac{\partial}{\partial a}\right)^2 a = \delta b = [\lambda_1 B + \lambda_2(BA+AB) + \ldots]c$$

is also, clearly, on substitution for $B$, of the form $F(A)c$, and the same is similarly true of every expression $\delta^m a$; thus

$$\epsilon\delta^2 a = \epsilon\delta(\delta a) = \epsilon\delta b = \delta\epsilon(\delta a) = \delta(\epsilon\delta a) = \delta(\delta\epsilon a) = \delta^2\epsilon a,$$

$$\epsilon\delta^3 a = \epsilon\delta(\delta^2 a) = \delta\epsilon(\delta^2 a) = \delta(\epsilon\delta^2 a) = \delta^3\epsilon a,$$

and so on; and in general $\epsilon\delta^m a = \delta^m\epsilon a$.

Now, with the same $b$, let $t$ be an ordinary number, and put

$$f = \left[1 + t\left(b\frac{\partial}{\partial a}\right) + \frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2 + \ldots\right]a,$$

so that

$$F = \epsilon a + t\epsilon\delta a + \frac{t^2}{2!}\epsilon\delta^2 a + \ldots = \left[1 + t\left(b\frac{\partial}{\partial a}\right) + \frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2 + \ldots\right]A.$$

Put, further,
$$\nabla = \mu_0 + \mu_1 F + \mu_2 F^2 + \ldots,$$

so that, after substitution for $F$ and arrangement according to powers of $t$,

$$\nabla = \nabla^{t=0} + t\left(\frac{d\nabla}{dt}\right)_{t=0} + \frac{t^2}{2!}\left(\frac{d^2\nabla}{dt^2}\right)_{t=0} + \dots .$$

We have $\dfrac{dF}{dt} = [\delta + t\delta^2 + \dots]A = \delta[1 + t\delta + \dots]A = \delta F,$

and thus

$$\frac{d\nabla}{dt} = \mu_1\frac{dF}{dt} + \mu_2\left(\frac{dF}{dt}F + F\frac{dF}{dt}\right) + \mu_3\left(\frac{dF}{dt}F^2 + F\frac{dF}{dt}F + F^2\frac{dF}{dt}\right) + \dots$$

$$= \mu_1\delta F + \mu_2(\delta F.F + F.\delta F) + \mu_3(\delta F.F^2 + F\delta F.F + F^2\delta F) + \dots = \delta\nabla,$$

from which $\quad \left(\dfrac{d^m\nabla}{dt^m}\right)_{t=0} = (\delta^m\nabla)_{t=0} = \delta^m\nabla_{t=0}.$

Therefore, the substitution

$$F = \left[1 + t\left(b\frac{\partial}{\partial a}\right) + \frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2 + \dots\right]A$$

in

$$\nabla = \mu_0 + \mu_1 F + \mu_2 F^2 + \dots$$

gives

$$\nabla = \left[1 + t\left(b\frac{\partial}{\partial a}\right) + \frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2 + \dots\right]\nabla_0$$

where

$$\nabla_0 = \mu_0 + \mu_1 A + \mu_2 A^2 + \dots .$$

Here $b$ is any series of the form

$$b = (\lambda_0 + \lambda_1 A + \lambda_2 A^2 + \dots)c.$$

We shall presently utilise this result. It is convenient to give here another particular result. If $b$ reduce to $Ac$, and $F(A, C)$, or simply $F$, denote any sum of terms each of which is a product of powers of $A$ and $C$, with a numerical coefficient, we have, as we shall now prove,

$$\left(b\frac{\partial}{\partial a}\right)F(A, C) = FC - CF = (F, C).$$

In particular, if $F(A, C)$ reduce to $A$,

$$\left(b\frac{\partial}{\partial a}\right)A = B = (A, C).$$

To prove the result, notice

$$\left(b\frac{\partial}{\partial a}\right)A^2 = (A, C)A + A(A, C) = A^2C - CA^2 = (A^2, C),$$

and, if $\quad \left(b\dfrac{\partial}{\partial a}\right)H = (H, C), \qquad \left(b\dfrac{\partial}{\partial a}\right)K = (K, C),$

then $\left(b\frac{\partial}{\partial a}\right)C^m HC^n = C^m(H, C)C^n = C^m HC^{n+1} - C^{m+1}HC^n = (C^m HC^n, C)$

and $\left(b\frac{\partial}{\partial a}\right)HK = (H, C)K + H(K, C) = HKC - CHK = (HK, C)$.

Another result to be utilised follows hence. We have

$$(A_2C) = (A, (A, C)) = A^2C - 2ACA + CA^2,$$

and in general, easily, by induction,

$$(A_rC) = \sum_{s=0}^{r}(-1)^s\binom{r}{s}A^{r-s}CA^s,$$

and so

$$\begin{aligned}
&(A_1C)\binom{m}{1}A^{m-1} + (A_2C)\binom{m}{2}A^{m-2} + (A_3C)\binom{m}{3}A^{m-3} + \ldots + (A_mC)\\
&= (AC - CA)\binom{m}{1}A^{m-1} + (A^2C - 2ACA + CA^2)\binom{m}{2}A^{m-2}\\
&\qquad + (A^3C - 3A^2CA + 3ACA^2 - CA^3)\binom{m}{3}A^{m-3} + \ldots\\
&\qquad + \left[A^mC - \binom{m}{1}A^{m-1}CA + \ldots + (-1)^{m-1}\binom{m}{1}ACA^{m-1} + (-1)^m CA^m\right]\\
&= CA^m[(1-1)^m - 1] + \binom{m}{1}ACA^{m-1}[(1-1)^{m-1}]\\
&\qquad\qquad + \binom{m}{2}A^2CA^{m-2}[(1-1)^{m-2}] + \ldots + A^mC\\
&= (A^m, C) = \left(b\frac{\partial}{\partial a}\right)A^m,
\end{aligned}$$

and thus, if $\phi(A) = \nu_0 + \nu_1 A + \nu_2 A^2 + \ldots$, $b = Ac$,

$$\left(b\frac{\partial}{\partial a}\right)\phi(A) = (A, C)\frac{\phi'(A)}{1!} + (A_2C)\frac{\phi''(A)}{2!} + (A_3C)\frac{\phi'''(A)}{3!} + \ldots,$$

where $\phi'(A) = \nu_1 + 2\nu_2 A + 3\nu_3 A^2 + \ldots$, $\phi''(A) = 2\nu_2 + 3.2.\nu_3 A + \ldots$.

This is the same as

$$\left(b\frac{\partial}{\partial a}\right)\phi(A) = B\frac{\phi'(A)}{1!} + (A_1B)\frac{\phi''(A)}{2!} + (A_2B)\frac{\phi'''(A)}{3!} + \ldots,$$

and this result, though proved for $B = (A_1C)$, can easily be shown to hold for arbitrary $B$. A particular case, given by Mr. Campbell, *Proc. London Math. Soc.*, Vol. xxix., 1897, p. 16, is

$$\begin{aligned}
BA^{m-1} + ABA^{m-2} + \ldots + A^{m-1}B = \binom{m}{1}BA^{m-1} &+ \binom{m}{2}(A_1B)A^{m-2}\\
&+ \binom{m}{3}(A_2B)A^{m-3} + \ldots + (A_{m-1}B).
\end{aligned}$$

### 3. *The Exponential Theorem.*

Applying the theorem just obtained for arbitrary $A$ and $B$,

$$\left(b\frac{\partial}{\partial a}\right)\phi(A) = B\frac{\phi'(A)}{1!}+(A_1B)\frac{\phi''(A)}{2!}+(A_2B)\frac{\phi'''(A)}{3!}+\dots,$$

to the case where for $\phi(A)$ we have

$$\Delta = 1+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\dots,$$

so that each of $\phi'(A)$, $\phi''(A)$, ... is also $\Delta$, we obtain

$$\left(b\frac{\partial}{\partial a}\right)\Delta = \left(B+\frac{(A,B)}{2!}+\frac{(A,(A,B))}{3!}+\dots\right)\Delta.$$

Now put, with arbitrary $a'$,

$$b = \left(1+\frac{A}{2!}+\frac{A^2}{3!}+\dots\right)^{-1}a' = \left(1-\frac{A}{2}+\frac{\varpi_1}{2!}A^2+\frac{\varpi_2}{4!}A^4+\dots\right)a',$$

where $\varpi_1 = \frac{1}{6}$, $\varpi_2 = -\frac{1}{30}$, ... are Bernoulli's numbers with associated signs; then, as

$$B+\frac{(A,B)}{2!}+\frac{(A,(A,B))}{3!}+\dots$$

is the derivative of $\left(1+\frac{A}{2!}+\frac{A^2}{3!}+\dots\right)b$

or $a'$, we shall obtain

$$B+\frac{(A,B)}{2!}+\frac{(A,(A,B))}{3!}+\dots = A'.$$

Thus, with the particular $b$ so defined from $A$ and $A'$, we have

$$\left(b\frac{\partial}{\partial a}\right)\Delta = A'\Delta,$$

and hence $\left(b\frac{\partial}{\partial a}\right)^m\Delta = A'^m\Delta.$

Next, put $a'' = \left[1+\left(b\frac{\partial}{\partial a}\right)+\frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\dots\right]a,$

and $\Delta'' = 1+A''+\frac{A''^2}{2!}+\dots;$

then, as follows from a result proved in the previous section (p. 33),

$$\Delta'' = \left[1 + \left(b\frac{\partial}{\partial a}\right) + \frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2 + \ldots\right]\Delta = \left(1 + A' + \frac{A'^2}{2!} + \ldots\right)\Delta,$$

which we appropriately write $\Delta'' = \Delta'\Delta$;

this is then the consequence of

$$a'' = \left[1 + \left(b\frac{\partial}{\partial a}\right) + \frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2 + \ldots\right]a,$$

where, $a$ and $a'$ being arbitrary,

$$b = \left(1 + \frac{A}{2!} + \frac{A^2}{3!} + \ldots\right)^{-1} a'.$$

This, however, gives

$$B = A' - \tfrac{1}{2}(A, A') + \frac{\varpi_1}{2!}(A_2A') + \frac{\varpi_2}{4!}(A_4A') + \ldots,$$

$$\left(b\frac{\partial}{\partial a}\right)b = \left\{-\frac{B}{2} + \frac{\varpi_1}{2}[BA] + \frac{\varpi_2}{4!}[BA^3] + \ldots\right\}a',$$

where $\qquad [BA^{m-1}] = BA^{m-1} + ABA^{m-2} + \ldots + A^{m-1}B,$

so that $\left(b\frac{\partial}{\partial a}\right)b$ is of the form $F(A, A')a'$, where $F(A, A')$ is a series of polynomials in $A$ and $A'$ with numerical coefficients. So therefore is every term $\left(b\frac{\partial}{\partial a}\right)^m a$. Hence, as $A'a' = 0$, we have an equation

$$a'' = a + a' + (H_0 + H_1 + H_2 + H_3 + \ldots)Aa',$$

where $H_r$ is a homogeneous polynomial in $A$ and $A'$, of dimensions $r$. As $A''$, the derivative of this, is a series of alternants, we infer therefore, with arbitrary non-commutative symbols $A$, $A'$, if

$$\Delta = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots, \qquad \Delta' = 1 + A' + \frac{A'^2}{2!} + \frac{A'^3}{3!} + \ldots,$$

and the expansion of

$$(\Delta'\Delta - 1) - \tfrac{1}{2}(\Delta'\Delta - 1)^2 + \tfrac{1}{3}(\Delta'\Delta - 1)^3 - \ldots$$

be arranged according to homogeneous polynomials in $A$ and $A'$ of ascending dimension, that each polynomial is an aggregate of alternants of $A$ and $A'$, any one such aggregate being the derivative of the terms of the same dimension in

$$a'' = a + a' + (H_0 + H_1 + H_2 + \ldots)Aa'.$$

For instance, as will be presently seen,

$$H_0 = -\tfrac{1}{2}, \quad H_1 = \tfrac{1}{12}(A-A'), \quad H_2 = \tfrac{1}{24}AA',$$

and we find on trial, as was remarked in the introduction, that to the fourth dimension the expansion in question is

$$A+A'-\tfrac{1}{2}(A, A')+\tfrac{1}{12}(A-A', (A, A'))+\tfrac{1}{24}(A, (A', (A, A'))).$$

The theorem has a converse. When $A$, $A'$ are perfectly general there exists no other derivative $A_{11}$ such that

$$\Delta_{11} = 1+A_{11}+\frac{A_{11}^2}{2!}+\ldots$$

is equal to $\Delta'\Delta$.

For otherwise $\Delta_{11} = \Delta''$, $\Delta_{11}^{-1}\Delta'' = 1$, and so, if we put

$$\psi(a, a') = a+a'+(H_0+H_1+H_2+\ldots)Aa',$$

and denote by $A_0$ the derivative of $\psi(a'', -a_{11})$, we shall have $\Delta_0 = 1$, namely,

$$\left(1+\frac{A_0}{2!}+\frac{A_0^2}{3!}+\ldots\right)A_0 = 0\,;$$

or, on multiplying by $1-\frac{1}{2}A_0+\frac{\varpi_1}{2!}A_0^2+\ldots$, we shall have $A_0 = 0$, namely, $\psi(a'', -a_{11}) = 0$. The only solution of this which holds for all values of $a''$ must be one for which the terms of any order are separately zero; in particular, from the terms of the first order $a''-a_{11} = 0$; while, conversely, if this be so, $\Delta'' = \Delta_{11}$.

### 4. *On certain Properties of $\psi(a, a')$, and the Determination of its Coefficients.*

The actual form of $\psi(a, a')$ can be determined from its definition

$$\left[1+\left(b\frac{\partial}{\partial a}\right)+\frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]a\,;$$

it is at once seen, however, that beyond the first few terms this is an intricate process, and, moreover, the terms then appear arranged according to ascending dimension in $a'$ only, and not, as will appear more proper, according to ascending dimension in $a$ and $a'$ jointly.

Some information appears on the face of the series. First we have

$$\psi(-a', -a) = -\psi(a, a').$$

For $\Delta^{-1}\Delta'^{-1}\Delta'\Delta = 1$, or $\Delta''^{-1} = \Delta^{-1}\Delta'^{-1}$, namely, as

$$\Delta''^{-1} = 1 - A'' + \frac{A''^2}{2!} - \dots,$$

we have $$-A'' = (\Delta^{-1}\Delta'^{-1} - 1) - \tfrac{1}{2}(\Delta^{-1}\Delta'^{-1} - 1)^2 + \dots.$$

Thus in the series

$$\psi(a, a') = a + a' + (H_0 + H_1 + H_2 + \dots)\, Aa'$$

$H_{2n}$ is symmetrical in $A$ and $A'$, while $H_{2n-1}$ is changed in sign by interchanging $A$ and $A'$. Next, the terms of the first dimension in $a'$ in $\psi(a, a')$ are

$$\left(b\frac{\partial}{\partial a}\right) a = \left(1 - \frac{A}{2} + \frac{\varpi_1}{2!}A^2 + \frac{\varpi_2}{4!}A^4 + \dots\right) a';$$

thus $H_0 = -\frac{1}{2}$, while $H_{2n}$ has no terms in $A^{2n}$ or $A'^{2n}$, and $H_{2n-1}$ contains the terms $\frac{\varpi_n}{(2n)!}(A^{2n-1} - A'^{2n-1})$. This gives $H_1 = \frac{1}{12}(A - A')$, and $H_2$ = numerical multiple of $AA' + A'A$; we have seen, however, that $(AA' - A'A)Aa' = 0$, so that $H_2$ is a numerical multiple of $AA'$.

To obtain further information we prove first two lemmas: ($\alpha$) if $a$, $a'$ be arbitrary and $c = \Delta a'$, then $C = \Delta A'\Delta^{-1}$, and hence

$$\Delta_c = 1 + C + \frac{C^2}{2!} + \dots = \Delta\Delta'\Delta^{-1};$$

($\beta$) we have $\psi(a', a) = \Delta\psi(a, a') = \psi(a, \Delta a')$, which are both included in

$$\psi(a, \Delta^\lambda a') = \Delta^{\lambda-1}\psi(a', a).$$

The result ($\alpha$) is obvious on forming the expression

$$\Delta A'\Delta^{-1} = \left(1 + A + \frac{A^2}{2!} + \dots\right) A' \left(1 - A + \frac{A^2}{2!} - \dots\right),$$

in which the terms of dimension $(n+1)$ are

$$\frac{A^n}{n!}A' - \frac{A^{n-1}}{(n-1)!}A'\frac{A}{1!} + \frac{A^{n-2}}{(n-2)!}A'\frac{A^2}{2!} - \dots + (-1)^n A'\frac{A^n}{n!},$$

which is $\frac{1}{n!}(A_n A')$, the derivative of $\frac{A^n a'}{n!}$. Or it follows thus: From $c = \Delta a'$, we have

$$C\Delta = \left(A' + \frac{(A, A')}{1!} + \frac{(A_2 A')}{2!} + \dots\right)\Delta$$

$$= A'\Delta + \left(B + \frac{(A, B)}{2!} + \frac{(A_2 B)}{3!} + \dots\right)\Delta,$$

where $b = Aa'$; by previous theorems (§ 2), this is

$$A'\Delta + \left(b\frac{\partial}{\partial a}\right)\Delta = A'\Delta + (\Delta, A') = \Delta A',$$

so that $C = \Delta A'\Delta^{-1}$. To prove the result $\psi(a', a) = \Delta\psi(a, a')$ of lemma ($\beta$) we have only to notice that, if $a_{11} = \Delta a''$, then $\Delta_{11} = \Delta\Delta''\Delta^{-1}$, and so, if $a'' = \psi(a, a')$, and therefore $a_{11} = \Delta\psi(a, a')$, we have

$$\Delta_{11} = \Delta\Delta'\Delta\Delta^{-1} = \Delta\Delta',$$

which establishes $a_{11} = \psi(a', a)$. The same proof stated differently gives, from $a_{11} = \Delta\psi(a, a')$, the series

$$\begin{aligned} A_{11} &= \Delta A''\Delta^{-1} = \Delta(\Delta'\Delta - 1)\Delta^{-1} - \tfrac{1}{2}\Delta(\Delta'\Delta - 1)^2\Delta^{-1} + \dots \\ &= (\Delta\Delta' - 1) - \tfrac{1}{2}(\Delta\Delta' - 1)^2 + \dots, \end{aligned}$$

which shows that $a_{11} = \psi(a', a)$. The result $\psi(a, \Delta a') = \Delta\psi(a, a')$ of lemma ($\beta$) is obvious from the form

$$\psi(a, a') = a + a' + (H_0 + H_1 + H_2 + \dots)Aa'.$$

For consider any term $P = A^{\lambda}A'^{\lambda'}A^{\mu}A'^{\mu'} \dots Aa'$;

if herein we replace $a'$ by $\Delta a'$, and therefore $A'$ by $\Delta A'\Delta^{-1}$, it becomes

$$A^{\lambda}\Delta A'^{\lambda'}\Delta^{-1}A^{\mu}\Delta A'^{\mu'}\Delta^{-1} \dots A\Delta a',$$

or, since $\Delta$ and $A$ are commutable,

$$\Delta A^{\lambda}A'^{\lambda'}A^{\mu}A'^{\mu'} \dots \Delta^{-1}\Delta Aa',$$

or $\Delta P$: while $a + a'$ becomes $\Delta a + \Delta a'$, it being noticed that in virtue of $Aa = 0$ we have $\Delta a = a$; thus $\psi(a, \Delta a') = \Delta\psi(a, a')$.

Consider now the equation

$$-\psi(-a, -a') = \Delta\psi(a, a');$$

it gives

$$\begin{aligned} &a + a' - (H_0 - H_1 + H_2 - \dots)Aa' \\ &\qquad = \left\{1 + A + \frac{A^2}{2!} + \dots\right\}\{a + a' + (H_0 + H_1 + H_2 + \dots)Aa'\}; \end{aligned}$$

comparing here terms of second dimension, we have

$$-H_0Aa' = H_0Aa' + A(a + a') = (H_0 + 1)Aa',$$

so that $H_0 = -\frac{1}{2}$; comparing terms of third dimension,

$$H_1Aa' = H_1Aa' + (H_0 + \tfrac{1}{2})A^2a',$$

giving also $H_0 = -\frac{1}{2}$; comparing terms of $(n+2)$-th dimension, we have

$$(-1)^{n-1}H_n Aa' = \left(H_n + AH_{n-1} + \frac{A^2}{2!}H_{n-2} + \ldots + \frac{A^n}{n!}H_0\right)Aa' + \frac{A^{n+1}a'}{(n+1)!};$$

thus when $n$ is even, $= 2m$, say, we obtain

$$\left[2H_{2m} + AH_{2m-1} + \frac{A^2}{2!}H_{2m-2} + \ldots + \frac{A^{2m}}{(2m)!}\left(H_0 + \frac{1}{2m+1}\right)\right]Aa' = 0,$$

from which $H_{2m}Aa'$ can at once be calculated when all the preceding terms are known; for $n = 2m+1$ we obtain a less convenient equation in which the term of highest dimension is again the term involving $H_{2m}$; if, however, in the equation just obtained we interchange $a$ and $a'$, and subtract the result from this equation, we obtain

$$\left[(A+A')H_{2m-1} + \frac{A^2 - A'^2}{2!}H_{2m-2} + \ldots + \frac{A^{2m} - A'^{2m}}{(2m)!}\left(H_0 + \frac{1}{2m+1}\right)\right]Aa' = 0;$$

we proceed to show that this equation determines $H_{2m-1}Aa'$ when all preceding terms are known. To explain this, let $K_n$ denote what $H_n$ becomes when $A - A'$ is put for $A$; if $K_n$ were determined, $H_n$ could then be found by putting $A + A'$ for $A$; we have thus, since $(A-A')a' = Aa'$,

$$\left[AK_{2m-1} + \frac{(A-A')^2 - A'^2}{2!}K_{2m-2} + \ldots + \frac{(A-A')^{2m} - A'^{2m}}{(2m)!}\left(H_0 + \frac{1}{2m+1}\right)\right]$$

$$\times Aa' = 0;$$

this gives $AK_{2m-1}Aa'$ when preceding terms are known; we say two solutions for $K_{2m-1}$ are not then possible; for, if so, and $P$ were their difference, there would exist an equation $APAa' = 0$, in which $P$ is an integral polynomial in $A$ and $A'$; we have remarked the existence of identities $QAa' = 0$, in which $Q$ is such an integral polynomial; we say that $Q$ is incapable of the form $AP$; for let $PAa' = f$, so that $APAa' = 0$ is equivalent to $Af = 0$ or $AF = FA$; let $F = A^kG$, where $G$ is incapable of a form $AG_1$, in which $G_1$ is an integral polynomial in $A$ and $A'$; then $A^{k+1}G = A^kGA$, clearly an impossible result, since the highest powers of $A$ occurring as left-side factors are different for these two quantities.

As illustrations of these formulæ we proceed to calculate $H_1$, $H_2$, $H_3$, $H_4$. For $H_1$ we have

$$\left[(A+A')H_1 + \frac{A^2 - A'^2}{2}\left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right]Aa' = 0,$$

or, in the notation above,

$$\left(AK_1 - \frac{A^2 - AA' - A'A}{12}\right) Aa' = 0;$$

in virtue of $(AA' - A'A)Aa' = 0$, this gives $H_1 Aa' = \frac{1}{12}(A - A')Aa'$. For $H_2$ we then have

$$\left[2H_2 + A\,\frac{A - A'}{12} + \frac{A^2}{2}\left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right] Aa' = 0,$$

and hence at once $H_2 Aa' = \frac{1}{24} AA'.Aa'$. Hence we have for $K_3$

$$\left[AK_3 + \frac{(A - A')^2 - A'^2}{2}\,\frac{(A - A')A'}{24} + \frac{(A - A')^3 + A'^3}{6}\,\frac{A - 2A'}{12}\right.$$
$$\left. + \frac{(A - A')^4 - A'^4}{24}\left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right] Aa' = 0.$$

Taking account of $(AA' - A'A)Aa' = 0$, this is found to be

$$[720AK_3 + A^4 + 3A^3A' - 4A^2A'^2 + 4AA'^3 - 3AA'^2A - AA'A^2 + Q]Aa' = 0,$$

where $QAa' = [4A'AA'^2 - 3A'A^2A' - A'A^3 + A'^2A^2 - 2A'^3A]Aa'$,

which must then be capable of the form $APAa'$; in fact the previously given identities (§ 1) lead to

$$(A'^3A - 2A'AA'^2)Aa' = -AA'^3Aa',$$
$$(A'^2A^2 - 3A'A^2A')Aa' = (A^2A'^2 - 3AA'^2A)Aa',$$
$$A'A^3Aa' = (-A^3A' + 2AA'A^2)Aa',$$

and $QAa'$ then takes the form in question; putting then $A + A'$ for $A$, we find

$$H_3Aa' = -\tfrac{1}{720}[A^3 - A'^3 + 6(A^2A' - A'^2A) + 2(AA'^2 - A'A^2)]Aa',$$

in which $H_3$ is changed in sign when $A$, $A'$ are interchanged, and has $\varpi_2/4!$ for coefficient of $A^3$. Lastly, for $H_4$ we obtain, on substituting for $H_1, H_2, H_3$,

$$[-1440H_4 + A^3A' + 2A^2A'^2 - 6AA'^2A - 2AA'A^2 - AA'^3]Aa' = 0;$$

putting here, from the identities in question,

$$(A^3A' - 2AA'A^2)Aa' = -A'A^3Aa',$$

and $(2A^2A'^2 - 6AA'^2A)Aa' = (A^2A'^2 - 3AA'^2A + A'^2A^2 - 3A'A^2A')Aa'$, we find

$$H_4Aa' = \tfrac{1}{1440}[A^2A'^2+A'^2A^2-3AA'^2A-3A'A^2A'-AA'^3-A'A^3]Aa',$$

wherein $H_4$ is symmetrical in form. We have therefore, to the terms found,

$$\begin{aligned} a'' = a+a'+\{-\tfrac{1}{2}+\tfrac{1}{12}(A-A')+\tfrac{1}{24}AA' \\ -\tfrac{1}{720}(A^3-A'^3+6A^2A'-6A'^2A+2AA'^2-2A'A^2) \\ -\tfrac{1}{1440}(A'A^3+AA'^3+3A'A^2A'+3AA'^2A-A^2A'^2-A'^2A^2)\}Aa'. \end{aligned}$$

By taking the derivative of this we have therefore the expansion by simple alternants of $H(a, a') = (\Delta'\Delta-1)-\tfrac{1}{2}(\Delta'\Delta-1)^2+\ldots$, up to terms of the sixth dimension in $A$ and $A'$. And the law of succession for the calculation of the terms of $\psi(a, a')$ corresponds of course to a law for the terms of $H(a, a')$. In fact, $H(a', a) = -H(-a, -a') = \Delta H(a, a')\Delta^{-1}$, say

$$C_1-C_2+C_3-C_4+\ldots = \left(1+A+\frac{A^2}{2!}+\ldots\right)(C_1+C_2+\ldots)\left(1-A+\frac{A^2}{2!}-\ldots\right),$$

giving, if we notice $(A_m C) = \sum\limits_{s=0} (-1)^s \binom{m}{s} A^{m-s}CA^s$, on equating terms of dimension $r$,

$$(-1)^{r-1}C_r = C_r+(A, C_{r-1})+\frac{1}{2!}(A, (A, C_{r-2}))+\ldots+\frac{1}{(r-1)!}(A_{r-1}C_1).$$

It is manifest, however, that the application of this law is more laborious than the rule we have followed for the bases of the alternants.

In what has preceded the derivatives $A, A'$ have been perfectly general. But the fact that the series $H(a, a')$ is obtainable by forming the derivative of the series $\psi(a, a')$ remains true when $A$, $A'$ are less general. One of the most natural ways in which we may allow restrictions is to suppose that the powers and products of $A$ and $A'$ are also derivatives. We carry out a particular example of this suggestion, of a very simple kind, which is yet general enough to give a great deal of information in regard to the expansion of $\psi(a, a')$. We suppose $A^2$ to be a derivative, in fact equal to $\mu A$, where $\mu$ is a number; and similarly $A'^2 = \mu' A'$. Then $A^2a' = \mu Aa'$, and therefore

$$(A, (A, A')) = \mu(A, A'), \quad \text{or} \quad A^2A'-2AA'A+A'A^2 = \mu(AA'-A'A),$$

which gives $\mu A'A = AA'A$; and similariy $\mu' AA' = A'AA'$. It is consistent with these to suppose $AA' = \mu A'$ and $A'A = \mu' A$. These four equations give, if $B = pA + qA'$, whatever numbers $p$ and $q$ may be, $B^2 = (p\mu + q\mu')B$, and conversely. It is then easily seen that

$$H_m(A, A')Aa' = H_m(\mu, \mu')Aa',$$

and we have

$$\psi(a, a') = a + a' + [H_0 + H_1(\mu, \mu') + H_2(\mu, \mu') + \dots]Aa' = a + a' + \lambda Aa',$$

say, where $\lambda$ is a number, giving $A'' = (1 - \lambda\mu')A + (1 + \lambda\mu)A'$. On the other hand, from $A^m = \mu^{m-1}A$, we have $\Delta = 1 + \frac{1}{\sigma(\mu)}A$, where

$$\sigma(\mu) = \left(1 + \frac{\mu}{2!} + \frac{\mu^2}{3!} + \dots\right)^{-1} = 1 - \frac{\mu}{2} + \frac{\varpi_1}{2!}\mu^2 + \frac{\varpi_2}{4!}\mu^4 + \dots = \frac{\mu}{e^\mu - 1};$$

thus $\Delta'\Delta = \left(1 + \frac{A'}{\sigma(\mu')}\right)\left(1 + \frac{A}{\sigma(\mu)}\right) = 1 + \frac{A'}{\sigma(\mu')} + \frac{1}{\sigma(\mu)}\left(1 + \frac{\mu'}{\sigma(\mu')}\right)A;$

also, from the form $A'' = (1 - \lambda\mu')A + (1 + \lambda\mu)A'$, we have

$$A''^2 = (\mu + \mu')A'',$$

and

$$\Delta'' = 1 + \frac{A''}{\sigma(\mu + \mu')};$$

putting this equal to $\Delta'\Delta$, we have

$$1 - \lambda\mu' = \frac{\sigma(\mu + \mu')}{\sigma(\mu)}\left[1 + \frac{\mu'}{\sigma(\mu')}\right], \qquad 1 + \lambda\mu = \frac{\sigma(\mu + \mu')}{\sigma(\mu')},$$

which are consistent in virtue of

$$\frac{\mu + \mu'}{\sigma(\mu + \mu')} = \frac{\mu}{\sigma(\mu)} + \frac{\mu'}{\sigma(\mu')}\left(1 + \frac{\mu}{\sigma(\mu)}\right),$$

or

$$e^{\mu + \mu'} - 1 = e^\mu - 1 + (e^{\mu'} - 1)e^\mu,$$

and give $\lambda$; namely, in this case $H_m(\mu, \mu')$ consists of the terms of order $m$ in the expansion of $[\sigma(\mu')]^{-1}\frac{\sigma(\mu + \mu') - \sigma(\mu')}{\mu}$. We find

$$\frac{1}{\sigma(\mu')}\,\frac{\sigma(\mu + \mu') - \sigma(\mu')}{\mu} = -\tfrac{1}{2} + \frac{\mu - \mu'}{12} + \frac{\mu\mu'}{24} - \tfrac{1}{720}(\mu^3 - \mu'^3 + 4\mu^2\mu' - 4\mu'^2\mu)$$

$$- \tfrac{1}{1440}(\mu^3\mu' + \mu'^3\mu + 4\mu^2\mu'^2) + \dots;$$

thus we have at once the values of $H_0$, $H_1$, $H_2$ and infer, putting down for $H_3(A, A')$, $H_4(A, A')$ the most general possible respectively unsymmetrical and symmetrical polynomials of dimensions three and four certain relations among the numerical coefficients in these.

### *An important Property of the Series $\psi(a, a')$.*

Denoting $1+A+A^2/2!+\dots$ by $\Delta_a$, we manifestly have

$$\Delta_{a''}\Delta_{\psi(n, a')} = \Delta_{a''}\Delta_{a'}\Delta_a = \Delta_{\psi(a', a'')}\Delta_a,$$

leading to $$\psi[\psi(a, a'), a''] = \psi[a, \psi(a', a'')],$$

which can be verified directly from the series $\psi(a, a')$.

### 5. *Some Applications to the Theory of Groups.*

The applications we shall notice are of two kinds: the first, that which was in view in Mr. Campbell's paper referred to in the introduction to this paper, when the capitals $A$, $B$, ... are linear operators of the form

$$X = \xi_1(x_1 \dots x_n)\frac{\partial}{\partial x_1}+\dots+\xi_n(x_1 \dots x_n)\frac{\partial}{\partial x_n};$$

the second when the capitals are certain matrices. As regards the first, it is easy to show that, if

$$x_i' = \left(1+tX+\frac{t^2}{2!}X^2+\dots\right)x_i = e^{tX}x_i,$$

where $t$ is a number, then, at least in a neighbourhood where the analytic function $F(x_1 \dots x_n)$ is regular, a similar condition being assumed for $\xi_1(x_1 \dots x_n) \dots \xi_n(x_1 \dots x_n)$,

$$F(x_1' \dots x_n') = \left(1+tX+\frac{t^2}{2!}X^2+\dots\right)F(x_1 \dots x_n).$$

If then we put $$F_1(x) = F(e^{tX_1}x) = e^{tX_1}F(x),$$

where $X_1$ is one linear operator, and define similarly $F_2$ from another operator $X_2$, the exponential theorem proved above gives

$$e^{tX_1}e^{tX_2} = e^{tX_3},$$

where $$tX_3 = (e^{tX_1}e^{tX_2}-1)-\tfrac{1}{2}(e^{tX_1}e^{tX_2}-1)^2+\dots$$

is a series of alternants and therefore also a linear operator, and therefore

$$F_3(x) = e^{tX_1}e^{tX_2}F(x) = e^{tX_1}F_2(x) = F_2(e^{tX_1}x).$$

In particular, putting $F(x) = x_i$, and, with linear operators $P_1, \dots, P_r$,

$$X_1 = e_1P_1+\dots+e_rP_r, \qquad X_2 = e_1'P_1+\dots+e_r'P_r,$$

where $e_1, \ldots, e_r, e'_1, \ldots, e'_r$ are numerical constants, we have, if

$$e^{X_1}x_i = f_i(x, e), \qquad e^{X_2}x_i = f_i(x, e'),$$

the equation

$$e^{X_3}x_i = f_i[f(x, e), e'].$$

If, further, every alternant of two of the operators $P_1, \ldots, P_r$ is expressible linearly by $P_1 \ldots P_r$, say in the form

$$(P_\rho, P_\sigma) = c_{\rho\sigma 1}P_1 + \ldots + c_{\rho\sigma r}P_r \quad (\rho, \sigma = 1, 2, \ldots, r),$$

then $X_3$ will be also so expressible, say in the form $e''_1P_1 + \ldots + e''_rP_r$, and we shall have

$$f_i[f(x, e), e'] = f_i(x, e''),$$

the fundamental equation of Lie's group theory.

Now we have seen (§ 3) that

$$X_3 = X_1 + X_2 + \tfrac{1}{2}(X_1, X_2) + \tfrac{1}{12}(X_1 - X_2, (X_1, X_2)) - \tfrac{1}{24}(X_1, (X_2, (X_1, X_2))) + \ldots ;$$

and we have

$$(X_1, X_2) = (e_1P_1 + \ldots + e_rP_r,\ e'_1P_1 + \ldots + e'_rP_r) = \sum_\tau \sum_\rho \sum_\sigma c_{\rho\sigma\tau} e_\rho e'_\sigma P_\tau$$
$$(\rho, \sigma, \tau = 1, 2, \ldots, r).$$

We introduce now a matrix of $r$ rows and columns whose general element is

$$E_{\rho\sigma} = \sum_\tau c_{\sigma\tau\rho} e_\tau \quad (\tau = 1, \ldots, r),$$

so that each element is a linear function of $e_1, \ldots, e_r$ with coefficients chosen from the constants $c_{\rho\sigma\tau}$; then we have, since from $(P_\rho, P_\sigma) = -(P_\sigma, P_\rho)$ follows $c_{\rho\sigma\tau} = -c_{\sigma\rho\tau}$,

$$(X_1, X_2) = -\sum_\tau \sum_\sigma E_{\tau\sigma} e'_\sigma P_\tau = -\sum_\tau (Ee')_\tau P_\tau,$$

where $Ee'$ is a symbol for the set of $r$ quantities

$$(Ee')_\tau = E_{\tau_1}e'_1 + E_{\tau_2}e'_2 + \ldots + E_{\tau_r}e'_r,$$

and clearly $Ee' = -E'e$, if $E'$ be the same function of $e_1, \ldots, e'_r$ as is $E$ of $e_1 \ldots e_r$. Hence it follows, if $(Ee')_\tau$ be momentarily denoted by $f_\tau$,

$$(X_1, (X_1, X_2)) = -(e_1P_1 + \ldots + e_rP_r,\ f_1P_1 + \ldots + f_rP_r) = \sum_\tau (Ef)_\tau P_\tau,$$

$$(X_2, (X_1, X_2)) = -(e'_1P_1 + \ldots + e'_rP_r,\ f_1P_1 + \ldots + f_rP_r) = \sum_\tau (E'f)_\tau P_\tau,$$

and therefore again, if $(E'f)$ be momentarily denoted by $g_\tau$,

$$(X_1, (X_2, (X_1, X_2))) = (e_1P_1 + \ldots + e_rP_r,\ g_1P_1 + \ldots + g_rP_r) = -\sum_\tau (Eg)_\tau P_\tau;$$

but $Ef = E^2e'$, $E'f = E'Ee'$, $Eg = EE'f = EE'Ee'$; thus we have

$$X_3 = \sum_{\tau=1}^{r} \left(e_\tau + e'_\tau - \tfrac{1}{2}(Ee')_\tau + \tfrac{1}{12}(E^2e')_\tau - \tfrac{1}{12}(E'Ee')_\tau + \tfrac{1}{24}(EE'Ee')_\tau + \ldots\right) P_\tau;$$

or, since $X_3 = \sum_{\tau=1}^{r} e''_\tau P_\tau$, we have $r$ equations

$$e''_\tau = e_\tau + e'_\tau - \tfrac{1}{2}(Ee')_\tau + \tfrac{1}{12}(E^2e')_\tau - \tfrac{1}{12}(E'Ee')_\tau + \tfrac{1}{24}(EE'Ee')_\tau + \ldots,$$

all of which are represented by

$$e'' = e + e' + \left[-\tfrac{1}{2} + \tfrac{1}{12}(E - E') + \tfrac{1}{24}EE' + \ldots\right] Ee',$$

and this is the series we have previously denoted by $\psi(e, e')$, there being $r$ functions $\psi_1(e, e'), \ldots, \psi_r(e, e')$. Thus it appears that the fundamental equations of Lie's group theory are

$$f_i[f(x, e), e'] = f_i[x, \psi(e, e')],$$

and we have proved, to employ the ordinary nomenclature, not only that the equations of the parameter group have the form given by $e'' = \psi(e, e')$ expressible by the matrices $E$, $E'$, but that there is a one to one correspondence between the terms of this, and the terms, which are alternants, in the expansion of

$$e^A e^B - 1 - \tfrac{1}{2}(e^A e^B - 1)^2 + \ldots.$$

And this brings us to the second application of our preceding general theory to the theory of groups. It is, in fact, easily recognised that the laws fundamental for derivatives and bases which were set out at the beginning of this paper hold for matrices $E$, such as have been here introduced, and for the sets $e_1, \ldots, e_r$. To see this it is only necessary to notice (1) the fact proved above that $Ee' + E'e = 0$, (2) the fact that the $E$ matrix of the set $Ee'$ is $EE' - E'E$. This last follows from the equation

$$EE'e'' + E'E''e + E''Ee' = 0,$$

which expresses that relation among the constants of structure which follow from the identity

$$(X_1, (X_2, X_3)) + (X_2, (X_3, X_1)) + (X_3, (X_1, X_2)) = 0;$$

for this equation is $\quad (EE' - E'E)e'' + E''Ee' = 0.$

It must be remarked, however, that it is only for groups of general form that the equation $E = 0$ involves $e_1 = 0, \ldots, e_r = 0$, or $e = 0$. The group must have no special transformations.

Conversely that intricate part of the algebra of the theory of groups which is concerned with the theory of structure may conveniently be stated in the general terms of the earlier part of this paper. As the theory is there set out, the primary set of capitals may be of infinite number, and in any case the alternants formed from them of successive dimensions form an indefinitely extended system from which there is no return to the original capitals, save perhaps by reverting an infinite series

$$\lambda_0 B+\lambda_1(A, B)+\lambda_2(A, (A, B))+\dots .$$

If we wish to have a finite system of elements, we may then naturally suppose, to speak in terms of the bases of the alternants, that there is but a finite number of bases, say $a_1$, $a_2$, ..., $a_r$, in terms of which every other base is expressible linearly with numerical coefficients. We shall thus have, for instance, equations such as

$$A_\rho a_\sigma = -(c_{\rho\sigma 1} a_1+\dots+c_{\rho\sigma r} a_r).$$

It is at once evident that this leads to equations $F(A_\rho) a_\sigma = 0$, where $F(A_\rho)$ is an integral polynomial in $A_\rho$ with numerical coefficients; and so on. It is interesting to consider from this point of view such propositions, for instance, as one, due to Cartan, relating to the necessary and sufficient condition that a group be integrable; with our proposition that the $E$ matrix of $Ee'$ is $(E, E')$, this condition is that the constants of structure should be such that for some positive integral $\lambda$ we should have, for every $e$ and $e'$, an equation $(EE'-E'E)^\lambda = 0$.

# ON THE FAILURE OF CONVERGENCE OF FOURIER'S SERIES

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[Read December 8th, 1904.—Received December 14th, 1904.]

IT is well known that, if $f(x)$ is a limited integrable function, defined for the interval $(-\pi, \pi)$, the corresponding Fourier's series, at any point $x$ at which it converges, has for its limiting sum $\underset{\epsilon=0}{\mathrm{L}}\ \tfrac{1}{2}\{f(x+\epsilon)+f(x-\epsilon)\}$, provided this limit exists at the point. It is known that the nature of the function in an arbitrarily small neighbourhood of the point $x$ determines whether the function converges at $x$ or not; various sufficient conditions have been found that this convergence may take place. In the present communication a general theorem is obtained relating to the most general distribution of the points of the interval at which the series either converges to the value of the function, or can be made so to converge by bracketing the terms of the series in a suitable manner. In the latter case the series is an oscillating one: some preliminary remarks are accordingly made on the subject of oscillating series. An example, due to Schwarz, of a series which represents a continuous function, and fails to converge at a particular point, is considered in detail, and it is shown that the series is in reality an oscillating one, at the point, and that by introducing a suitable system of brackets, but without altering the order of the terms, the series can be made to converge to the value of the function. Finally, a function is constructed which fails to converge at points in every interval contained in the interval $(-\pi, \pi)$. A more complicated example of such a function has been given[*] by P. Du Bois Reymond.

## *On Oscillating Series.*

1. Let us suppose that $u_1, u_2, \ldots, u_n, \ldots$ is an unending sequence of numbers, such that $u_n$ has for each value of $n$ a definite numerical value assigned by means of a prescribed law; let $s_n$ denote the sum $u_1+u_2+\ldots+u_n$, and let us consider the aggregate $(s_1, s_2, \ldots, s_n, \ldots)$, the elements of which may be denoted in the usual manner by points on a

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[*] Abhandlungen der bayerischen Akademie, Vol. XII.

straight line. This enumerable set of points will be denoted by $G$; the following cases may arise:—

(1) The set $G$ may consist of points all of which lie between two fixed points $A$, $B$, and the derivative $G'$ of the set $G$ may consist of a single point $s$. In this case the series $u_1+u_2+\ldots+u_n+\ldots$ is convergent, and $s$ is its limiting sum.

(2) The set $G$ may be unlimited in one direction or in both directions, and the derivative $G'$ may be non-existent. It may be said that $G$ has the improper limiting point $+\infty$, or the improper limiting point $-\infty$; it may have both $+\infty$ and $-\infty$ as improper limiting points. In either case $|s_n|$ has no upper limit, and the series $u_1+u_2+\ldots+u_n+\ldots$ is said to be *divergent*. There are thus two species of divergent series: for example, the series $1/1+1/2+\ldots+1/n+\ldots$ is divergent and has the improper limiting point $+\infty$, whereas the series

$$1-2+3-4+\ldots+(2n-1)-2n+\ldots$$

has the two improper limiting points $+\infty$, $-\infty$.

(3) The set $G$ may consist of points all lying between two fixed points $A$, $B$, and the derivative $G'$ may consist of more than one point; in this case the series is said to be an *oscillating* series. The set $G'$ may contain a finite or an infinite number of points, but is, in any case, in accordance with a well-known theorem, a closed set, and consequently has an upper limit $U$ and a lower limit $L$. These limits $U$ and $L$ are called the *limits of indeterminacy* of the series $\Sigma u_n$.

It is always possible to find a sequence $(s_{n_1}, s_{n_2}, s_{n_3}, \ldots)$ of partial sums, where $n_1<n_2<n_3\ldots$, which converges to the limit $U$, and another such sequence which converges to the limit $L$, or one which converges to any prescribed point of $G'$. It thus appears that, by means of a suitable system of brackets, the oscillating series $\Sigma u_n$ may be converted into a convergent series of which the sum is a prescribed point of $G'$, the terms in each bracket being amalgamated. The set $G'$ may be non-dense in the interval $(L, U)$, or it may consist of all the points of that interval, or it may consist of a closed set of the most general type which is dense in some parts of the interval $(L, U)$.

For example, the series $1-1+1-1+1-\ldots$ has the points 1, 0 for the upper and lower limits of indeterminacy, and $G'$ consists of these two points.

Again, it is easy to construct a series which oscillates between the limits of indeterminacy 0, 1, and such that $G'$ consists of the whole interval (0, 1).

Let $s_1 = \frac{1}{2},\quad s_2 = \frac{1}{3},\quad s_3 = \frac{1}{4},\quad s_4 = \frac{2}{3},\quad s_5 = \frac{1}{5},$

$s_6 = \frac{2}{4},\quad s_7 = \frac{1}{6},\quad s_8 = \frac{2}{5},\quad s_9 = \frac{3}{4},\quad s_{10} = \frac{1}{7},\quad \ldots,$

and generally

$$s_{m(m+1)+1} = \frac{1}{2m+2},\qquad s_{m(m+1)+2} = \frac{2}{2m+1},\qquad \ldots,\qquad s_{(m+1)^2} = \frac{m+1}{m+2},$$

$$s_{(m+1)^2+1} = \frac{1}{2m+3},\qquad s_{(m+1)^2+2} = \frac{2}{2m+2},\qquad \ldots,\qquad s_{(m+1)(m+2)} = \frac{m+1}{m+3}.$$

The set $G$ consists of all the rational numbers between 0 and 1, and thus $G'$ consists of the whole interval (0, 1).

It follows that the series

$$\frac{1}{2} - \frac{1}{2.3} - \frac{1}{3.4} + \frac{5}{3.4} - \frac{7}{3.5} + \ldots$$

where

$$u_{m(m+1)+1} = \frac{1}{2m+2} - \frac{m}{m+2},\qquad u_{m(m+1)+2} = \frac{2}{2m+1} - \frac{1}{2m+2},$$

$$u_{m(m+1)+3} = \frac{3}{2m} - \frac{2}{2m+1},\qquad \ldots,\qquad u_{(m+1)^2} = \frac{m+1}{m+2} - \frac{m}{m+3},$$

$$u_{(m+1)^2+1} = \frac{1}{2m+3} - \frac{m+1}{m+2},\qquad u_{(m+1)^2+2} = \frac{2}{2m+2} - \frac{1}{2m+3},\qquad \ldots$$

can have its terms so bracketed that the sum of the resulting series is any prescribed number in the interval (0, 1).

(4) The derivative $G'$ may exist, but it may be unlimited in one or in both directions: thus $U$ may have the improper value $+\infty$, or $L$ may have the improper value $-\infty$, or both these cases may arise simultaneously. In this case the series oscillates between limits one or both of which are indefinitely great. The series may be made to diverge by introducing a suitable system of brackets, or it may be made to converge to any point of $G'$.

For example, a series may be constructed which oscillates between infinite limits of indeterminacy, but which, by introducing a suitable system of brackets, may be made to converge to any prescribed number whatever.

If $x' = \dfrac{2x-1}{\sqrt{\{x(1-x)\}}}$, where the positive sign is ascribed to the radical, the points $x$ of the interval (0, 1) have a (1, 1) correspondence with the points $x'$ of the interval $(-\infty, \infty)$; it is clear that a set of points in the interval (0, 1) corresponds to a set in the interval $(-\infty, \infty)$, the relative order of pairs of points in the one interval being the same as that of the corresponding points in the other interval. Further, a limiting point of a set of points in the interval (0, 1) corresponds to a limiting point of the corresponding set of points in the interval $(-\infty, \infty)$. The rational points within the interval (0, 1) correspond to a set of points everywhere dense in the unlimited line $(-\infty, +\infty)$. We may apply this method to transform the series obtained in (3) which oscillates between the limits of indeterminacy 0, 1, and which can be made by introducing suitable brackets to converge to any number in the interval (0, 1).

We find

$$s'_1 = 0, \quad s'_2 = -\frac{1}{\sqrt{2}}, \quad s'_3 = -\frac{2}{\sqrt{3}}, \quad s'_4 = \frac{1}{\sqrt{2}}, \quad s'_5 = -\frac{3}{2},$$

$$s'_6 = 0, \quad s'_7 = -\frac{4}{\sqrt{5}}, \quad s'_8 = -\frac{1}{\sqrt{6}}, \quad s'_9 = \frac{2}{\sqrt{3}}, \quad s'_{10} = -\frac{5}{\sqrt{6}}, \quad \dots,$$

and generally

$$s'_{m(m+1)+1} = -\frac{2m}{\sqrt{(2m+1)}}, \qquad s'_{m(m+1)+2} = -\frac{2m-3}{\sqrt{\{2(2m-1)\}}}, \qquad \dots,$$

$$s'_{(m+1)^2} = \frac{m}{\sqrt{(m+1)}}, \qquad s'_{(m+1)^2+1} = -\frac{2m+1}{\sqrt{(2m+2)}}, \qquad \dots,$$

$$s'_{(m+1)(m+2)} = \frac{m-1}{\sqrt{\{2(m+1)\}}}.$$

Thus the series

$$-\frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{2}}\right) - \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\right) + \frac{3}{2}$$

$$-\frac{4}{\sqrt{5}} + \left(\frac{4}{\sqrt{5}} - \frac{1}{\sqrt{6}}\right) + \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right) + \dots$$

has the required character: it may be made to converge to any value whatever by suitably bracketing the terms together and amalgamating the terms in each bracket, without altering the order of the terms.

The transformation $x' = \dfrac{2x-1}{\sqrt{\{x(1-x)\}}}$ is an example of an unlimited number of transformations by means of which sets of points in a finite interval may be made to correspond with sets of points in an unlimited straight line, the ordering of the sets being the same. Another simple transformation of this kind is $x' = \tan \pi x/2$, by which a set of points $x$ in the interval $(-1, +1)$ is transformed into a set of points $x'$ on an unlimited straight line.

### *The Points at which Fourier's Series does not converge.*

2. Let $f(x)$, $\phi(x)$ denote two functions which are limited and integrable in the interval $(-\pi, \pi)$, and let $a_s$, $b_s$ denote the Fourier's coefficients $\int_{-\pi}^{\pi} f(x) \cos sx\, dx$, $\int_{-\pi}^{\pi} f(x) \sin sx\, dx$, the corresponding coefficients for the function $\phi(x)$ being denoted by $a'_s$, $b'_s$. It has been well known for a long time that, provided the Fourier's series for the functions $f(x)$, $\phi(x)$ converge uniformly in the domain $(-\pi, \pi)$, the series $\frac{1}{2}a_0 a'_0 + \sum_1^n (a_s a'_s + b_s b'_s)$ converges, as $n$ is indefinitely increased, to the sum $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \phi(x)\, dx$. The theorem has, however, more recently been established,* that the series converges to the same sum, quite independently of the mode of convergence of the Fourier's series, and in fact independently of any assumption that these series converge at all.

In the particular case in which the two functions $f(x)$, $\phi(x)$ are identical, the theorem takes the form that for any limited integrable function $f(x)$, the series $\frac{1}{2}a_0^2 + \sum_1 (a_s^2 + b_s^2)$ is always convergent and has for its sum $\frac{1}{\pi}\int_{-\pi}^{\pi} \{f(x)\}^2 dx$, and that this is true independently of any assumption as to the convergence of the Fourier's series

$$\tfrac{1}{2}a_0 + \sum_1^\infty (a_s \cos sx + b_s \sin sx).$$

This theorem will be here applied to an examination of the properties of the Fourier's series.

The function $f(x)$ being limited and integrable in the interval $(-\pi, \pi)$, let $R_n$ denote $f(x) - \frac{1}{2}a_0 - \sum_1^n (a_s \cos sx + b_s \sin sx)$; we have

$$\int_{-\pi}^{\pi} R_n^2\, dx = \int_{-\pi}^{\pi} \{f(x)\}^2 dx - \pi\left[\tfrac{1}{2}a_0^2 + \sum_1^n (a_s^2 + b_s^2)\right].$$

The expression on the right-hand side is essentially positive, and converges to zero as $n$ is indefinitely increased. Let $\epsilon$, $m$ be two fixed numbers which may be so chosen that $\epsilon^2/4m$ is as small as we please; then an integer $N$ exists, dependent on $m$ and $\epsilon$, such that

$$\int_{-\pi}^{\pi} R_n^2\, dx < \frac{\epsilon^2}{4m}, \quad \text{provided } n \geqslant N.$$

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* The first proof was given by De la Vallée Poussin; see *Annales de la Soc. Scient. de Bruxelles*: other proofs have been given by Hurwitz, *Math. Annalen*, Vol. LVII., and by Fischer, *Monatshefte für Math. u. Physik*, Vol. XV.

Let us take a fixed value of $n$ which is $\geqslant N$; then, since $|R_n|$ is an integrable function, the interval $(-\pi, \pi)$ can be divided into a finite number of parts such that the sum of those parts in each of which the fluctuation of $|R_n|$ is $\geqslant \frac{1}{2}\epsilon$ is less than an arbitrarily chosen positive number $\eta$. In each of the other parts, the fluctuation of $|R_n|$ is $< \frac{1}{2}\epsilon$. Let these latter parts be denoted by $\delta_n$; then $\int R_n^2 dx$, taken through all the parts $\delta_n$, is $< \epsilon^2/4m$. If one of the intervals of $\delta_n$ contains a point at which $|R_n| \geqslant \epsilon$, then at every point in that interval $|R_n| \geqslant \frac{1}{2}\epsilon$; it is clear that the sum of those of the intervals $\delta_n$ for the whole of each of which $|R_n| \geqslant \frac{1}{2}\epsilon$ must be less than $1/m$. It has now been shown that there exists a finite set of intervals of which the sum is $> 2\pi - \eta - 1/m$, such that at every point in them the condition $|R_n| < \epsilon$ is satisfied; the set of intervals depends upon the particular value of $n$ chosen.

The number of intervals in the set depends upon the value of $\eta$, and may increase indefinitely as $\eta$ is indefinitely diminished; it follows that for each value of $n$ which is $\geqslant N$ there exists a measurable set of points such that at each point $|R_n| < \epsilon$, the measure of the set being

$$\geqslant 2\pi - 1/m.$$

It has now been established that a sequence $G_N, G_{N+1}, G_{N+2}, \ldots$ of measurable sets of points exists such that the measure of each set is $\geqslant 2\pi - 1/m$, and such that at any point of any one of them $G_n$ the condition $|R_n| < \epsilon$ is satisfied.

The following theorem* in the theory of sets of points will now be applied to the sequence $G_N, G_{N+1}, G_{N+2}, \ldots$ :—

If $P_1, P_2, \ldots, P_n, \ldots$ is a sequence of sets of points, each of which sets is a component of a closed set of finite content $l$, and if the interior measure of each of the sets is greater than a fixed number $C$, then there exists a set of points of interior measure $\geqslant C$, and of the power of the continuum, such that each point of the set belongs to an infinite number of the given sets.

In our case the sets $G_N, G_{N+1}, G_{N+2}, \ldots$ are all measurable and their measures are all $> 2\pi - 1/m - \eta$, however small $\eta$ may be; it follows that a set exists of interior measure $\geqslant 2\pi - 1/m - \eta$, each point of which belongs to an infinite number of the sets $G_N, G_{N+1}, \ldots$. Since $\eta$ is arbitrarily small, it follows that the interior measure of the set of points thus found is $\geqslant 2\pi - 1/m$. For each point of this set $G(\epsilon)$ there are an infinite number of values of $n$ such that $|R_n| < \epsilon$. Let us consider the set $H_\epsilon$ of points for each of which $|R_n| \geqslant \epsilon$, for all values of $n$ except for a finite number of such values, and let $a$ be the exterior measure of this set, and, if possible, let $a > 0$; for the complementary set which is of interior measure $2\pi - a$, there must be for each point an infinite number of values of $n$ for which $|R_n| < \epsilon$. Let us choose $m$ so large that $1/m < a$; then for each point of $G(\epsilon)$ which is of interior measure $2\pi - 1/m > 2\pi - a$ there are an infinite number of values of $n$ for which $|R_n| < \epsilon$, and these cannot all be included in the set complementary to $H_\epsilon$; it follows that it is impossible that $a > 0$; thus the set $H_\epsilon$ is measurable, and its measure is zero.

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* This theorem was stated and proved by W. H. Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, in his paper on "Open Sets and the Theory of Content." The theorem, so far as it relates to measurable sets, was stated without proof by Borel in the *Comptes Rendus* for December, 1903. The expression "measure" is employed in the present paper in accordance with the usage of Borel and Lebesgue, instead of the expression "content" employed by W. H. Young, because the latter has been used in another sense by Harnack, Cantor, and others. The word content is here restricted to the case of closed sets, in which case the "content" of Harnack and the "measure" of Borel are identical.

Since the set $H_\epsilon$ of those points for each of which $|R_n| < \epsilon$ at most for a finite number of values of $n$ has zero measure, it follows that the complementary set $K_\epsilon$, for each point of which $|R_n| < \epsilon$, for an infinite number of values of $n$, has the measure $2\pi$. Let us now take a sequence of diminishing values of $\epsilon$, say $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$, which converges to zero, and consider the sets $K_{\epsilon_1}, K_{\epsilon_2}, \ldots, K_{\epsilon_n}, \ldots$; each of these sets has the measure $2\pi$: it follows, by applying again the theorem in sets of points already employed, that there exists a set of points of measure $2\pi$ each point of which belongs to an infinite number of the sets $K_{\epsilon_1}, K_{\epsilon_2}, \ldots, K_{\epsilon_n}, \ldots$. This set $L$ is such that for any point $P$ of it a sequence $\epsilon_{p_1}, \epsilon_{p_2}, \epsilon_{p_3}, \ldots$ of values of $\epsilon$ belonging to the sequence $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ exists such that $|R_n| < \epsilon_{p_1}$ for an infinite number of values of $n$, $|R_n| < \epsilon_{p_2}$ for an infinite number of values of $n$, and so on. The sequence $\epsilon_{p_1}, \epsilon_{p_2}, \epsilon_{p_3}, \ldots$ converges to the limit zero; thus for any point of the set $L$ we have $\lim R_n = 0$, provided a properly chosen sequence of increasing values of $n$ is taken.

For any point of the set $L$, either the Fourier's series converges to the value $f(x)$, or else it oscillates between finite or infinite limits of indeterminacy, but so that the functional value at the point is a limiting point of the partial sums of the series. In the latter case the series can be made to converge to the value $f(x)$, by bracketing the terms in a suitable manner, the terms in any one bracket being regarded as amalgamated; no change is made in the order of the terms.

*If for any value of the variable $x$ the Fourier's series does not converge to the value $f(x)$, but can be made to converge to that value by introducing a suitable system of brackets, the terms in each bracket* *being amalgamated, but the order of the terms being unaltered, the series will be said to be* quasi-convergent *with* $f(x)$ *as its sum.*

The set complementary to $L$ has zero measure, and contains (1) all those points for which the Fourier's series is divergent, (2) points at which the function $f(x)$ is discontinuous and the series converges to a value different from $f(x)$, and (3) points at which the series oscillates but is not quasi-convergent with $f(x)$ as sum.

The following general theorem has now been established :—

*If* $f(x)$ *is a limited integrable function defined for the interval* $(-\pi, \pi)$, *there exists in this interval a set of points* $L$ *of which the measure is equal to that of the whole interval, such that at each point of* $L$ *the Fourier's series either converges to* $f(x)$, *or is quasi-convergent with* $f(x)$ *as sum. The complementary set is of zero measure, and contains all points at which the series is divergent, or converges to a value different from* $f(x)$, *or oscillates without being quasi-convergent.*

*The set* $L$ *is everywhere dense in the interval, and has the cardinal number of the continuum. The complementary set may or may not be everywhere dense.*

In the case in which $f(x)$ is a continuous function, the set complementary to $L$ consists of points at which the Fourier's series diverges or oscillates without being quasi-convergent.

The theorem leaves the possibility open that, even in the case of a continuous function, there may be no point in the interval at which the series is convergent.

### *A Continuous Function for which the Series does not converge.*

3. It is well known that the question as to whether a Fourier's series converges at a particular point must be answered by an examination of the limit of an integral of Dirichlet's type

$$\int_0^a \phi(z)\,\frac{\sin(2m+1)z}{\sin z}\,dz \qquad (0 < a \leqslant \tfrac{1}{2}\pi),$$

when the positive integer $m$ is indefinitely increased.

Let the product $1.3.5 \ldots (2\lambda+1)$ be denoted by $[2\lambda+1]$, and let the function $\phi(z)$ be defined for the interval $(0, a)$ in the following manner:—In the interval $(\pi/[\lambda-1], \pi/[\lambda])$ let $\phi(z) = c_\lambda \sin[\lambda]z$, where $c_\lambda$ is a constant dependent upon the value of $\lambda$; let $\lambda$ have all values $\lambda_1, \lambda_1+1, \lambda_1+2, \ldots$ where $\lambda_1$ is a fixed integer, and we may suppose $a$ so chosen that $a = \pi/[\lambda_1-1]$; also let $\phi(0) = 0$. If the sequence $c_{\lambda_1}, c_{\lambda_1+1}, c_{\lambda_1+2}, \ldots$ be so chosen that it converges to the limit zero, the function $\phi(z)$ is continuous at the point $z = 0$, but has an indefinitely great number of oscillations in an arbitrarily small neighbourhood of that point. It has been shown by Schwarz* that, if $c_\lambda$ is so chosen that $c_\lambda \log(2\lambda+1)$ becomes indefinitely great as $\lambda$ is indefinitely increased, the integral may become indefinitely great as $m$ is increased indefinitely, and thus that a Fourier's series exists which does not represent the given continuous function at a particular point.

It will here be shown that the series is in reality oscillatory, and that the point $z = 0$ is a point of quasi-convergence in the sense defined above. A simplified proof of Schwarz's result will first be given.

It is known that the integral $\int_0^a \phi(z) \dfrac{\sin(2m+1)z}{\sin z}\,dz$ may, for the purpose of the consideration of the limiting values when $m$ is indefinitely increased, be replaced by † $\int_0^a \phi(z) \dfrac{\sin(2m+1)z}{z}\,dz$. Thus we may consider the latter integral.

Let $$2m+1 = 1,\ 3,\ 5\ \ldots\ 2\mu+1 \equiv [\mu];$$

then $$\int_0^a \phi(z) \frac{\sin[\mu]z}{z}\,dz$$

$$= c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{\sin^2[\mu]z}{z}\,dz + \sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z \sin[\mu]z}{z}\,dz$$
$$+ \sum_{n=\mu+1}^{\infty} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z \sin[\mu]z}{z}\,dz;$$

the first integral on the right-hand side may be written in the form

$$\tfrac{1}{2}c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{1-\cos 2[\mu]z}{z}\,dz,$$

which is equivalent to

$$\tfrac{1}{2}c_\mu \log(2\mu+1) - \tfrac{1}{2}c_\mu \frac{[\mu]}{\pi} \int_{\pi/[\mu]}^{\beta} \cos 2[\mu]z\,dz,$$

where $\beta$ is a number between $\pi/[\mu]$ and $\pi/[\mu-1]$.

Now let $c_\mu \log(2\mu+1)$ increase indefinitely with $\mu$; this is consistent with $c_\mu$ having a zero limit, for we have only to take $c_\mu = \dfrac{1}{\{\log(2\mu+1)\}^s}$, where $s$ is some fixed positive number less than unity.

Since $c_\mu \dfrac{[\mu]}{\pi} \int_{\pi/[\mu]}^{\beta} \cos 2[\mu]z\,dz$ is numerically not greater than $c_\mu/\pi$, we see that, with the supposition made as to $c_\mu$, the expression

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* See the history of the theory of Fourier's series, by Sachs, *Schlömilch's Zeitschrift*, Supplement, Vol. xxv.

† A rigid proof of this is given by Brodén, *Math. Annalen*, Vol. LII., p. 220.

$$c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{\sin^2 [\mu] z}{z}\, dz$$

becomes indefinitely great as $\mu$ is increased indefinitely.

To evaluate
$$\sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin [n] z \sin [\mu] z}{z}\, dz,$$

we see, by writing $\sin [n] z \sin [\mu] z$ as half the difference of two cosines, and applying the second mean value theorem to each integral, that the absolute value of the expression is less than

$$\sum_{n=\lambda_1}^{\mu-1} c_n \frac{[n]}{\pi} \left\{ \frac{1}{[\mu]-[n]} + \frac{1}{[\mu]+[n]} \right\}$$

or than
$$\sum_{n=\lambda_1}^{\mu-1} \frac{c_n}{\pi} \frac{[n]}{[\mu-1]} \left\{ \frac{1}{2\mu+1-\dfrac{[n]}{[\mu-1]}} + \frac{1}{2\mu+1+\dfrac{[n]}{[\mu-1]}} \right\},$$

which is less than
$$\frac{c_{\lambda_1}}{\pi} \sum \frac{[n]}{[\mu-1]} \frac{1}{\mu},$$

and this is
$$\frac{c_{\lambda_1}}{\pi\mu} \left\{ 1 + \frac{1}{2\mu-1} + \frac{1}{(2\mu-1)(2\mu-3)} + \dots \right\}.$$

Thus the absolute value of the expression is less than $2c_\lambda/\pi\mu$, and this becomes indefinitely small as $\mu$ is indefinitely increased; thus the limiting value of the expression is zero.

Finally, we have to consider the expression

$$\sum_{n=\mu+1}^{\infty} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin [n] z \sin [\mu] z}{z}\, dz\ ;$$

since $\left| \dfrac{\sin [\mu] z}{z} \right| < [\mu]$, and $|\sin [n] z| \leqslant 1$, the absolute value of the expression is less than $c_{\mu+1} [\mu] \dfrac{\pi}{[\mu]}$, and this has the limit zero when $\mu$ is indefinitely increased.

Schwarz's theorem has now been established, that

$$\int_0^a \frac{\phi(z) \sin [\mu] z}{\sin z}\, dz$$

increases indefinitely as $\mu$ is indefinitely increased, where

$$[\mu] = 1 \,.\, 3 \,.\, 5 \,\dots\, (2\mu+1),$$

and $\phi(z)$ is defined by $\phi(0) = 0$, $\phi(z) = c_\lambda \sin [\lambda] z$ in the interval $(\pi/[\lambda],\ \pi/[\lambda-1])$ where $\lambda = \lambda_1, \lambda_1+1, \lambda_1+2, \dots$ and $a = \pi/[\lambda_1-1]$, provided $c_\lambda$ has the value $1/\{\log(2\lambda+1)\}^s$, where $0 < s < 1$.

4. We proceed to consider the case in which $2m+1 = (2p+1)[\mu-1]$, where $p$ is an integer which varies with $\mu$ in such a manner that it always lies between 0 and $\mu$.

In this case, as before, we divide the integral $\int_0^\infty \phi(z)\frac{\sin(2m+1)z}{z}\,dz$ into the three parts

$$c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \sin[\mu]z\,\frac{\sin(2p+1)[\mu-1]z}{z}\,dz,$$

$$+\sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z\sin(2p+1)[\mu-1]z}{z}\,dz,$$

$$+\sum_{n=\mu+1}^{\infty} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z\sin(2p+1)[\mu-1]z}{z}\,dz.$$

The first part is equal to

$$\frac{c_\mu[\mu]}{2\pi}\int_{\pi/[\mu]}^{\beta}\Big[\cos\{[\mu-1](2\mu-2p)\} - \cos\{[\mu-1](2\mu+2p+2)\}\Big]\,dz$$

where $\beta$ is a number between $\pi/[\mu]$ and $\pi/[\mu-1]$, and this expression is less in absolute value than

$$\frac{c_\mu[\mu]}{\pi}\left\{\frac{1}{[\mu-1](2\mu-2p)} + \frac{1}{[\mu-1](2\mu+2p+2)}\right\}$$

or than
$$\frac{c_\mu}{\pi}\left\{\frac{2\mu+1}{2\mu-2p} + \frac{2\mu+1}{2\mu+2p+2}\right\},$$

and this may be written

$$\frac{c_\mu}{\pi}\left\{\frac{1+\dfrac{1}{2\mu}}{1-\dfrac{p}{\mu}} + \frac{1+\dfrac{1}{2\mu}}{1+\dfrac{1}{\mu}+\dfrac{p}{\mu}}\right\}.$$

If, now, $p$ increases with $\mu$ in such a manner that $p/\mu$ is always less than some fixed number which is less than unity, this expression diminishes indefinitely as $\mu$ is indefinitely increased; it would also be sufficient that $p/\mu = 1-\kappa/\{\log(2\mu+1)\}^{s'}$, where $s' < s$, $c_\mu = 1/\{\log(2\mu+1)\}^{s}$, and $\kappa$ is finite.

Next, we have $\sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z\sin(2p+1)[\mu-1]z}{z}\,dz$ is less in absolute value than

$$\sum_{n=\lambda_1}^{\mu-1}\frac{c_n[n]}{\pi}\left\{\frac{1}{(2p+1)[\mu-1]-[n]} + \frac{1}{(2p+1)[\mu-1]+[n]}\right\}$$

or than
$$\frac{c_{\lambda_1}}{\pi}\Sigma\frac{[n]}{[\mu-1]}\left\{\frac{1}{2p+1-\dfrac{[n]}{[\mu-1]}}+\frac{1}{2p+1+\dfrac{[n]}{[\mu-1]}}\right\},$$

and this is less than

$$\frac{c_{\lambda_1}}{p\pi}\left\{1+\frac{1}{2\mu-1}+\frac{1}{(2\mu-1)(2\mu-3)}+\dots\right\}$$

or than $2c_{\lambda_1}/p\pi$; thus the expression becomes indefinitely small when $p$ is increased indefinitely.

That
$$\sum_{n=\mu+1}^{\infty}c_n\int_{\pi/[n]}^{\pi/[n-1]}\frac{\sin[n]z\sin(2p+1)[\mu-1]z}{z}\,dz$$

has the limit zero is seen from the fact that its absolute value is less than $c_{\mu+1}(2p+1)[\mu-1]\dfrac{\pi}{[\mu]}$, or than $\pi c_{\mu+1}\dfrac{2p+1}{2\mu+1}$.

We have now established that $\displaystyle\int_0^a\phi(z)\frac{\sin(2m+1)z}{z}\,dz$ has the limit zero if $2m+1$ increases indefinitely through a sequence of the form

$$[\mu_1-1](2p_1+1),\qquad[\mu_2-1](2p_2+1),\qquad[\mu_3-1](2p_3+1),\qquad\dots$$

where $\mu_1, \mu_2, \mu_3, \dots$ is an increasing sequence of integers, and $p_1, p_2, p_3, \dots$ are such that $p/\mu \leqslant 1-\kappa/\{\log(2\mu+1)\}^{\kappa'}$.

The limit of the same integral has been shown to be infinite if $2m+1$ increases indefinitely through a sequence of values $[\mu_1]$, $[\mu_2]$, $[\mu_3]$, ....

### *A Continuous Function for which the Series does not converge at a Dense Set of Points.*

5. Let the continuous function $f(x)$ be defined for the interval $(-\pi, \pi)$ as follows:—If $-\pi \leqslant x \leqslant \xi$, where $\xi$ is a fixed point in the interval, let $f(x)=0$; if $0 \leqslant x-\xi < 2a$, let $f(x)=\phi\left(\dfrac{x-\xi}{2}\right)$, where $\phi\left(\dfrac{x-\xi}{2}\right)$ is the function $\phi(z)$ which has been already discussed; in case $\xi+2a<\pi$, we take $f(x)=0$, for $\xi+2a<x\leqslant\pi$.

The limit of the sum of the first $2m+1$ terms of the Fourier's series for the function $f(x)$ is that of the expression

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x')\frac{\sin(2m+1)\dfrac{x'-x}{2}}{\sin\dfrac{x'-x}{2}}\,dx',$$

and at the point $\xi$ the limit depends upon that of

$$\frac{1}{\pi}\int_0^\pi \phi(z)\frac{\sin(2m+1)z}{\sin z}\,dz.$$

It has been shown that this limit is zero, or indefinitely great, according to the nature of the sequence of values through which $2m+1$ increases indefinitely. It follows that, at the point $\xi$, the Fourier's series is not convergent, but is quasi-convergent to the value $f(\xi)\equiv 0$ of the function.

Let us now denote the function $f(x)$ by $\mathsf{f}(x, \xi)$, and let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be an enumerable set of values of $\xi$, everywhere dense in the interval $(-\pi, \pi)$, and let us consider the function

$$F(x) = c_1\mathsf{f}(x, \xi_1)+c_2\mathsf{f}(x, \xi_2)+\ldots+c_n\mathsf{f}(x, \xi_n)+\ldots,$$

where $c_1, c_2, \ldots, c_n, \ldots$ are constants so chosen that the series

$$c_1+c_2+\ldots+c_n+\ldots$$

is absolutely convergent.

Since the upper limits of each of the functions $|\mathsf{f}(x, \xi)|$ has the same finite value, it follows that the series $c_1\mathsf{f}(x, \xi_1)+c_2\mathsf{f}(x, \xi_2)+\ldots$ is uniformly convergent in the interval $(-\pi, \pi)$, and thus the function $F(x)$ is continuous, and the expression

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(x')\frac{\sin(2m+1)\dfrac{x'-x}{2}}{\sin\dfrac{x'-x}{2}}\,dx'$$

is equal to the sum

$$\frac{1}{2\pi}\sum_1^\infty c_n\int_{-\pi}^{\pi} \mathsf{f}(x', \xi_n)\frac{\sin(2m+1)\dfrac{x'-x}{2}}{\dfrac{x'-x}{2}}\,dx',$$

which may be written in the form

$$c_1\chi_1(x, m)+c_2\chi_2(x, m)+\ldots+c_n\chi_n(x, m)+\ldots,$$

where $\underset{m=\infty}{\mathrm{L}}\chi_1(x, m)$ is zero, unless $x=\xi_1$, at which point the limit may be either 0 or $\infty$, according to the mode in which $m$ is indefinitely increased; a similar statement holds as regards $\chi_2(x, m)$ at the point $\xi_2$, and generally $\underset{m=\infty}{\mathrm{L}}\chi_n(x, m)$ is zero, unless $x=\xi_n$, in which case the limit depends upon the mode in which $m$ becomes infinite.

At the point $\xi_n$, the term $c_n\chi_n(x, m)$ has an infinite limit provided $m$ is increased indefinitely in a proper manner, but it might happen that the limit of

$$c_{n+1}\chi_{n+1}(\xi_n, m)+c_{n+2}\chi_{n+2}(\xi_n, m)+\ldots$$

is also infinite, although each separate term has a zero limit; in that case the limit of the whole expression for the sum of the series might be finite or zero, in whatever manner $m$ were made indefinitely great. If this happened for a particular set of values of the constants $c_1, c_2, \ldots, c_n, c_{n+1}, \ldots$, it would no longer happen if these constants were replaced by $c_1 e_1$, $c_2 e_1 e_2$, $c_3 e_1 e_2 e_3, \ldots, c_n e_1 e_2 \ldots e_n, \ldots$, where $e_1, e_2, e_3, \ldots$ is a sequence of descending positive numbers, provided they are properly chosen. For, if

$$c_n \chi_n(\xi_n, m) + \{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2}\chi_{n+2}(\xi_n, m) + \ldots\},$$

when $m$ is indefinitely increased, were finite, being dependent on the form $\infty - \infty$, the expression

$$e_1 e_2 \ldots e_n c_n \chi_n(\xi_n, m) + e_1 e_2 \ldots e_{n+1}\{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2} e_{n+2} \chi_{n-2}(\xi_n, m) + \ldots\}$$

would also be finite or zero, only in case

$$e_{n+1} \frac{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2} e_{n+2} \chi_{n+2}(\xi_n, m) + \ldots}{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2}\chi_{n+2}(\xi_n, m) + \ldots}$$

had as its limit unity, when $m$ is indefinitely increased. But this limit can be altered by changing $e_{n+1}$, without altering $e_{n+2}, e_{n+3}, \ldots$; and thus $e_{n+1}$ can certainly be chosen so that this expression does not converge to unity when $m$ is indefinitely increased.

It has thus been shown that, by choosing the numbers $e_1, e_2, \ldots$ properly, the limit of the ratio of

$$e_1 e_2 \ldots e_n e_{n+1}\{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2} e_{n+2} \chi_{n+2}(\xi_n, m) + \ldots\}$$

to

$$e_1 e_2 \ldots e_n c_n \chi_n(x, m)$$

will be different from what it was when all the $e$'s were equal to unity. It has therefore been shown that, by altering the numbers $c_1, c_2, c_3, \ldots$ in a suitable way, the infinite limit of $c_n \chi_n(\xi_n, m)$ when $m$ is indefinitely increased will no longer be removed by an infinite limit of the sum

$$c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2}\chi_{n+2}(\xi_n, m) + \ldots.$$

*It has thus been shown that it is possible to choose the numbers* $c_1, c_2, c_3, \ldots$ *in such a manner that the continuous function*

$$F(x) = \sum_1^\infty c_n \mathfrak{f}(x, \xi_n)$$

*is such that the Fourier's series fails to converge to the value of the function at each point of the everywhere dense set of points* $(\xi_1, \xi_2, \ldots \xi_n)$.

# ON CERTAIN CLASSES OF SYZYGIES

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[Communicated November 10th, 1904.—Received January 9th, 1905.]

THE object of this paper is to extend so far as possible the results obtained for perpetuant syzygies in a recent paper* by Mr. Wood and myself to syzygies between concomitants of binary forms of finite order. In using the symbolical notation for binary forms it is unnecessary to write down any symbolical factors involving the variables, for these may be supplied when the orders of the forms are known. In this case the symbolical expressions used in the paper referred to for covariants of forms of infinite order may also be regarded as covariants of forms

$$a_{1_x}^{n_1}, \quad a_{2_x}^{n_2}, \quad \ldots,$$

all of finite order; provided that in any such expressions no letter $a_r$ occurs to a degree greater than $n_r$, the order of the corresponding quantic.

The result to be proved is that the syzygies obtained for perpetuants are true when the orders of some or all of the quantics concerned are finite, if those forms in which any letter $a_r$ appears to a degree greater than the order $n_r$ of the corresponding quantic are ignored, except for forms which have a factor $(a_r a_s)^\lambda (a_s a_t)^{n_s-\lambda}$.

Thus, if $\Sigma P + \Sigma P' = 0$ be one of the perpetuant syzygies obtained in the former paper, where $\Sigma P$ is the sum of those products in which the letter $a_1$ does not appear to a degree greater than $n_1$, the letter $a_2$ does not appear to a degree greater than $n_2$, and so on, and $\Sigma P'$ is the sum of those products in which some letter $a_r$ appears to a degree greater than $n_r$, then for the quantics $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, ... we have a syzygy

$$\Sigma P = \Sigma Q$$

where each of the terms $Q$ has a factor of the form $(a_r a_s)^\lambda (a_s a_t)^{n_s-\lambda}$.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 221.

The result is exactly analogous to the extension of reducibility results for perpetuants to forms of finite order.* It will be convenient here to call covariants such as $Q$ forms of the second class, as was done in the paper referred to.

To obtain the result it is necessary to prove it true for Stroh's syzygies, and this constitutes practically the only difficulty.

1. Stroh's fundamental syzygy

$$\{(ab)+(cd)\}^w-\{(ad)+(cb)\}^w=0$$

is true for forms of finite order only when $w$ is less than the order of each of the forms concerned. We proceed to obtain a corresponding syzygy, true for all values of $w$; in fact, it is to be proved that

$$\Sigma\binom{w}{r}(ab)^r(cd)^{w-r}-\Sigma\binom{w}{r}(ad)^r(cb)^{w-r}=\Sigma Q$$

where each of the terms $Q$ on the right has a factor of the form

$$(a_r a_s)^\lambda (a_s a_t)^{n_s-\lambda}$$

and the sums on the left extend only to such terms as can possibly represent covariants of the quantics in question. Thus, if $a_x^{n_1}$, $b_x^{n_2}$, $c_x^{n_3}$, $d_x^{n_4}$ are the quantics, then $r \ngtr n_1$ or $n_2$ in the first sum, while it is $\nless w-n_3$ or $w-n_4$; and similarly for the second sum.

2. We shall first prove the identity ($n_4 < w$)

$$\sum_{r=0}^{n_4}\binom{w}{r}\left[(ab)^{w-r}(cd)^r-(cb)^{w-r}(ad)^r\right]$$

$$=\sum_{r=0}^{w-n_4-1}\binom{w}{r}(ac)^{w-n_4-r}(cb)^r\sum_{s=0}^{n_4}\binom{w-n_4-1-r+s}{s}(ad)^{n_4-s}(cd)^s$$

where it is supposed that none of $n_1$, $n_2$, $n_3$ are less than $w$.

Let us assume this true when $n_4 \geqslant a$, and prove it for the case when $n_4=a-1$.

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* A. Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 1.

Since it is true when $n_4 = a$, we have

$$
\begin{aligned}
\sum_{r=0}^{a-1} \binom{w}{r} &\left[(ab)^{w-r}(cd)^r - (cb)^{w-r}(ad)^r\right] \\
&= -\binom{w}{a}\left[(ab)^{w-a}(cd)^a - (cb)^{w-a}(ad)^a\right] \\
&\quad + \sum_{r=0}^{w-a-1} \binom{w}{r}(ac)^{w-a-r}(cb)^r \sum_{s=0}^{a} \binom{w-a-1-r+s}{s}(ad)^{a-s}(cd)^s \\
&= -\sum_{r=0}^{w-a-1} \binom{w}{a}\binom{w-a}{r}(ac)^{w-a-r}(cb)^r(cd)^a \\
&\quad + \binom{w}{a}(cb)^{w-a}(ac)\left[(ad)^{a-1} + (ad)^{a-2}(cd) + \ldots + (cd)^{a-1}\right] \\
&\quad + \sum_{r=0}^{w-a-1} \binom{w}{r}(ac)^{w-a-r}(cb)^r \sum_{s=0}^{a} \binom{w-a-1-r+s}{s}(ad)^{a-s}(cd)^s \\
&= \binom{w}{a}(cb)^{w-a}(ac)\sum_{s=0}^{a-1}(ad)^{a-1-s}(cd)^s \\
&\quad + \sum_{r=0}^{w-a-1} \binom{w}{r}(ac)^{w-a-r}(cb)^r \sum_{s=0}^{a} \binom{w-a-1-r+s}{w-a-1-r}\left[(ad)^{a-s}(cd)^s - (cd)^a\right]^* \\
&= \binom{w}{a}(cb)^{w-a}(ac)\sum_{s=0}^{a-1}(ad)^{a-1-s}(cd)^s \\
&\quad + \sum_{r=0}^{w-a-1} \binom{w}{r}(ac)^{w-a-r+1}(cb)^r \sum_{s=0}^{a-1}(ad)^{a-s-1}(cd)^s \sum_{t=0}^{s} \binom{w-a-1-r+t}{w-a-1-r} \\
&= \sum_{r=0}^{w-a} \binom{w}{r}(ac)^{w-a+1-r}(cb)^r \sum_{s=0}^{a-1} \binom{w-a-r+s}{w-a-r}(ad)^{a-s-1}(cd)^s.
\end{aligned}
$$

Dividing this result by $a_x$, we see that the theorem is true for $n_4 = a-1$.

Hence, if the relation is true when $n_4 = a$, it is true when $n_4 = a-1$.

It is easily verified when $n_4 = w-1$; it is true for $n_4 = w$; hence it is always true.

3. In exactly the same way it may be shown that, if $n_2$ and $n_4$ are each less than $w$, but neither $n_1$ nor $n_4$ is less than $w$, then

$$
\begin{aligned}
\sum_{r=w-n_2}^{n_4} \binom{w}{r} &\left[(ab)^{w-r}(cd)^r - (cb)^{w-r}(ad)^r\right] \\
&= \sum_{r=0}^{w-n_4-1}\sum_{s=0}^{n_4} \binom{w}{r}\binom{w-n_4-1-r+s}{s}(ac)^{w-n_4-r}(cb)^r(ad)^{n_4-s}(cd)^s \\
&\quad - \sum_{r=0}^{w-n_2-1}\sum_{s=0}^{n_2} \binom{w}{r}\binom{w-n_2-1-r+s}{s}(ac)^{w-n_2-r}(cd)^r(ab)^{n_2-s}(cb)^s.
\end{aligned}
$$

---

* For $\sum_{t=0}^{s} \binom{w-a-1-r+s}{w-a-1-r} = \binom{w-r}{w-a-r} = \dfrac{\binom{w}{a}\binom{w-a}{r}}{\binom{w}{r}}$.

4. To extend the result of § 3 to the case in which $n_1$, $n_2$, $n_4$ are all less than $w$, but $n_3 \geqslant w$, we start with the result of § 3, where $n_1$ may be supposed equal to $w$, and then transform the relation to one which has a factor $a_x$; this factor being removed, we have a relation for the case $n_1 = w-1$. Proceeding thus step by step, a relation is established for all cases when $n_1 < w$.

LEMMA.—The following relation is required:—

$$(ac)^{w-n_4-r}(cb)^r(ad)^{n_4-s}(cd)^s$$

$$= \sum_{t=0}^{\lambda}\binom{n_4-s}{t}(ac)^{w-r-s-t}(cb)^r(cd)^{s+t}$$

$$+(-)^{\lambda+1}\sum_{t=\lambda+1}^{n_4-s}(-)^t\binom{t-1}{\lambda}\binom{n_4-s}{t}(ac)^{w-n_4-r}(cb)^r(cd)^{s+t}(ad)^{n_4-s-t}.$$

We assume that this is true for a particular value of $\lambda$, and then shew that it is true for $\lambda+1$.

Thus, on this assumption,

$$(ac)^{w-n_4-r}(cb)^r(ad)^{n_4-s}(cd)^s$$

$$= \sum_{t=0}^{\lambda}\binom{n_4-s}{t}(ac)^{w-r-s-t}(cb)^r(cd)^{s+t}$$

$$+\binom{n_4-s}{\lambda+1}(ac)^{w-n_4-r}(cb)^r(cd)^{s+\lambda+1}\{(ac)^{n_4-s-\lambda-1}-[(ad)-(cd)]^{n_4-s-\lambda-1}$$
$$+(ad)^{n_4-s-\lambda-1}\}$$

$$+(-)^{\lambda+1}\sum_{t=\lambda+2}^{n_4-s}(-)^t\binom{t-1}{\lambda}\binom{n_4-s}{t}(ac)^{w-n_4-r}(cb)^r(cd)^{s+t}(ad)^{n_4-s-t}$$

$$= \sum_{t=0}^{\lambda+1}\binom{n_4-s}{t}(ac)^{w-r-s-t}(cb)^r(cd)^{s+t}$$

$$+(-)^{\lambda+2}\sum_{t=\lambda+2}^{n_4-s}(-)^t\left\{\binom{n_4-s}{\lambda+1}\binom{n_4-s-\lambda-1}{t-\lambda-1}-\binom{t-1}{\lambda}\binom{n_4-s}{t}\right\}$$
$$\times(ac)^{w-n_4-r}(cb)^r(cd)^{s+t}(ad)^{n_4-s-t}$$

$$= \sum_{t=0}^{\lambda+1}\binom{n_4-s}{t}(ac)^{w-r-s-t}(cb)^r(cd)^{s+t}$$

$$+(-)^{\lambda+2}\sum_{t=\lambda+2}^{n_4-s}(-)^t\binom{t-1}{\lambda+1}\binom{n_4-s}{t}(ac)^{w-n_4-r}(cb)^r(cd)^{s+t}(ad)^{n_4-s-t}.$$

Hence the lemma is true for $\lambda+1$ if it is true for any particular value of $\lambda$. But it is evidently true when $\lambda = 0$; hence it is always true.

It follows in exactly the same way that

$$(ac)^{w-n_2-r}(cd)^r(ab)^{n_2-s}(cb)^s$$

$$= \sum_{t=0}^{\lambda}\binom{n_2-s}{t}(ac)^{w-r-s-t}(cd)^r(cb)^{s+t}$$

$$+(-)^{\lambda+1}\sum_{t=\lambda+1}^{n_2-s}(-)^t\binom{t-1}{\lambda}\binom{n_2-s}{t}(ac)^{w-n_2-r}(cd)^r(cb)^{s+t}(ab)^{n_2-s-t}.$$

5. Now apply these results to modify the relation of § 3, so that it involves not more than $n_1$ $a$'s in any term on the right. Then

$$\sum_{r=w-n_2}^{n_4}\binom{w}{r}\left[(ab)^{w-r}(cd)^r-(cb)^{w-r}(ad)^r\right]$$

$$= \sum_{r=0}^{w-n_4-1}\ \sum_{s=0}^{w-r-n_1-1}\ \sum_{t=w-r-s-n_1}^{n_4-s}(-)^{t+w-r-s-n_1}\binom{w}{r}\binom{w-n_4-1-r+s}{s}$$

$$\times\binom{t-1}{w-r-s-n_1-1}\binom{n_4-s}{t}(ac)^{w-n_4-r}(cb)^r(ad)^{n_4-s-t}(cd)^{s+t}$$

$$+\sum_{r=0}^{w-n_4-1}\ \sum_{s=w-r-n_1\text{ or }0}^{n_4}\binom{w}{r}\binom{w-n_4-1-r+s}{s}(ac)^{w-n_4-r}(cb)^r(ad)^{n_4-s}(cd)^s$$

$$-\sum_{r=0}^{w-n_2-1}\ \sum_{s=0}^{w-r-n_1-1}\ \sum_{t=w-r-s-n_1}^{n_2-s}(-)^{t+w-r-s-n_1}\binom{w}{r}\binom{w-n_2-1-r+s}{s}$$

$$\times\binom{t-1}{w-r-s-n_1-1}\binom{n_2-s}{t}(ac)^{w-n_2-r}(cd)^r(ab)^{n_2-s-t}(cb)^{s+t}$$

$$-\sum_{r=0}^{w-n_2-1}\ \sum_{s=w-r-n_1\text{ or }0}^{n_2}\binom{w}{r}\binom{w-n_2-1-r+s}{s}(ac)^{w-n_2-r}(cd)^r(ab)^{n_2-s}(cb)^s$$

$$+\sum_{r=0}^{w-n_4-1}\ \sum_{s=0}^{w-r-n_1-1}\ \sum_{t=0}^{w-r-s-n_1-1}\binom{w}{r}\binom{w-n_4-1-r+s}{s}\binom{n_4-s}{t}$$

$$\times(ac)^{w-r-s-t}(cb)^r(cd)^{s+t}$$

$$-\sum_{r=0}^{w-n_2-1}\ \sum_{s=0}^{w-r-n_1-1}\ \sum_{t=0}^{w-r-s-n_1-1}\binom{w}{r}\binom{w-n_2-1-r+s}{s}\binom{n_2-s}{t}$$

$$\times(ac)^{w-r-s-t}(cd)^r(cb)^{s+t}.$$

Let us first suppose that $n_1 \geqslant$ both $n_2$ and $n_4$. Then the left-hand side of the relation will require no alteration. Now the relation is an identity; therefore, if a $c$ is introduced into every determinant factor by means of relations of the form $(ab) = (ac)+(cb)$, then the sum of the coefficients of each of the resulting terms $(ac)^\lambda(cb)^\mu(cd)^\nu$ must be separately zero.

In particular, all those terms which have a factor $(ac)^{n_1+1}$ must vanish separately; now no such terms can arise from any part of either side of the relation except in its last two lines, and in these lines every term contains a factor $(ac)^{n_1+1}$. Moreover these terms are unaffected by the operation of introducing $c$ into every determinant factor; their sum is therefore zero.

Hence, when $n_1 > n_2$ and $n_1 > n_4$,

$$\sum_{r=w-n_2}^{n_4} \binom{w}{r} \left[(ab)^{w-r}(cd)^r - (cb)^{w-r}(ad)^r\right]$$

$$= \sum_{r=0}^{w-n_4-1} \sum_{s=0}^{w-r-n_1-1} \sum_{t=w-r-s-n_1}^{n_4-s} (-)^{t+w-r-s-n_1} \binom{w}{r} \binom{w-n_4-1-r+s}{s}$$
$$\times \binom{t-1}{w-r-s-n_1-1} \binom{n_4-s}{t} (ac)^{w-n_4-r} (cb)^r (ad)^{n_4-s-t} (cd)^{s+t}$$
$$+ \sum_{r=0}^{w-n_4-1} \sum_{s=w-r-n_1 \text{ or } 0}^{n_4} \binom{w}{r} \binom{w-n_4-1-r+s}{s} (ac)^{w-n_4-r} (cb)^r (ad)^{n_4-s} (cd)^s$$
$$- \sum_{r=0}^{w-n_2-1} \sum_{s=0}^{w-r-n_1-1} \sum_{t=w-r-s-n_1}^{n_2-s} (-)^{t+w-r-s-n_1} \binom{w}{r} \binom{w-n_2-1-r+s}{s}$$
$$\times \binom{t-1}{w-r-s-n_1-1} \binom{n_2-s}{t} (ac)^{w-n_2-r} (cd)^r (ab)^{n_2-s-t} (cb)^{s+t}$$
$$- \sum_{r=0}^{w-n_2-1} \sum_{s=w-r-n_1 \text{ or } 0}^{n_2} \binom{w}{r} \binom{w-n_2-1-r+s}{s} (ac)^{w-n_2-r} (cd)^r (ab)^{n_2-s} (cb)^s.$$

If $n_1 < n_2$, there will be terms on the right which when $c$ is introduced into every determinant factor will yield terms having a factor $(ac)^{n_1+1}$.

We may bring over these terms to the right-hand side, and, using the results of § 4, we may replace them by others by means of the relation

$$\sum_{r=w-n_2}^{w-n_1-1} \binom{w}{r} (ab)^{w-r} (cd)^r$$

$$= \sum_{r=w-n_2}^{w-n_1-1} \sum_{t=0}^{w-r-n_1-1} \binom{w}{r} \binom{w-r}{t} (ac)^{w-r-t} (cd)^r (cb)^t$$
$$+ \sum_{r=w-n_2}^{w-n_1-1} \sum_{t=w-r-n_1}^{w-r} (-)^{t+w-r-n_1} \binom{w}{r} \binom{t-1}{w-r-n_1-1} \binom{w-r}{t}$$
$$\times (cd)^r (cb)^t (ab)^{w-r-t}.$$

The first of these two sums will disappear with the last two lines of the former sums, and so the relation holds good with an alteration of limits on the left and with the subtraction of the second of the above sums from the right.

Each term of this second sum can be written in the form

$$\Sigma N(cd)^r (ac)^s (ab)^{n_1-s}(cb)^{w-r-n_1}.$$

If $n_1 < n_4$, another alteration in the limits must be made on the left, and terms of the form

$$\Sigma N(cb)^{w-r}(ac)^s(ad)^{n_1-s}(cd)^{r-n_1}$$

introduced on the right.

6. We have next to consider the case when $w$ is greater than the order of each of the quantics. The method of proof here does not require any assumption as to the relative magnitudes of $w$, $n_1$, $n_2$, $n_3$, $n_4$; so that the theorem is proved again for the three cases already discussed; on the other hand, the proof given here affords no information as to what are the terms of the second class on the right-hand side of Stroh's syzygy.

We shall assume that the theorem has been proved when the total weight of the syzygy is $w$ and the orders of the quantics are $n_1$, $n_2$, $n_3+1$, $n_4$; and then shew that it is true when the weight of the syzygy is $w$ and the orders of the quantics are $n_1$, $n_2$, $n_3$, $n_4$.

We have then a syzygy

$$\text{(I.)} \qquad \Sigma \binom{w}{r}(ab)^{w-r}(cd)^r - \Sigma\binom{w}{r}(cb)^{w-r}(ad)^r = \Sigma T$$

where $T$ is a covariant of $a_x^{n_1}$, $b_x^{n_2}$, $c_x^{n_3+1}$, $d_x^{n_4}$ of the second class; and the sums on the left extend only to such symbolical products as may be interpreted.

In will in general happen that when the order of $c$ becomes $n_3$ either one or two terms on the left of (I.) cease to be interpretable. Let $T'$ be the sum of these terms; they must be removed to the right-hand side of the relation.

Now (I.) is an algebraical identity between the symbols $a, b, c, d, x$; hence, if we can express the forms $\Sigma T - T'$ in terms of a sum of symbolical products which represent covariants of $a_x^{n_1}$, $b_x^{n_2}$, $c_x^{n_3}$, $d_x^{n_4}$ of the second class multiplied by a factor $c_x$, we shall have a new relation, from which a factor $c_x$ may be removed: the result is then the syzygy required.

We proceed to prove that this can be done. Those factors of a symbolical product which make the covariant belong to the second class are here referred to as the second class factors. No covariant of $\Sigma T - T'$ need be considered which has already a factor $c_x$.

If a covariant of $\Sigma T - T'$ has a factor $(\epsilon\zeta)$, which does not appear amongst its second class factors, and also a factor $b_x$, we will use the identity for $(\epsilon\zeta)\, b_x$, viz., $\quad (\epsilon\zeta) b_x = (\epsilon b)\zeta_x + (b\zeta)\epsilon_x$.

Thus all the covariants of $\Sigma T - T'$ can be expressed in terms of—

(1) Covariants of $a_x^{n_1}$, $b_x^{n_2}$, $c_x^{n_3}$, $d_x^{n_4}$ of the second class multiplied by $c_x$.

(2) Covariants of $a_x^{n_1}$, $b_x^{n_2}$, $c_x^{n_3+1}$, $d_x^{n_4}$ of the second class which have no factor $b_x$ or $c_x$.

(3) Covariants of $a_x^{n_1}$, $b_x^{n_2}$, $c_x^{n_3+1}$, $d_x^{n_4}$ of the second class which contain the letter $b$ in all factors other than the second class factors and have no factor $c_x$.

7. The covariants can be further restricted. Covariants of set (3) contain a factor $(bc)^{n_3+1}$ unless the letter $c$ appears in the original second class factors.

If the second class factors were originally $(c\epsilon)^{\lambda}(\epsilon\zeta)^{n-\lambda}$ (where $n$ is the order of $\epsilon$), we have now a factor $(bc)^{n_3+1-\lambda}(c\epsilon)^{\lambda}$. By means of repeated use of the identity for $(c\epsilon)\, b_x$ such a covariant can be expressed in terms of covariants (1), (2), and

(4) Covariants which have a factor $(bc)^{n_3+1}$.

It remains to consider covariants which have a factor $(ac)^{\lambda}(cd)^{n_3+1-\lambda}$. Here we begin by introducing $d$ into as many determinant factors as possible; the covariant is thus expressed in terms of covariants (1), of covariants which have no factor $c_x$ or $d_x$, and of covariants which have a factor $(cd)^{n_3+1}$.

By means of the identity for $(ac)(bd)$ we may express those covariants which have no factor $c_x$ or $d_x$ in terms of covariants which have one or other of the factors $(bc)^{\lambda}(cd)^{n_3+1-\lambda}$, $(cd)^{\lambda}(da)^{n_4-\lambda}$.

Covariants having either one of these factors or else the factor $(cd)^{n_3+1}$ have already been discussed.

We are thus left with covariants (1), (2), and (4).

Covariants of set (2) may be written

$$(bc)^{\lambda}(cd)^{\mu}(da)^{\nu}(ab)^{\rho}(bd)^{\sigma}(ac)^{\tau};$$

where $\qquad \lambda+\mu+\tau = n_3+1, \qquad \lambda+\rho+\sigma = n_2,$

and one of the indices is zero.

By means of the identity for $(ab)(cd)$ we may express this in terms of similar covariants in which either $\rho = 0$ or $\mu = 0$. When $\rho = 0$ there is a factor $(bc)^{\lambda_1}(bd)^{n_2-\lambda_1}$, and by means of the identity for $(ac)\, d_x$ we may express the covariant in terms of covariants (1) and of

(5) Covariants which have no factor $c_x$, $b_x$, or $d_x$;

(6) Covariants which have no factor $c_x$ or $b_x$ and which contain the letter $a$ only in factors $(ad)$, $a_x$.

When $\mu = 0$ there is a factor $(bc)^{\lambda}(ca)^{n_3+1-\lambda}$, and by means of repeated use of the identity

$$(ac)(ab)\,d_x = (ac)(db)\,a_x + (bc)(ad)\,a_x + (ab)(ad)\,c_x$$

the covariant can be expressed in terms of covariants (1), (4), (5) and of covariants of the same kind for which $\rho = 0$.

Thus we can express all covariants of the sum $\Sigma T - T'$ in terms of covariants (1), (4), (5), (6).

Now the relation from which we started may be regarded as an identity expressing $\Sigma T - T'$ as a sum of terms having a factor $c_x$. Hence the sum of the covariants of sets (4), (5), and (6) which now appear on the right-hand side of our relation must have a factor $c_x$. It is shewn in the next paragraph that this means that the sum is zero; hence $\Sigma T - T'$ is equal to a sum of covariants (1).

The theorem is then true when the order of $c$ is $n_3$, provided it is true when the order of $c$ is $n_3+1$. Hence, if it is true when the order of $c$ is $w$, it is always true. But it is true when the order of no one of the quantics is less than $w$: hence by repeated use of the induction it is true always.

8. *A sum of covariants, linear in the coefficients of each of the four quantics*

$$a_x^{n_1},\quad b_x^{n_2},\quad c_x^{n_3+1},\quad d_x^{n_4},$$

*each of which is of one of the following kinds*:—

(1) *Covariants which have no factor $c_x$, $b_x$, or $d_x$;*

(2) *Covariants which have no factor $c_x$, $b_x$ and in which the letter $a$ appears only in factors $(ad)$, $a_x$;*

(3) *Covariants which have a factor $(bc)^{n_3+1}$;*—

*cannot be expressed as a sum of symbolical products having a factor $c_x$, unless this sum is zero.*

The covariants (1) can be expressed as transvectants

$$\left[(cb)^{\lambda_1}(cd)^{\lambda_2}(bd)^{\lambda_3},\ a_x^{n_1}\right]^{n_2+n_3+n_4-2\lambda_1-2\lambda_2-2\lambda_3},$$

each of which only contains a single term. These transvectants may be expressed in terms of the transvectants

$$\left[(cb)^{\lambda_1}(cd)^{n_3-\lambda_1}(bd)^{\lambda_2},\ a_x^{n_1}\right]^{n_2+n_4-n_3-2\lambda_2},$$

and of transvectants $\left[(cb)^{\lambda_1}(cd)^{\lambda_2},\ a_x^{n_1}\right]^{n_2+n_3+n_4-2\lambda_1-2\lambda_2}$.

The covariants (2) can be expressed as transvectants

$$\left[(cb)^{\lambda_1}(cd)^{n_3-\lambda_1}(bd)^{n_2-\lambda_1},\ a_x^{n_1}\right]^{\rho},$$

each of which contains only a single term.

The covariants (8) can be expressed in terms of transvectants

$$\left[(cb)^{n_3+1}(bd)^{\lambda_1},\ a_x^{n_1}\right]^{\rho}.$$

Now transvectants of $a_x^{n_1}$ which have not the same index are independent; hence, if the sum of covariants has a factor $c_x$, the sums of the equivalent transvectants of $a_x^{n_1}$ for each index must have a factor $c_x$. Hence in all cases in which the index of the transvectant is equal to the order of the covariant on the right the sum of the transvectants must be zero. For other cases, if $w$ is the weight of the covariant, and $\rho$ the index of the transvectant, then the index of $(bd)$ is known to be $w-\rho-n_3-1$, and the index of $(cb)$ is the least of the numbers $n_2-(w-\rho-n_3-1)$ and $n_3+1$; so that there is only one such term. This term does not contain a factor $c_x$; so that, if the sum of covariants considered has a factor $c_x$, the coefficient of each of these transvectants must be zero; *i.e.*, the sum must be zero.

COROLLARY.—*The same is true if one or more of the letters a, b, c, d refers not to a single fundamental quantic, but to either of two quantics: thus we may suppose that a is written for a or a′ [and in this case it may be supposed that the covariant has a factor* $(aa')^{\mu_1}$*], provided further that there is no factor* $a_x$ *or else that the letter a′ appears only in factors* $(aa')$, $a'_x$, *and that this restriction is to be applied in every case where a letter does not refer to a single fundamental quantic.*

The method of proof is nearly identical with that which precedes. The covariants (1) when

$$\nu_1-\mu_1 \geqslant n_3+n_2+n_4-2\lambda_1-2\lambda_2-2\lambda_3 \geqslant \rho, \text{ say,}$$

can be expressed as transvectants

$$\left[(cb)^{\lambda_1}(cd)^{\lambda_2}(bd)^{\lambda_3},\ (aa')^{\mu_1}a_x^{\nu_1-\mu_1}a_y'^{\nu'_1-\mu_1}\right]^{\rho}_{y=x};$$

and when $\nu_1-\mu_1 < \rho$ these covariants can be expressed as transvectants

$$\left[(cb)^{\lambda_1}(cd)^{\lambda_2}(bd)^{\lambda_3},\ (aa')^{\mu_1}a_x^{\nu_1-\mu_1}a_x'^{\rho+\mu_1-\nu_1}a_y'^{\nu'_1+\nu_1-2\mu_1-\rho}\right]^{\rho}_{y=x}.$$

The letters on the left-hand side of the transvectants may here be simple or composite: it does not affect matters.

Similarly covariants (2) can be expressed as transvectants

$$(P,\ Q)^{\rho}_{y=x},$$

where

$$P = (cb)^{\lambda_1}(cd)^{n_3-\lambda_1}(bd)^{n_2-\lambda_1}\big[d_x^{n_4+2\lambda_1-n_2-n_3},\ (\delta\delta')^{\mu_4}\delta_x^{\nu_4+2\lambda_1-n_2-n_3-\mu_4}\delta_y'^{\nu'-\mu_4},$$
$$(\delta\delta')^{\mu_4}\delta_x^{\nu_4+2\lambda_1-n_2-n_3-n_4}\delta_x'^{\rho+n_2+n_3+n_4-\nu_4-2\lambda_1}\delta_y'^{\cdots},$$
$$\text{or}\quad (\delta\delta')^{\mu_4}\delta_x'^{\rho}\delta_y'^{\nu'_4-\rho-\mu_4}\big],$$

according to the magnitude of $\nu_4$, and where

$$Q = (aa')^{\mu_1}(a_x^{\nu_1-\mu_1}a_y'^{\nu'_1-\mu_1} \text{ or } a_x^{\nu_1-\mu_1}a_x'^{\rho+\mu_1-\nu_1}a_y'^{\nu'_1+\nu_1-\rho-2\mu_1}).$$

The covariants (8) can be expressed in terms of transvectants in an exactly similar way. When this has been done the argument is the same as in the preceding theorem.

If $a$ represents a fundamental quantic, the theorem follows exactly in the same way.

9. When one of the letters $a$, $b$, $c$, $d$ refers to a covariant of the original quantics Stroh's syzygies for perpetuants are true as before. Unfortunately forms which are of the second class when $c$ is one of the original quantics are not necessarily of the second class when $c$ is a covariant. For, although a form having a factor $(ac)^{\lambda}(cb)^{n_3-\lambda}$ can be proved to be really of the second class when $c$ is a covariant of order $n_3$, yet forms having a factor $(ab)^{\lambda}(bc)^{n_2-\lambda}$ are by no means necessarily of the second class. As we shall require Stroh's syzygy in the cases when some or all of $a$, $b$, $c$, $d$ refer to covariants, it is necessary to establish it in these cases. The method of proof is essentially the same as that used in §§ 6, 7.

We shall first confine our attention to the cases when $a$, $b$, $c$, $d$ represent either covariants of the second degree or else fundamental quantics.

If $c = (\gamma\gamma')^{\mu_3}\gamma_x^{\nu_3+1-\mu_3}\gamma_x'^{\nu'_3-\mu_3}$, Stroh's syzygy will be written

$$\Sigma\binom{\lambda}{r}(ab)^r(\gamma\gamma')^{\mu_3}(\gamma d)^{\lambda-r}-\Sigma\binom{\lambda}{r}(ad)^r(\gamma\gamma')^{\mu_3}(\gamma b)^{\lambda-r} = \Sigma T.$$

In the same way, if $a = (aa')^{\mu_1}$, the letter $a'$ will appear in the factors $(aa')$, $a'_x$ only on the left-hand side.

It will be assumed that the syzygy is true when the order of $\gamma$ is $\nu_3+1$, and then it will be shewn to be true when the order of $\gamma$ is $\nu_3$. It is, however, necessary to further limit the forms $\Sigma T$ which are of the second class. The precise limitations imposed depend on the number of the quantics $a$, $b$, $c$, $a$ which are fundamental, and are stated later in §§ 12–15.

It is sufficient here to notice that in the beginning of the induction, when $a$, $b$, $d$ are fundamental quantics and $\lambda = \nu_3+1$, then the terms $\Sigma T - T'$ only contain $\gamma'$ in their second class factors when it appears in the factor $(\gamma\gamma')$.

10. LEMMA.—All covariants linear in the coefficients of each of the quantics $a_x^{n_1}$, $b_x^{n_2}$, $c_x^{n_3+1}$, $d_x^{n_4}$ [where $c_x^{n_3+1} \equiv (\gamma\gamma')^{\mu_3}\gamma_x^{\nu_3+1-\mu_3}\gamma'^{\,\nu'_3-\mu_3}$ and $a$, $b$, $d$ are fundamental quantics or else covariants of degree 2], which belong to the second class owing to factors of the form $(\epsilon\zeta)^\lambda(\zeta\eta)^{n-\lambda}$, or of the form $(\epsilon\zeta)^\lambda(\zeta\zeta')^{n-\lambda}$, where $n$ is the order of the quantic $\zeta$, and $\epsilon$, $\zeta$, $\eta$ refer to any of the quantics $a_x^{n_1}$, $b_x^{\nu_2}$, $\gamma_x^{\nu_3+1}$, $d_x^{n_4}$ if fundamental (or if one or more of these are covariants to $\alpha_x^{\nu_1}$, $\beta_x^{\nu_2}$, or $\delta_x^{\nu_4}$ as the case may be), can be expressed in terms of—

(1) Covariants of the same character when $\gamma_x^{\nu_3}$ is written for $\gamma_x^{\nu_3+1}$, which have a factor $\gamma_x$.

(2) Covariants which have no factor $c_x$, $b_x$, or $d_x$; where, for instance, $c_x$ stands for either $\gamma_x$ or $\gamma'_x$.

(3) Covariants which have no factor $c_x$, $b_x$ and in which the letters $a$ only appear in factors $(ad)$, $a_x$.

(4) Covariants which have a factor $(\beta\gamma')^{\nu'_3-\lambda}(\gamma'\gamma)^\lambda$ or a factor $(\beta\gamma')^{\nu_3-\lambda}(\beta\beta')^\lambda$, or, if $b$ is a fundamental quantic, a factor $(b\gamma')^{\nu'_3-\lambda}(\gamma'\gamma)^\lambda$. Provided further that, if $b \equiv (\beta\beta')^{\mu_2}\beta_x^{\nu_2-\mu_2}\beta_x'^{\,\nu'_2-\mu_2}$, then $\nu'_2 \geqslant \nu_2$, and that in the similar cases the orders of $\alpha$ and $\delta$ are not greater than those of $\alpha'$, $\delta'$ respectively.

In the course of the proof additional factors $(\alpha\alpha')$, $(\beta\beta')$, $(\gamma\gamma')$, $(\delta\delta')$ will from time to time make their appearance; it will be regarded as a reduction whenever such an additional factor appears without destroying the restrictions imposed on the covariants under consideration: none of the operations employed involve the loss of such a factor.

We shall first show that any covariant of the kind considered which has a factor $\gamma'_x$ may be expressed in terms of covariants (1) and of covariants of the same kind which have an additional factor $(\gamma\gamma')$. All such covariants may then be neglected.

For consider such a covariant. Either there is a factor $(\gamma\gamma')^{\mu_3}(\gamma\epsilon)^{\nu_3+1-\mu_3}$, or else there is a factor $(\gamma\eta)$ which does not belong to the second class factors.

In the first case a reduction is obtained by means of the identity for $(\gamma\epsilon)\gamma'_x$; in the second case by the identity for $(\gamma\eta)\gamma'_x$. In either case the statement is true.

The next step is to remove all factors $b_x$. If there is a factor $b_x$, any factor $(\epsilon\zeta)$ which does not contain $b$ and does not belong to the second class factors may be removed by the identity for $(\epsilon\zeta)\,b_x$; and every time a factor $c_x$ appears there is a reduction. Thus we are eventually left with covariants (1), with covariants which have no factor $b_x$ or $c_x$, and with covariants which have the letter $b$ ($\beta$ or $\beta'$) in every determinant factor except the second class factors. If $b$ is a fundamental quantic, all such covariants have a factor $(b\gamma')^{\nu'_3-\mu_3}(\gamma\gamma')^{\mu_3}$, and are therefore included in (4). When $b\equiv(\beta\beta')^{\mu_2}$, the covariants can be expressed in terms of such as have a factor $(\beta\gamma')^{\nu'_3-\mu_3}(\gamma\gamma')^{\mu'}$ or a factor $(\beta\gamma')^{\nu_3-\mu_3}(\beta\beta')^{\mu_3}$, and of covariants of exactly the same nature as to the second class factors which have additional factors $(\beta\beta')$. Thus, in addition to covariants (1) and (4), we have only to consider covariants which have no factor $b_x$ or $c_x$.

Let us suppose that $d_x\equiv(\delta\delta')^{\mu_4}$, that there is a factor $(\epsilon\delta')$ where $\epsilon\neq\delta$, and that there is a factor $\delta_x$. By means of the identity for $(\epsilon\delta')\delta_x$, all factors $\delta_x$ may be removed, or else all determinant factors except $(\delta\delta')$ in which $\delta'$ appeared; it will be assumed that this operation has been carried out, as also the corresponding operation affecting the letters $a$.

We have now to consider covariants

$$\text{(I.)}\qquad (bc)^{\lambda}(cd)^{\mu}(da)^{\nu}(ab)^{\rho}(bd)^{\sigma}(ac)^{\tau},$$

where
$$\lambda+\mu+\tau=n_3+1,\qquad \lambda+\rho+\sigma=n_2,$$

and where the letter $c$ stands for either of the letters $\gamma$, $\gamma'$; the letter $b$ stands for either of the letters $\beta$, $\beta'$; and so on.

By means of the identity for $(bd)(ac)$, these covariants can be expressed in terms of similar covariants, in which either $\sigma=0$ or $\tau=0$.

*When* $\sigma=0$, $\qquad\rho+\lambda=n_2$.

If $a=(aa')^{\mu_1}$, either there is no factor $a_x$ or every letter $a$ in the factors $(ab)$, $(ad)$, $(ac)$ may be taken to be $a$. In the first case we may use the identities

$$(a'b)(a\epsilon)-(ab)(a'\epsilon)=-(aa')(b\epsilon),$$

and hence these covariants can be expressed in terms of covariants which have a factor $(ba)^{\nu_1-\mu_1}(aa')^{\mu_1}$, of covariants of the above form in which all the factors $(ab)$ are $(ab)$, and of covariants of exactly the same nature which have additional factors $(aa')$.

The first set of covariants can be expressed in terms of covariants having a factor $(\beta a)^{\nu_1-\mu_1}(aa')^{\mu_1}$, or of covariants having a factor $(\beta a)^{\nu_2-\mu_2}(\beta\beta')^{\mu_2}$—there being no factor $b_x$ or $c_x$ in either case—and of covariants having no factor $b_x$ or $c_x$ but having additional factors $(\beta\beta')$. When there is a factor $(\beta a)^{\nu_1-\mu_1}(aa')^{\mu_1}$ or a factor $(\beta a)^{\nu_2-\mu_2}(\beta\beta')^{\mu_2}$ we may use the identity for $(ac)\,d_x$ to express the covariant in terms of covariants (2), of covariants (I.) in which $\tau = 0$, and of covariants (1).

We need only then consider the case in which the factor $(ab)^\rho\,(bc)^\lambda$ is $(\alpha b)^\rho\,(bc)^\lambda$.

Reasoning as before, it may be shown that we may further suppose this factor to be $(ab)^\rho\,(b\gamma)^\lambda$.

If $b$ represents a fundamental quantic, we may at once express such a covariant in terms of covariants (2), of covariants (I.) for which $\tau = 0$, and of covariants (1), by using the identity for $(ac)\,d_x$. The same is true when $b = (\beta\beta')^{\mu_2}$, for it is evident (since $\lambda+\rho = n_2$) that we may express such covariants in terms of covariants (I.) which have either a factor $(a\beta)^{\nu_2-\mu'_2}(\beta\beta')^{\mu'_2}$ or a factor $(\gamma\beta)^{\nu_2-\mu'_2}(\beta\beta')^{\mu'_2}$, where $\mu'_2 \geqslant \mu_2$, the order of $\beta$ having been assumed to be equal to or less than that of $\beta'$.

*When* $\tau = 0$, $\qquad\qquad \lambda+\mu = n_3+1.$

Proceeding as before, we can express all such covariants in terms of covariants (I.) which have one of the following factors

$$(\beta\beta')^{\mu_2}(\beta\gamma)^{\nu_2-\mu_2}, \quad (\gamma\gamma')^{\mu_3}(\gamma\beta)^{\nu_3+1-\mu_3}, \quad (\gamma\gamma')^{\mu_3}(\gamma\delta)^{\nu_3+1-\mu_3}, \quad (\delta\delta')^{\mu_4}(\delta\gamma)^{\nu_4-\mu_4},$$

and of covariants (I.) for which $\tau = 0$, but which have additional factors $(aa')$, ....

In the first two cases we may remove factors $(ab)\,d_x$ one pair at a time; one of the resulting forms has a factor $b_x$, and this may be removed by the identity for $(cd)\,b_x$ unless there is a factor $(bc)^{n_3+1}$. Thus we are eventually left with covariants (1), (2), (3) and covariants which have a factor $(bc)^{n_3+1}$.

When there is a factor $(\gamma\gamma')^{\mu_3}(\gamma\delta)^{\nu_3+1-\mu_3}$ or a factor $(\delta\delta')^{\mu_4}(\gamma\delta)^{\nu_4-\mu_4}$ we may proceed just as in the first two cases, if there is a factor $(\gamma'\delta)$, $(\gamma\delta')$, or $(\gamma'\delta')$. Otherwise there is a factor $(\gamma\gamma')^{\mu_3}(\beta\gamma')^{\nu'_3-\mu_3}$, and the covariant is one of the forms required. This completes the proof of the lemma, for covariants which have a factor $(bc)^{n_3+1}$ can be expressed in terms of covariants (4) and covariants (I.) which have additional factors $(\beta\beta')$.

11. *Covariants which have a factor* $(\beta\gamma')^{\nu'_3-\mu_3}(\gamma\gamma')^{\mu_3}$ *or a factor* $(\beta\gamma')^{\nu_3-\mu_3}(\beta\beta')^{\mu_3}$, *or, if* $b$ *represents a fundamental quantic, a factor* $(b\gamma')^{\nu'_3-\mu_3}(\gamma'\gamma)^{\mu_3}$, *can be expressed in terms of*

(1) *Covariants of the same character which possess also a factor* $\gamma_x$.

(2) *Covariants which have no factor* $c_x$, $b_x$, *or* $d_x$.

(3) *Covariants which have no factor* $c_x$ *or* $b_x$ *and in which the letters* $a$ *only appear in factor* $(ad)$, $a_x$.

(4) *Covariants which have a factor* $(cb)^{n_3+1}$.

To prove this, take a covariant which has one of the three given factors. Any factor $(\epsilon\zeta)\beta_x$, $[\epsilon \neq \beta,\ \zeta \neq \beta]$ may be removed by means of the identity. We then obtain covariants (1), (4) and covariants of the kind considered which have no factor $\beta_x$. Let us suppose that one of these latter covariants has a factor $(\epsilon\zeta)$ which contains neither $\beta$ nor $\beta'$; then, by means of the identity for $(\epsilon\zeta)\beta'_x$, this may be expressed in terms of covariants (1), (4) and of covariants which still have the original factor, but have no factor $b_x$. The identity for $(ab)(cd)$ must then be used to reduce the index of $(ab)$ or that of $(cd)$ to zero in the last case. If the index of $(ab)$ is zero, we may express the covariants in terms of covariants (1), (2), (3) by means of the identity for $(ac)d_x$. If the index of $(cd)$ is zero, the same identity may be used to reduce the index of $(ac)$ or of $d_x$ to zero; this leaves us with covariants (1), (2) and covariants which have no factor $(ac)$. In these latter the identity for $(ab)(cd)$ must be used again: the process is one which may be repeated, for the index of $(ab)$ is continually being decreased. Thus we are eventually left with covariants (1), (2), (3) and covariants which have no factors $(ac)$ or $(cd)$, and these are covariants (4).

12. Consider now the case when $a$, $b$, $d$ refer to fundamental quantics and $c = (\gamma\gamma')^{\mu_3}$. We shall assume that when the order of $\gamma$ is $\nu_3+1$ the terms $\Sigma T$ on the right-hand side of the syzygy (see § 9) are of the second class and only contain the letter $\gamma'$ in second class factors, when it appears in the factor $(\gamma\gamma')$; or else contain a factor $(b\gamma')^{\nu_3'-\mu_3}(\gamma\gamma')^{\mu_3}$. But, by §§ 10, 11, these terms can be expressed in terms of

(1) Covariants (1) of § 10.

(2) Covariants which have no factor $c_x$, $b_x$, or $d_x$.

(3) Covariants which have no factor $c_x$ or $b_x$ and in which the letters $a$ only appear in the factors $(ad)$, $a_x$.

(4) Covariants which have a factor $(cb)^{\mu_3+1}$.

Now the assumed syzygy tells us that the sum of the covariants (2), (3), (4) has a factor $c_x$; hence by § 8 this sum is zero.

We may then divide each side of the resulting relation by $\gamma_x$, and thus obtain a relation of exactly the same form as that from which we started; but in which the order of $\gamma$ is $\nu_3$.

The assumption made is true when the induction begins, and hence is always true.

13. Let $a$, $d$ represent fundamental quantics, $b = (\beta\beta')^{\mu_2}$, $c = (\gamma\gamma')^{\mu_3}$. The induction proceeds on the same lines as before; at each step the

order of $\gamma$ is decreased by unity. At the commencement of the induction the order of $\gamma$ is equal to the weight of the syzygy. Now when this is the case the letter $\gamma'$ may be ignored; so that practically $c$ may be regarded as a fundamental quantic. In order to see that in this case the syzygy is true, and also to find the character of the terms of the second class on the right, we must interchange $b$ and $c$ in the result of § 12. Thus the terms on the right only contain the letter $\beta'$ in their second class factors, when it appears in the factor $(\beta\beta')$, or else contain a factor

$$(\gamma\beta')^{\nu'_2-\mu_2}(\beta\beta')^{\mu_2}.$$

Covariants which have a factor

$$(\gamma\beta')^{\nu'_2-\mu_2}(\beta\beta')^{\mu_2}$$

can be expressed in terms of

(7) Covariants which have a factor

$$(\gamma\beta')^{\nu'_2-\mu_2}(\beta\beta')^{\mu_2}\gamma_x,$$

of covariants (1), (5), (6) of §§ 10, 11, and of covariants which have no factors $b_x$ or $c_x$.

These latter covariants have been discussed in § 10. Hence, if at any stage of the induction the covariants on the right are of the kind discussed in the lemma of § 10 or else have one or other of the factors

$$(\beta\gamma')^{\nu'_3-\mu_3}(\gamma\gamma')^{\mu_3},\quad (\beta\gamma')^{\nu_2-\mu_2}(\beta\beta')^{\mu_2},\quad (\gamma\beta')^{\nu'_2-\mu_2}(\beta\beta')^{\mu_2},$$

then these covariants can be expressed in terms of covariants belonging to the sets (1), (2), (3), (5), (6), (7) of §§ 10, 11, 13.

This sum can be expressed by § 3 as a sum of covariants of the sets (1), (5), and (7), and hence the covariants on the right are of the same nature when we diminish the order of $\gamma$ by unity.

The theorem is then true in this case.

14. Let $a$ represent a fundamental quantic, and

$$b = (\beta\beta')^{\mu_2},\qquad c = (\gamma\gamma')^{\mu_3},\qquad d = (\delta\delta')^{\mu_4}.$$

The induction proceeds as before, commencing when the order of $\gamma$ is so large that $c$ may be regarded as a fundamental quantic. The terms now on the right-hand side of the syzygy must be obtained by interchanging $c$ and $d$ in § 13. They must then consist of covariants of the kind discussed in the lemma of § 10 and covariants which have one or other of the factors

$$(\beta\delta')^{\nu'_4-\mu_4}(\delta\delta')^{\mu_4},\qquad (\beta\delta')^{\nu_2-\mu_2}(\beta\beta')^{\mu_2},\qquad (\delta\beta')^{\nu'_2-\mu_2}(\beta\beta')^{\mu_2}.$$

It will then be assumed that always in this case the covariants on the right are of the kind discussed in § 10 or contain one of the following factors

$$(\beta\delta')^{\nu'_4-\mu_4}(\delta\delta')^{\mu_4},\quad (\beta\delta')^{\nu_2-\mu_2}(\beta\beta')^{\mu_2},\quad (\delta\beta')^{\nu'_2-\mu_2}(\beta\beta')^{\mu_2},$$

$$(\beta\gamma')^{\nu'_3-\mu_3}(\gamma\gamma')^{\mu_3},\quad (\beta\gamma')^{\nu_2-\mu_2}(\beta\beta')^{\mu_2},\quad (\gamma\beta')^{\nu'_2-\mu_2}(\beta\beta')^{\mu_2}.$$

The induction may now be established as before.

15. Let each of the letters $a$, $b$, $c$, $d$ represent a covariant of degree 2. Then the process of proof is the same as before, the covariants on the right being of the same kind as the covariants on the right in the case of § 14, or else containing one of the factors

$$(\beta a')^{\nu'_1-\mu_1}(aa')^{\mu_1},\quad (\beta a')^{\nu_2-\mu_2}(\beta\beta')^{\mu_2},\quad (a\beta')^{\nu'_2-\mu_2}(aa')^{\mu_2}.$$

16. Finally it may be shown in the same way that, whatever covariants $a$, $b$, $c$, $d$ may represent, the theorem is true.

For, if $c$ is a covariant, it may always be written as a sum of terms of the second class and of covariants of the form

$$(\gamma\gamma_1)^{\lambda_1}(\gamma_1\gamma_2)^{\lambda_2}\ldots(\gamma_k\gamma')^{\lambda_{k+1}},$$

the sequence of the letters being fixed, and possibly one or more of the indices zero.

The other covariants $a$, $b$, $d$ being similarly expressed, the theorem is proved in exactly the same manner as the more elementary cases proved in §§ 8–15; the alterations required are practically only verbal.

17. It is necessary to extend the work in the paper on "Perpetuant Syzygies," step by step, to the case when the orders of the quantics concerned are finite. The generating functions found for perpetuant products cannot be true for products of covariants of forms of finite order, for the corresponding generating functions when expanded cannot go to infinity; all reference to the generating functions is then omitted.

Essentially the discussion of perpetuant products proceeded as ollows:—

(i.) All products were arranged in a fixed sequence, defined in Section III., § 5; and a product was defined as reducible if it could be expressed in terms of products which came after it in the fixed sequence.

(ii.) The discussion of products $(ab)^{\lambda}C_{\kappa}$ in Section III., §§ 6–10.

(iii.) The extension of the perpetuant type theorem, Section I.

(iv.) The application of (ii.) and (iii.) to any product $C_{\kappa}C_{\lambda}$ in Section VI.

The rest of the paper consisted of applications of these results to particular forms of products.

When the orders of the fundamental quantics are finite, there is nothing to prevent our taking the products in the sequence defined in (i.). A product will then be called reducible if it can be expressed in terms of products which come after it and of covariants of the second class.

(ii.) The introduction of differential operators in § 6, when writing Stroh's syzygy, may be made in the same way when the orders are finite; it being understood that symbolical products which have not a proper meaning (owing to a letter occurring to too great a degree) must be omitted; and that forms of the second class are neglected. It is useful to remark that, if $P$ be a symbolical product of degree $n_4$ in $a_4$, we have a syzygy

$$e^{(a_1a_2)A_4} A_4^{-\rho} P = R,$$

in which the first $\rho$ terms must be omitted.

The argument of §§ 7, 8, 9 simply concerns a set of linear equations, and holds good when the orders are finite, all non-interpretable forms arising from Stroh's syzygies being kept until the argument is finished. In the same way § 10 holds good.

18. (iii.) The method of proof of the extension of the perpetuant type theorem was identical with Grace's proof of the original theorem.* To prove the corresponding theorem for forms of finite order, a method identical with that used to extend the original theorem to forms of finite order† may be employed. But, as no new point arises in the course of proof, it is thought unnecessary to reproduce it.

(iv.) Having established the results (ii.) and (iii.) for forms of finite order, the argument of Section VI. for products $C_\kappa C_\lambda$ follows as well.

Thus we arrive at the conclusion that all products which would be reducible as products of perpetuants are reducible as products of covariants of forms of finite order.

19. It is possible, at least in a few of the simplest cases, to calculate generating functions for irreducible products, when the orders are finite.

Thus the generating function for forms $C_2 = (a_1a_2)^\lambda$ $(n_1 \not> n_2)$ is $\dfrac{x-x^{n_1}}{1-x}$, the form $(a_1a_2)^{n_1}$ being of the second class.

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* *Proc. London Math. Soc.*, Vol. xxxv.

† A. Young, *Proc. London Math. Soc.*. Ser. 2, Vol. 1.

*Consider next the forms* $C_2^2$:—When $\lambda > n_2-2$ and $n_2 \ngtr n_3$ the first term of the syzygy

$$\Sigma \binom{\lambda}{r} (a_1a_4)^r (a_2a_3)^{\lambda-r} = R$$

which is interpretable is

$$\binom{\lambda}{\lambda-n_2} (a_1a_4)^{\lambda-n_2} (a_2a_3)^{n_2},$$

and this is of the second class. Thus we have a reduction for $(a_1a_4)^{\lambda-n_2+1}(a_2a_3)^{n_2-1}$. And from the syzygy

$$\Sigma \binom{\lambda}{r} (a_1a_4)^r (a_3a_2)^{\lambda-r} = R$$

we have also a reduction for $(a_1a_4)^{\lambda-n_2+2}(a_2a_3)^{n_2-2}$.

Hence, for all values of $\lambda$,

$$(a_1a_4)^{\lambda}(a_2a_3)^{\mu} = R$$

when $\mu > n_2-3$.

Similarly $(a_1a_3)^{\lambda}(a_2a_4)^{\mu} = R$, $n_2 \ngtr n_4$ when $\mu > n_2-2$.

Consider the generating function for $(a_1a_2)^{\lambda}(a_3a_4)^{\mu}$, $n_1 \ngtr n_2$, $n_3 \ngtr n_4$. We first have to include all terms having a factor $(a_1a_2)^2(a_3a_4)^2$, then to exclude those which have a factor $(a_1a_2)^2(a_3a_4)^{n_3}$, then to exclude those which have a factor $(a_1a_2)^{n_1}(a_3a_4)^2$, and finally to include those which have a factor $(a_1a_2)^{n_1}(a_3a_4)^{n_3}$ and have been excluded twice.

Thus the generating function is

$$\frac{x^4}{(1-x)^2} - \frac{x^{n_3+2}}{(1-x)^2} - \frac{x^{n_1+2}}{(1-x)^2} + \frac{x^{n_1+n_3}}{(1-x)^2} = \frac{x^4(1-x^{n_1-2})(1-x^{n_3-2})}{(1-x)^2}.$$

Assuming that $n_1, n_2, n_3, n_4$ are in ascending order of magnitude, we see that the generating function for $(a_1a_3)^{\lambda}(a_2a_4)^{\mu}$ is

$$\frac{x^4(1-x^{n_1-3})(1-x^{n_2-2})}{(1-x)^2}$$

and that that for $(a_1a_4)^{\lambda}(a_2a_3)^{\mu}$ is

$$\frac{x^5(1-x^{n_1-4})(1-x^{n_2-3})}{(1-x)^2}.$$

Hence the generating function for all products $C_2^2$ is

$$\frac{x^4(1-x^{n_1-2})(1-x^{n_3-2})}{(1-x)^2} + \frac{x^5(1-x^{n_1-3})(1-x^{n_2-3})}{(1-x)^2} + \frac{x^6(1-x^{n_1-4})(1-x^{n_2-4})}{(1-x)^2}$$

The generating functions for products $C_2^m$ may be calculated in the same way, *e.g.*, that for $(a_1a_2)^\lambda(a_2a_3)^\mu(a_4a_5)^\nu$ is

$$\frac{x^9(1-x^{n_1-3})(1-x^{n_2-5})(1-x^{n_4-4})}{(1-x)^3},$$

for this product is always reducible if $\lambda \geqslant n_1$, if $\mu \geqslant n_2-1$, or if $\nu \geqslant n_4-2$.

20. A covariant of degree 3 can be expressed in terms of members of the second class if its weight is greater than the order of any one of the quantics concerned. The generating function of covariants $C_3$, the quantics concerned being $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, $a_{3_x}^{n_3}$, $n_1 \not> n_2 \not> n_3$, is

$$\frac{x^3-(n_1-2)x^{n_1}+(n_1-3)x^{n_1+1}}{(1-x)^2}.$$

Consider products $C_2C_3$, in particular the set

$$(a_1a_2)^\nu(a_3a_4)^\lambda(a_3a_5)^\mu,$$

where $n_1$, $n_2$, $n_3$, $n_4$, $n_5$ are in ascending order of magnitude. The six syzygies

$$e^{\pm(a_1a_2)A_3}A_3^{-\rho}(a_3a_4)^\lambda(a_3a_5)^{n_3-\lambda} = R,$$

$$e^{\pm(a_1a_2)A_4}A_4^{-\rho}(a_3a_4)^\lambda(a_3a_5)^{n_3-\lambda} = R,$$

$$e^{\pm(a_1a_2)A_5}A_5^{-\rho}(a_3a_4)^\lambda(a_3a_5)^{n_3-\lambda} = R$$

reduce all forms $(a_1a_2)^{\rho+\sigma}(a_3a_4)^{\lambda-\sigma}(a_3a_5)^{n_3-\lambda}$ .

where $\sigma = 1, 2, 3, 4, 5, 6$. The argument for the linear independence of these syzygies is identical with that for the independence of perpetuant syzygies. We thus see that all products

$$(a_1a_2)^\nu(a_3a_4)^\lambda(a_3a_5)^\mu$$

are reducible for which $\lambda+\mu > n_3-7$.

The products when the letters are arranged in any one of the other possible manners may be treated in the same way. In § 12 (p. 240) of the paper on "Perpetuant Syzygies" a table is given of the limits of the indices $\nu$, $\lambda$, $\mu$ for irreducibility in the various cases. The general result here is that, if the product is reducible when $\nu \lessdot \nu_1$, then it is also reducible when $\lambda+\mu > n-\nu_1$, where $n$ is the order of the quantic of lowest order that occurs in $C_3$.

To prove this fact, it is merely necessary to remark that the syzygies which give the reductions in the first case are the same (except for their weight) as those which give the reductions in the latter case.

21. The main theorem of this paper applied to a single quantic shews that its covariants may be treated as perpetuants so far as the known results for perpetuants as regards reducibility or syzygies are concerned, provided that forms having a factor $(ab)^{\frac{1}{2}n}$ are neglected. The method by which this result has been arrived at is of such a general nature that it would appear almost certain that when the, as yet unknown, syzygies of degree 9 and of greater degree are discovered these also may be extended to covariants of forms of finite order by the same process.

# ON A CLASS OF EXPANSIONS IN OSCILLATING FUNCTIONS*

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[Received December 25th, 1904.—Read January 12th, 1905.]

IN memoirs published in the first two volumes of *Liouville's Journal*, Liouville and Sturm considered transcendents satisfying differential equations of a certain type, and gave a proof that an arbitrary function could be expanded in a series of multiples of such functions. Many of the expansions used in physics are of the type in question. The proof given is not now accepted (see Burkhardt's *Report*, p. 759), and the object of the present paper is to give a sound proof, at least for functions which are analytical throughout the proposed range of validity of the expansions: the proof is actually of more general application than this, but there is no attempt to examine whether any function expansible in a Fourier series can also be expanded in the more general form. Another limitation is that the range of validity of the expansions is not supposed to reach to any critical point of the differential equation satisfied by the general term.

On the other hand, the expansions have a greater degree of generality than those of Liouville on account of the presence of four arbitrary real constants (restricted by an inequality) in the place of his two ($h$, $H$).

In the simple case of Fourier's series the oscillating series

$$\Sigma (\cos nx \cos nt + \sin nx \sin nt)$$

is multiplied by $F(t)$ and integrated with respect to $t$. The resulting series converges, and use is made of an expression for the sum to $n$ terms to find the sum to infinity. The same idea is carried out here for the wider class of expansions treated. There is a fundamental oscillating series $\Sigma\Phi$, whose terms are symmetric functions of $x$ and $t$; an expression is found for the sum of $n$ terms of this series by means of a contour integral; the oscillating series is multiplied by $F(t)$ and integrated, and thus turned into a convergent series whose sum is $F(x)$.

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* For references on the subject see a report on "Entwickelungen nach oscillirenden Functionen," by H. Burkhardt, in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, now being issued.

2. In the differential equation

$$\frac{d^2y}{dx^2}+(X+\lambda)y = 0, \tag{1}$$

let $\lambda$ be a parameter, $X$ a function of $x$ independent of $\lambda$. Suppose that, for real values of $x$ between 0 and 1 inclusive, $X$ has no singularity and that its values are real throughout that range.

Derivatives with respect to $x$ and $\lambda$ will occur; they will be indicated by dashes and dots respectively.

Let $\phi$, $\psi$, $\chi$, $\omega$[*] be functions satisfying (1) and distinguished by the following initial conditions:—

when $x = 0$, $\quad \phi = 0, \quad \phi' = 1, \quad \psi = 1, \quad \psi' = 0;$

when $x = 1$, $\quad \chi = 0, \quad \chi' = 1, \quad \omega = 1, \quad \omega' = 0.$

Then, by the known theory of differential equations of the second order, $\phi'\psi-\phi\psi'$, $\chi'\omega-\chi\omega'$ are constants, and, in fact, each of them is equal to 1.

Also $\phi\chi'-\phi'\chi$, $\psi\chi'-\psi'\chi$, $\omega\phi'-\omega'\phi$, $\omega\psi'-\omega'\psi$ are constants, that is, depend on $\lambda$ only: let their values be $e$, $g$, $h$, $k$. Then

$$e = \phi(1) = -\chi(0), \qquad h = \phi'(1) = \omega(0),$$

$$g = \psi(1) = \chi'(0), \qquad k = \psi'(1) = -\omega'(0),$$

and

$$gh-ek = 1.$$

We also have $\quad \chi = g\phi-e\psi, \qquad \omega = -k\phi+h\psi,$

$$\phi = h\chi+e\omega, \qquad \psi = k\chi+g\omega.$$

3. From the theory of differential equations[†] we know that $\phi$, $\psi$, $\chi$, $\omega$ are analytical functions of $\lambda$ as well as of $x$, and that therefore $e$, $g$, $h$, $k$ are analytical functions of $\lambda$. This is true for all finite values of $\lambda$. There is, of course, an essential singularity at infinity, as will appear more fully from results that are to follow.

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[*] Written $\phi$, $\psi$, … or $\phi(x)$, $\psi(x)$, … or $\phi(x, \lambda)$, … when necessary.

[†] See especially Poincaré, *Acta Mathematica*, Vol. XIII., pp. 15, 16. Other references are given in Forsyth's *Theory of Differential Equations*, Vol. II., chap. ii.

4. The more general equation

$$\frac{d^2y}{dx^2}+(X+\lambda U)y=0$$

may be reduced to the form (1) by a change of variables, if $U$ is a holomorphic function of $x$ only and is constantly positive, not zero, between the limits 0, 1. The reduction is made by taking $U^{\frac{1}{4}}y$ as the new dependent variable and $\int_0^x U^{\frac{1}{2}}dx \Big/ \int_0^1 U^{\frac{1}{2}}dx$ as the new independent.

5. Suppose $\lambda_1$ to be a new value of $\lambda$ and $\phi_1, \psi_1, \ldots, e_1, g_1, \ldots$ to be what $\phi, \psi, \ldots, e, g, \ldots$ become when $\lambda_1$ takes the place of $\lambda$. Then

$$\phi''+(X+\lambda)\phi=0, \qquad \chi_1''+(X+\lambda_1)\chi_1=0.$$

$$\phi''\chi_1-\phi\chi_1''=(\lambda_1-\lambda)\phi\chi_1, \qquad \phi'\chi_1-\phi\chi_1'=(\lambda_1-\lambda)\int\phi\chi_1\,dx.$$

Take the limits to be 0 and 1; thus

$$\left.\begin{aligned}
&(\lambda_1-\lambda)\int_0^1\phi\chi_1\,dx = -\phi(1)-\chi_1(0)=e_1-e.\\
\text{Similarly}\qquad &(\lambda_1-\lambda)\int_0^1\phi\omega_1\,dx = \phi'(1)-\omega_1(0)=h-h_1,\\
&(\lambda_1-\lambda)\int_0^1\psi\chi_1\,dx = -\psi(1)+\chi_1'(0)=g_1-g,\\
&(\lambda_1-\lambda)\int_0^1\psi\omega_1\,dx = \psi'(1)+\omega_1'(0)=k-k_1.
\end{aligned}\right\}\qquad(2)$$

If in these we diminish $\lambda_1-\lambda$ without limit, we have

$$\left.\begin{aligned}
\dot{e}&=\int_0^1\phi\chi\,dx, & \dot{g}&=\int_0^1\psi\chi\,dx,\\
\dot{h}&=-\int_0^1\phi\omega\,dx, & \dot{k}&=-\int_0^1\psi\omega\,dx.
\end{aligned}\right\}\qquad(3)$$

These results could have been found by differentiating the equations $\phi''+(X+\lambda)\phi=0, \ldots$ with respect to $\lambda$. Thus

$$\dot{\phi}''+(X+\lambda)\dot{\phi}=-\phi.$$

But

$$\chi''+(X+\lambda)\chi=0,$$

so that

$$\chi\dot{\phi}''-\chi''\dot{\phi}=-\phi\chi.$$

Integrating between 0 and 1, we have

$$\dot{e} = \int_0^1 \phi\chi\,dx,$$

and so for the rest.

Other useful results are

$$\left.\begin{aligned}
(\lambda_1-\lambda)\int_0^1 \phi\phi_1\,dx &= e_1 h - e h_1 \\
(\lambda_1-\lambda)\int_0^1 \phi\psi_1\,dx &= g_1 h - e k_1 - 1 \\
(\lambda_1-\lambda)\int_0^1 \phi_1\psi\,dx &= -g h_1 + e_1 k + 1 \\
(\lambda_1-\lambda)\int_0^1 \psi\psi_1\,dx &= g_1 k - g k_1 \\
(\lambda_1-\lambda)\int_0^1 \chi\chi_1\,dx &= g e_1 - e g_1 \\
(\lambda_1-\lambda)\int_0^1 \omega\chi_1\,dx &= g_1 h - e_1 k - 1 \\
(\lambda_1-\lambda)\int_0^1 \chi\omega_1\,dx &= -g h_1 + e k_1 + 1 \\
(\lambda_1-\lambda)\int_0^1 \omega\omega_1\,dx &= k h_1 - k_1 h \\
\int_0^1 \phi^2\,dx &= \dot{e}h - e\dot{h} \\
\int_0^1 \phi\psi\,dx &= \dot{g}h - e\dot{k}, \;\dots
\end{aligned}\right\}. \tag{4}$$

6. Now let $E$, $G$, $H$, $K$, $L$ be real constants, and suppose $\lambda$, $\lambda_1$ to be values satisfying the condition*

$$Ee + Gg + Hh + Kk = 2L. \tag{5}$$

We shall show that, if a certain condition of inequality is fulfilled by $E$, $G$, $H$, $K$, $L$, the values of $\lambda$ satisfying (5) are all real.

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* The reader will perhaps find it a help to take first one or two special cases, for instance, that in which $G$, $H$, $K$, $L$ vanish, and that in which $E = K = 0$, $G = H = L = 1$. It is hardly necessary to recommend comparison with the trigonometrical expansions which arise when $X = 0$.

For, by the formulæ of § 5,

$$\begin{aligned}(\lambda_1-\lambda)\int_0^1 &(E\chi_1-H\omega_1+L\psi_1)(E\chi-H\omega+L\psi)\,dx\\ &= E^2(ge_1-eg_1)-EH(g_1h-e_1k-gh_1+ek_1)\\ &\qquad +H^2(kh_1-k_1h)+LE(2g_1-2g)-LH(2k-2k_1)+L^2(g_1k-gk_1)\\ &= (Ee_1+Hh_1-2L)(Eg+Hk)-(Ee+Hh-2L)(Eg_1+Hk_1)\\ &\qquad +L^2(g_1k-gk_1)\\ &= -(Gg_1+Kk_1)(Eg+Hk)+(Gg+Kk)(Eg_1+Hk_1)+L^2(g_1k-gk_1)\\ &= (L^2+EK-GH)(g_1k-gk_1).\end{aligned}$$

Hence, unless $\lambda_1 = \lambda$,

$$\int_0^1\{(E\chi_1-H\omega_1+L\psi_1)(E\chi-H\omega+L\psi)+(GH-EK-L^2)\psi\psi_1\}\,dx = 0. \quad (6)$$

Suppose then that $\lambda$ is complex; the conjugate quantity will also satisfy the condition (5), and may be taken as $\lambda_1$. Thus $\chi$, $\chi_1$ and $\omega$, $\omega_1$ and $\psi$, $\psi_1$ are also conjugate, and, if

$$GH > EK+L^2,$$

the equation (6) is impossible, the expression on the left having a positive value.

If $GH = EK+L^2$, (6) cannot now be true unless

$$E\chi-H\omega+L\psi = 0;$$

that is, $\quad E+kL = 0, \qquad -H+gL = 0.$

Thus $g$, $k$ are real and are equal to $g_1$, $k_1$. But

$$g_1k-gk_1 = (\lambda_1-\lambda)\int_0^1 \psi\psi_1\,dx,$$

a purely imaginary quantity, not zero. Hence $\lambda$ cannot be complex, even when $GH = EK+L^2$.

When $E, H, L$ all vanish, this argument fails, but the same result may be established by considering $\int_0^1 (G\chi_1-K\omega_1-L\phi_1)(G\chi-K\omega-L\phi)\,dx$.

7. We shall then suppose $GH \nless EK+L^2$, so that the values of $\lambda$ satisfying the condition (5) must be real. They will generally be distinct also. If not, we should have

$$E\dot{e}+G\dot{g}+H\dot{h}+K\dot{k} = 0,$$

that is, $$\int_0^1 \{E\phi\chi+G\psi\chi-H\phi\omega-K\psi\omega\}\,dx = 0.$$

The subject of integration is here

$$(E\phi+G\psi)(g\phi-e\psi)+(H\phi+K\psi)(k\phi-h\psi)$$

or $$\phi^2(Eg+Hk)+\phi\psi(Gg-Ee+Kk-Hh)-\psi^2(Ge+Kh),$$

and is of constant sign if

$$(Gg-Ee+Kk-Hh)^2 < -4(Eg+Hk)(Ge+Kh),$$

or if

$$(Gg+Ee+Kk+Hh)^2 < 4(Gg+Kk)(Ee+Hh)-4(Eg+Hk)(Ge+Kh)$$

or $$4(GH-EK)(gh-ek),$$

that is, if $$L^2 < GH-EK.$$

Hence, in this case, a double value of λ cannot occur.

If $L^2 = GH-EK$, a double value of λ will occur only when the same subject of integration vanishes identically, that is, when

$$Eg+Hk = 0 = Ge+Kh, \qquad Gg+Kk = Ee+Hh = L.$$

These conditions give

$$Le = (Gg+Kk)e-(Ge+Kh)g = -K,$$

and similarly $$Lg = H, \quad Lh = G, \quad Lk = -E.$$

Hence this case cannot occur when $L = 0$. It does happen when $X$ is constant and $$E = K = 0, \qquad G = H = \pm L.$$

A triple value of λ cannot occur. This would mean the further condition

$$E\ddot{e}+G\ddot{g}+H\ddot{h}+K\ddot{k} = 0,$$

which may be written $$\ddot{g}h+g\ddot{h}-\ddot{e}k-e\ddot{k} = 0$$

or $$\frac{\partial^2}{\partial\lambda^2}(gh-ek) = 2(\dot{g}\dot{h}-\dot{e}\dot{k}).$$

Thus we should have $\dot{g}\dot{h}-\dot{e}\dot{k} = 0$, that is,

$$\tfrac{1}{2}\int_0^1\int_0^1 (\phi x\,\psi t-\phi t\,\psi x)(\chi x\,\omega t-\chi t\,\omega x)\,dx\,dt = 0,$$

on account of the formulæ (3).

But $$\chi x\,\omega t-\chi t\,\omega x = \phi x\,\psi t-\phi t\,\psi x,$$

and therefore this double integral is positive and cannot vanish unless, for all values of $x$ and $t$,

$$\phi x/\psi x = \phi t/\psi t = \text{a constant},$$

which is impossible, since $\phi'\psi - \phi\psi' = 1$.

8. Let $x$, $t$ be two quantities, real, positive, and $< 1$. Consider the value of $\int \Omega(x, t, \lambda)\,d\lambda$ where $\Omega(x, t, \lambda)$ stands for

$$\frac{E\phi x\chi t + G\psi x\chi t - H\phi x\omega t - K\psi x\omega t + L(\phi x\psi t - \psi x\phi t)}{Ee + Gg + Hh + Kk - 2L}$$

and the integral is taken round a closed curve in the $\lambda$ plane which does not pass through any point at which the denominator vanishes. The only singularities of the subject of integration for finite values of $\lambda$ are poles at the points where

$$Ee + Gg + Hh + Kk = 2L.$$

At any such point, supposed to be a simple zero of the denominator, the derivative of the denominator is

$$\int_0^1 (E\phi\chi + G\psi\chi - H\phi\omega - K\psi\omega)\,dx,$$

and the numerator is a symmetric function of $x$, $t$, since it may be written

$$(Eg + Hk)\phi x\phi t + (-Ee - Hh + L)\phi x\psi t + (Gg + Kk - L)\psi x\phi t$$
$$-(Ge + Kh)\psi x\psi t.$$

These expressions are respectively the denominator and numerator of the residue. In the exceptional case (see § 7) when the denominator has a double zero, the numerator vanishes also, as may be seen by taking it in the form last written. Thus the subject of integration has only a simple pole, the residue being

$$2\,\frac{(E\dot{g} + H\dot{k})\phi x\phi t + (G\dot{g} + K\dot{k})(\phi x\psi t + \psi x\phi t) - (G\dot{e} + K\dot{h})\psi x\psi t}{E\ddot{e} + G\ddot{g} + H\ddot{h} + K\ddot{k}}.$$

But now $$E : G : H : K :: -k : h : g : -e,$$

and $$\ddot{g}h + g\ddot{h} - \ddot{e}k - e\ddot{k} = 2(\dot{e}\dot{k} - \dot{g}\dot{h}).$$

The residue is therefore

$$\frac{(\dot{g}k - g\dot{k})\phi x\phi t + (e\dot{k} - h\dot{g})(\phi x\psi t + \phi t\psi x) + (\dot{e}h - e\dot{h})\psi x\psi t}{\dot{g}\dot{h} - \dot{e}\dot{k}}$$

or $$\frac{\phi x\phi t\int_0^1 \psi^2\,dx - (\phi x\psi t + \phi t\psi x)\int_0^1 \phi\psi\,dx + \psi x\psi t\int_0^1 \phi^2\,dx}{\frac{1}{2}\int_0^1\int_0^1 (\phi x\psi t - \phi t\psi x)^2\,dx\,dt}. \tag{7}$$

Hence $$\int \Omega(x, t, \lambda)\, d\lambda = 2\iota\pi \Sigma \Phi(x, t, \lambda)$$

where $\Phi(x, t, \lambda)$ stands for

$$\frac{(Eg+Hk)\,\phi x\,\phi t+(Gg+Kk-L)(\phi x\,\psi t+\psi x\,\phi t)-(Ge+Kh)\,\psi x\,\psi t}{\int_0^1 (E\phi\chi+G\psi\chi-H\phi\omega-K\psi\omega)\,dx},$$

and the summation covers all the values of $\lambda$, within the contour of integration, at which $$Ee+Gg+Hh+Kk = 2L,$$

and, if two of these values of $\lambda$ coincide, the corresponding terms in the summation are to be replaced by the expression (7). If we suppose the values of $\lambda$ at which poles of $\Omega$ occur, in ascending order of numerical magnitude, to be $\lambda_1, \lambda_2, \lambda_3, \ldots$, we thus have two expressions for the sum of $n$ terms of the series

$$\Phi(x, t, \lambda_1)+\Phi(x, t, \lambda_2)+\ldots,$$

namely, $(2\iota\pi)^{-1}\int \Omega(x, t, \lambda)\, d\lambda$ and $(2\iota\pi)^{-1}\int \Omega(t, x, \lambda)\, d\lambda$, the integrals being taken along some closed path, say the circle $|\lambda| = l$, which encloses $\lambda_1, \lambda_2, \ldots, \lambda_n$, and not $\lambda_{n+1}, \lambda_{n+2}, \ldots$. In order to test the convergency of the series we may examine further these expressions for the sum to $n$ terms. It will be found that the series does not converge as it stands.

9. We need to know the orders of magnitude of $\phi, \psi, \ldots$ when $\lambda$ is a great complex quantity. Now when $X = 0$ and $\lambda = (a+\iota\beta)^2$ the most general solution of the equation (1) is

$$y = A \cos\{(a+\iota\beta)x+a+\iota b\}$$

where $A$, $a$, $b$ are constants, $A$ being complex and $a$, $b$ real. Here then

$$|y|^2 = |A|^2\{\cosh^2(\beta x+b)-\sin^2(ax+a)\}$$
$$= |A|^2\{\sinh^2(\beta x+b)+\cos^2(ax+a)\}.$$

Thus $|y|$ oscillates between $|A|\cosh(\beta x+b)$ and $|A|\sinh(\beta x+b)$.

Again, $$|y'|^2 = |\lambda|\,|A|^2\{\sinh^2(\beta x+b)+\sin^2(ax+a)\}$$
$$= |\lambda|\,|A|^2\{\cosh^2(\beta x+b)-\cos^2(ax+a)\},$$

which oscillates in like manner.

But $$|\lambda y^2|+|y'|^2 = |\lambda A^2|\cosh(2\beta x+2b),$$

which increases steadily with $x$. On the other hand, $|\lambda y^2|-|y'|^2$ oscillates between $\pm|\lambda A^2|$, and in fact $\lambda y^2+y'^2$ is constant and equal to $\lambda A^2$. These results are approximately true when $X$ is variable, as will now be shown.

The following lemma will be used. If $w$ is defined by the differential equation

$$w'' = 4\mu^2 w,$$

where $\mu$ is a function of $x$ that lies between two positive constants $\mu_1$, $\mu_2$ $(\mu_1 > \mu_2)$ so long as $x$ is between 0 and 1, and by the further conditions that when $x = 0$, $w = 1$, $w' = 0$, then $w$ is constantly between $\cosh 2\mu_1 x$ and $\cosh 2\mu_2 x$ so long as $x$ is between 0 and 1.

For let $$W = \cosh 2\mu_1 x - w.$$

Then $$W'' = 4(\mu_1^2 - \mu^2)\cosh 2\mu_1 x + 4\mu^2 W,$$

and therefore $W'$ increases with $x$ so long as $W \not< 0$, while $W$ increases with $x$ so long as $W' > 0$. Initially $W$ and $W'$ both vanish and therefore both increase with $x$ and are always positive, that is, $\cosh 2\mu_1 x > w$. Similarly $w > \cosh 2\mu_2 x$.

10. Let $\lambda = \rho + \iota\sigma$, $|X+\lambda| = Q$, and put $y = u \exp \iota v$, so that $\rho$, $\sigma$, $u$, $v$ are real, and $\rho = \alpha^2 - \beta^2$, $|\lambda| = l = \alpha^2 + \beta^2$. Then

$$y'' = (u'' - uv'^2 + 2\iota u'v' + \iota uv'')e^{\iota v},$$

and therefore $u'' - uv'^2 + 2\iota u'v' + \iota uv'' + u(X+\rho+\iota\sigma) = 0.$

Separating the real and imaginary parts, we have

$$u'' - uv'^2 + u(X+\rho) = 0, \qquad 2u'v' + uv'' + u\sigma = 0.$$

Now put $$w = |(X+\lambda)y^2| + |y'|^2 = Qu^2 + u'^2 + u^2v'^2.$$

Then
$$\begin{aligned} w' &= Q'u^2 + 2u'(u'' + Qu - uv'^2) + 2uv'(uv'' + 2u'v') \\ &= Q'u^2 + 2uu'(Q - X - \rho) - 2\sigma u^2 v'\,; \end{aligned}$$
$$\begin{aligned} w'' &= Q''u^2 + 2uu'(2Q' - X') + 2(Q - X - \rho)(uu'' + u'^2) - 2\sigma(u^2v'' + 2uu'v') \\ &= Q''u^2 + 2uu'(2Q' - X') + 2(Q - X - \rho)w, \end{aligned}$$

after substitution for $u''$, $v''$ and reduction.

Now the terms $Q''u^2 + 2uu'(2Q' - X')$ are generally insignificant compared with the others, for $$Q' = X'(X+\rho)/Q,$$

which is less than the finite quantity $X'$, and

$$Q'' = X''(X+\rho)/Q + \sigma^2 X'^2/Q^3,$$

which is similarly finite.

Let $\theta$ denote the greatest value of $\{Q''u^2 + 2uu'(2Q' - X')\}/w$ taken as a fraction whose numerator and denominator are homogeneous quadratics in $u$, $u'$, $uv'$, the denominator being essentially positive. By the ordinary methods we find that $Q\theta^2-Q''\theta-(2Q'-X')^2=0$. The roots of this quadratic are small quantities of the order of $Q^{-\frac{1}{2}}$, that is, of the order of $l^{-\frac{1}{2}}$.

Hence $w''=2w(Q-X-\rho+\epsilon)$ where $\epsilon$ is a small quantity comparable with $l^{-\frac{1}{2}}$. Further,

$$Q-l=|X+\lambda|-|\lambda|=\frac{2\rho X+X^2}{|X+\lambda|+|\lambda|}$$

$$=\rho X/l+\text{a small quantity comparable with } l^{-1}.$$

Hence $$Q-X-\rho=(l-\rho)(1-X/l)+\text{same error.}$$

Now $$l-\rho=2\beta^2;$$

so that $$w''=(4\beta^2+\eta)w,$$

where $\eta$ is finite even when $l$ is made infinitely great. We shall suppose for the present that $\beta$ becomes infinite with $l$, which will happen if the amplitude of the complex quantity $\lambda$ is not too small.

Let $4\mu_1^2$ and $4\mu_2^2$ be the greatest and least values of $4\beta^2+\eta$; then $\mu_1^2-\mu_2^2$ is always finite, and therefore $\mu_1-\mu_2$, $\mu_1-\beta$, $\beta-\mu_2$ tend to the limit zero as $\beta$ is increased, if $\mu_1$, $\mu_2$ have the same sign as $\beta$.

11. Now $y$ may stand for $\phi$ or $\chi$; in the former case the values of $w, w'$ when $x=0$ are $1, 0$; so that, by the lemma (§ 9), $w$ must lie between $\cosh 2\mu_1 x$ and $\cosh 2\mu_2 x$ and therefore $w$, or

$$|(X+\lambda)\phi^2|+|\phi'|^2=\tfrac{1}{2}(1+\epsilon_1)\exp 2\beta x\,^*$$

where $\epsilon_1$ tends to zero as $\beta$ is increased.

Similarly, $|(X+\lambda)\chi^2|+|\chi'|^2=\frac{1}{2}(1+\epsilon_2)\exp 2\beta(1-x)$,

where $\epsilon_2$ also tends to zero.

To find similar results for $\psi$ and $\omega$, let

$$w_1=|y^2|+|y'^2/(X+\lambda)|.$$

Then

$$w_1''=2w_1\{Q-X-\rho-Q''/2Q+Q'^2/Q^2\}+u^2\{Q''/Q-2Q'^2/Q^2\}$$
$$+4\sigma Q'u^2v'/Q^2+2(Q^2-2\sigma^2)X'uu'/Q^3.$$

Hence it comes out in the same way that

$$|\psi^2|+|\psi'^2/(X+\lambda)|=\tfrac{1}{2}(1+\epsilon_3)\exp 2\beta x,$$
$$|\omega^2|+|\omega'^2/(X+\lambda)|=\tfrac{1}{2}(1+\epsilon_4)\exp 2\beta(1-x),$$

where $\epsilon_3$, $\epsilon_4$ tend to the limit zero as $\beta$ is increased.

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* Here $\beta$ is taken positive; when $\beta$ is negative the value is $\frac{1}{2}(1+\epsilon_1)\exp(-2\beta x)$, and so throughout what follows.

12. Again, multiply the equation (1) by $2y'$ and integrate. Thus

$$y'^2+(X+\lambda)\,y^2 = \int X'y^2\,dx+\text{const.}$$

Taking $\phi$ for $y$, we have

$$\phi'^2+(X+\lambda)\,\phi^2 = 1+\int_0^x X'\phi^2\,dx,$$

whence
$$|\phi'^2+(X+\lambda)\,\phi^2| < 1+\xi\int_0^x \{|\phi'|^2+|(X+\lambda)\,\phi^2|\}\,dx \tag{8}$$

where $\xi$ is the greatest numerical value of $X'/Q$, and is therefore of the order of magnitude of $l^{-1}$.

Now $|\phi'|^2+|(X+\lambda)\,\phi^2|$ increases with $x$, from 0 upwards, since it satisfies the equation $w'' = (4\beta^2+\eta)\,w$ and the initial conditions $w = 1$, $w' = 0$. Hence in the integral in (8) the subject of integration is greatest at the upper limit, and

$$|\phi'^2+(X+\lambda)\,\phi^2| < 1+\xi x\,\{|\phi'|^2+|(X+\lambda)\,\phi^2|\}.$$

Thus $\dfrac{|\phi'^2+(X+\lambda)\,\phi^2|}{|\phi'|^2+|(X+\lambda)\,\phi^2|}$ consists of two parts, one of the order of magnitude of $l^{-1}$ and the other of that of $\exp(-2\beta x)$; both diminish indefinitely as $\beta$ increases, and therefore the ratio $|(X+\lambda)\,\phi^2|/|\phi'^2|$ tends to the limit unity.

We may therefore say that $|\phi'|^2$ and $|(X+\lambda)\,\phi^2|$, or in fact $|\lambda\phi^2|$, lie between the limits $\tfrac{1}{4}(1\pm\epsilon)\exp 2\beta x$, where $\epsilon$ diminishes indefinitely as $\beta$ increases. In the same way we have for $|\chi'|^2$ and $|\lambda\chi^2|$ limits of the form $\tfrac{1}{4}(1\pm\epsilon)\exp 2\beta(1-x)$. Similarly, from the equation

$$\frac{\psi'^2}{X+\lambda}+\psi^2 = 1-\int_0^x \frac{X'}{(X+\lambda)^2}\,\psi'^2\,dx$$

we deduce that
$$\left|\frac{\psi'^2}{X+\lambda}+\psi^2\right| < 1+\xi\int_0^x\left\{\left|\frac{\psi'^2}{X+\lambda}\right|+|\psi^2|\right\}dx$$

and that $|\psi^2|$ and $|\psi'^2/\lambda|$ lie between limits of the form $\tfrac{1}{4}(1\pm\epsilon)\exp 2\beta x$; similarly the limits for $|\omega^2|$ and $|\omega'^2/\lambda|$ are of the same form as those for $|\chi'|^2$ and $|\lambda\chi^2|$.*

By putting $x = 1$ in the results relating to $\phi$ and $\psi$ we find that $h^2$, $g^2$, $le^2$, and $k^2/l$ lie between limits of the form $\tfrac{1}{4}(1\pm\epsilon)\exp 2\beta$.

13. The expression $\phi x\,\psi t-\phi t\,\psi x$ has also to be considered. As a function of $x$ it satisfies the equation (1) and the further conditions that when $x = t$, $y = 0$, $y' = 1$. Hence, by analogy with the result found for $\phi$,

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* For another method of proof see Note II.

$|\lambda(\phi x\,\psi t-\phi t\,\psi x)^2|$ lies between limits of the form $\tfrac{1}{2}(1\pm\epsilon)\cosh 2\beta(x-t)$, the hyperbolic function being used because the sign of $x-t$ is uncertain.

$|(\phi' x\,\psi t-\psi' x\,\phi t)^2|$ also lies between limits of the same form.

Again, when $X=0$, $\psi=\phi'$ and $g=h$. When $X$ is variable we have

$$\phi'''+(X+\lambda)\phi'+X'\phi=0,$$

and therefore

$$(\psi''-\phi''')+(X+\lambda)(\psi-\phi')=X'\phi,$$

$$(\psi''-\phi''')\phi-(\psi-\phi')\phi''=X'\phi^2,$$

$$(\psi''-\phi''')\psi-(\psi-\phi')\psi''=X'\phi\psi.$$

Thus

$$(\psi'-\phi'')\phi-(\psi-\phi')\phi'=\int_0^x X'\phi^2\,dx,$$

$$(\psi'-\phi'')\psi-(\psi-\phi')\psi'=\int_0^x X'\phi\psi\,dx,$$

$$\psi-\phi'=\phi\int_0^x X'\phi\psi\,dx-\psi\int_0^x X'\phi^2\,dx=\int_0^x T'\phi t(\phi x\,\psi t-\phi t\,\psi x)\,dt$$

where $T'$ is what $X'$ becomes when $t$ is put for $x$.

But by our former work $|T'\phi t(\phi x\,\psi t-\phi t\,\psi x)|$, when $x>t$, does not exceed $C\exp\beta t\exp\beta(x-t)\div l$ where $C$ is a fixed finite quantity.

Hence $\psi-\phi'$ does not exceed $Cx\exp\beta x\div l$: it therefore vanishes in the limit in comparison with $\psi$ or $\phi'$, these being of the order of magnitude of $\exp\beta x$: it follows that $g-h$ vanishes in comparison with $g$ or $h$.

14. Thus in the numerator of $\Omega(x,t,\lambda)$ the quantities $\phi x\,\chi t$, $\psi x\,\chi t$, $\phi x\,\omega t$, $\psi x\,\omega t$, $\phi x\,\psi t-\psi x\,\phi t$ are respectively of the same orders of magnitude as $l^{-1}\exp\beta(1+x-t)$, $l^{-\frac{1}{2}}\exp\beta(1+x-t)$, $l^{-\frac{1}{2}}\exp\beta(1+x-t)$, $\exp\beta(1+x-t)$, $l^{-\frac{1}{2}}\cosh\beta(x-t)$, while in the denominator $e$, $g$, $h$, $k$ are of the same orders as $l^{-\frac{1}{2}}\exp\beta$, $\exp\beta$, $\exp\beta$, $l^{\frac{1}{2}}\exp\beta$. Thus when $x<t$ the terms of the numerator are small compared with those of the denominator, since $1+x-t$ and $t-x$ are both less than 1: the numerator is therefore small in comparison with the denominator unless the terms of the latter neutralize each other. This cannot happen unless $K=0$, since $k$ is great compared with $e$, $g$, $h$. When $K=0$ the sum of the terms $Gg+Hh$ will only be of lower order than $g$, $h$ if $G=-H$, since we have proved that $g-h$ is small compared with $g$ or $h$: this is, however, impossible, since, when $K=0$, $GH \nless L^2$, and therefore $GH$ cannot be negative. If then $K=0$, $G=-H$, we must have $G=H=L=0$, and only the terms containing $E$ are left.

Thus, when $x<t$, $\Omega(x,t,\lambda)$ tends to the limit zero as $\beta$ increases

without limit: it is, in fact, comparable at most with the greater of the two[*] quantities $l^{-\frac{1}{2}} \exp \beta(x-t)$, $l^{-\frac{1}{2}} \exp \beta(t-x-1)$.

The length of the contour of integration is comparable with $l$, and therefore $\int \Omega(x, t, \lambda)\, d\lambda$ tends to zero for those parts of the path where the amplitude of $\lambda$ is not nearly 0 or $2\pi$, so that $\beta/a$ is not small.

The part where the amplitude of $\lambda$ is nearly 0 or $2\pi$ must be treated separately. When $X = 0$, and the amplitude of $\lambda$ is 0, $\phi$ and $\psi$ are simple trigonometrical functions. We need to know within what range the trigonometrical forms give a satisfactory approximation.

15. From the equation $\phi'' + (X+\lambda)\phi = 0$

we have $$\phi' \sin x\sqrt{\lambda} - \sqrt{\lambda}\,\phi \cos x\sqrt{\lambda} = -\int_0^x X\phi \sin x\sqrt{\lambda}\, dx$$

and $$\phi' \cos x\sqrt{\lambda} + \sqrt{\lambda}\,\phi \sin x\sqrt{\lambda} = 1 - \int_0^x X\phi \cos x\sqrt{\lambda}\, dx$$

by multiplying by $\sin x\sqrt{\lambda}$ and $\cos x\sqrt{\lambda}$ and integrating.

Eliminate $\phi'$; then

$$\sqrt{\lambda}\,\phi = \sin x\sqrt{\lambda} - \int_0^x T\phi(t) \sin(x-t)\sqrt{\lambda}\, dt$$

where $T$ stands for the same function of $t$ that $X$ is of $x$.

Suppose $\kappa$ to be a superior limit to $|\sqrt{\lambda}\,\phi - \sin x\sqrt{\lambda}|$. We have

$$|\sin x\sqrt{\lambda}| \not> \cosh \beta x$$

where, as before, $$\lambda = (a+\iota\beta)^2.$$

Hence $|\phi\sqrt{\lambda}| \not> \kappa + \cosh \beta x$ and $|\phi| \not> l^{-\frac{1}{2}}(\kappa + \cosh \beta)$, since $x \not> 1$.

Let $B$ be the greatest numerical value of $T$ between $t = 0$ and $t = 1$. Then

$$|\phi\sqrt{\lambda} - \sin x\sqrt{\lambda}| = \left|\int_0^x T\phi(t) \sin(x-t)\sqrt{\lambda}\, dt\right| \not> B\, l^{-\frac{1}{2}}(\kappa + \cosh \beta) \cosh \beta.$$

Thus $$\kappa l^{\frac{1}{2}} \not> B(\kappa + \cosh \beta) \cosh \beta$$

and we may take $$\kappa = B \cosh^2 \beta / (l^{\frac{1}{2}} - B \cosh \beta).$$

This diminishes indefinitely as $l$ increases if $\cosh^2 \beta$ becomes infinitely small in comparison with $l^{\frac{1}{2}}$, or with $a$. This would happen if $\beta$ were $\not> \frac{1}{2} \log a$. We have also

$$\phi' = \cos x\sqrt{\lambda} - \int_0^x T\phi(t) \cos(t-x)\sqrt{\lambda}\, dt,$$

---

[*] The second must be taken when the term multiplied by $L$ is the most important in the numerator. Since $L$ does not vanish, $G$, $H$, $K$ cannot all vanish; so that the least favourable case is when $K = 0$. When $K$ does not vanish $\Omega(x, t, \lambda)$ is comparable with $l^{-\frac{1}{2}} \exp \beta(x-t)$ or else $l^{-1} \exp \beta(t-x-1)$.

so that $$|\phi' - \cos x\sqrt{\lambda}| \not> B l^{-\frac{1}{2}}(\kappa + \cosh\beta)\cosh\beta,$$

that is, $\kappa$.

The same quantity $\kappa$ is a superior limit to $|\chi\sqrt{\lambda} + \sin(1-x)\sqrt{\lambda}|$ and $|\chi' - \cos(1-x)\sqrt{\lambda}|$, and also to $|\psi - \cos x\sqrt{\lambda}|$, $|\psi'\lambda^{-\frac{1}{2}} + \sin x\sqrt{\lambda}|$, $|\omega - \cos(1-x)\sqrt{\lambda}|$, $|\omega'\lambda^{-\frac{1}{2}} - \sin(1-x)\sqrt{\lambda}|$. Hence, by putting $X = 0$ we get approximations to $\phi, \chi, \psi, \omega$ which hold good at least so long as $\beta$ is between $\pm\frac{1}{3}\log a$. When $\beta$ does not lie between these limits the work of §§ 10–14 is applicable, since $\beta$ increases indefinitely with $l$.

16. The path of integration may thus be divided into two parts, a minor arc on which $a$ is great and positive and $\beta$ lies between $\pm\frac{1}{3}\log a$, and a major arc on which $\beta$ lies without these limits. The value of the integral taken over the minor arc shows no sign of being infinitesimal, for the length of the arc becomes infinite and the subject of integration is of the order of magnitude of $l^{-\frac{1}{2}}$. Thus the series does not converge; in fact the successive terms of the series do not tend to zero as a limit.

If, however, we multiply the terms of the series by an analytical function of $t$, say, $F(t)$, and integrate with respect to $t$ between limits included in the range from 0 to 1, the result is a convergent series whose sum to infinity can be found.

17. If $a$, $b$ are such that $0 \leqslant a < x < b \leqslant 1$, we have, choosing the expression for $\Sigma\Phi$ in which the contribution of the major arc is small,

$$2\iota\pi \sum_{r=1}^{r=n} \int_a^b \Phi(x, t, \lambda_r)\, F(t)\, dt$$
$$= \iint_a^x \Omega(t, x, \lambda) F(t)\, dt\, d\lambda + \iint_x^b \Omega(x, t, \lambda) F(t)\, dt\, d\lambda,$$

the integration with respect to $\lambda$ being again over the circle $|\lambda| = l$, which encloses $\lambda_1, \lambda_2, \ldots, \lambda_n$.

For $\Omega$ we may write $-(T+\lambda)^{-1} d^2\Omega/dt^2$ and integrate by parts with respect to $t$. The expression becomes

$$\iint_a^x \frac{d}{dt}\Omega(t, x, \lambda) \frac{d}{dt}\, \frac{F(t)}{T+\lambda}\, dt\, d\lambda + \iint_x^b \frac{d}{dt}\Omega(x, t, \lambda)\frac{d}{dt}\, \frac{F(t)}{T+\lambda}\, dt\, d\lambda$$
$$-\int \left[ \Big/_{t=a}^{t=x} \left\{ \frac{d}{dt}\Omega(t, x, \lambda)\frac{F(t)}{T+\lambda} \right\} + \Big/_{t=x}^{t=b} \left\{ \frac{d}{dt}\Omega(x, t, \lambda)\frac{F(t)}{T+\lambda} \right\} \right] d\lambda.$$

In the single integral pick out the terms given by substituting $x$ for $t$.

They are

$$-\int \frac{F(x)}{X+\lambda}\left\{\frac{E\phi'\chi+G\psi'\chi-H\phi'\omega-K\psi'\omega+L(\phi'\psi-\phi\psi')}{Ee+Gg+Hh+Kk-2L}\right.$$
$$\left.-\frac{E\phi\chi'+G\psi\chi'-H\phi\omega'-K\psi\omega'+L(\phi\psi'-\phi'\psi)}{Ee+Gg+Hh+Kk-2L}\right\}d\lambda,$$

or $\int (X+\lambda)^{-1}F(x)\,d\lambda$ or $2\iota\pi F(x)$, since the point where $\lambda = -X$ falls within the contour of integration. It is now to be proved that the rest of the expression tends to zero when the radius $l$ is increased indefinitely. The major and minor arcs as defined above (§ 16) must be considered separately.

18. Writing still $a+\iota\beta$ for $\sqrt{\lambda}$, we have on the major arc $a$ positive and $\beta > \frac{1}{3}\log a$ numerically, and therefore increasing indefinitely with $l$. Then

$$\frac{d}{dt}\Omega(t,x,\lambda) = [E\phi'(t)\chi(x)+G\psi'(t)\chi(x)-H\phi'(t)\omega(x)$$
$$-K\psi'(t)\omega(x)+L\{\phi'(t)\psi(x)-\psi'(t)\phi(x)\}]$$
$$\div(Ee+Gg+Hh+Kk-2L),$$

which is at most* of the order of magnitude of $\exp\beta(x-t-1)$ or $\exp\beta(t-x)$, whichever is the higher, and therefore tends to zero if $t<x$. Also $F(t)$ is finite and $d\lambda/(T+\lambda)$ may be written

$$[\lambda/(\lambda+T)]\,(d\lambda/\lambda),$$

in which expression the first factor is always finite, and the second is $\iota\,d\theta$ where $\lambda = l\exp\iota\theta$. The range of values of $\theta$ in the integration is less than $2\pi$, and hence, for the major arc,

$$\int\left\{\frac{d}{dt}\Omega(t,x,\lambda)\frac{F(t)}{T+\lambda}\right\}_{t=a}d\lambda$$

tends to the limit zero.

Similarly $$\int\left\{\frac{d}{dt}\Omega(x,t,\lambda)\frac{F(t)}{T+\lambda}\right\}_{t=b}d\lambda,$$

for the major arc, tends to zero, since $b > x$.

The first double integral may be written

$$\iota\iint_a^x \frac{d}{dt}\Omega(t,x,\lambda)\frac{d}{dt}\left\{\frac{\lambda}{\lambda+T}F(t)\right\}dt\,d\theta.$$

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* See note to § 14.

The factor $\frac{d}{dt}\left\{\frac{\lambda}{\lambda+T}F(t)\right\}$ is always finite; let $A$ be a superior limit to its absolute value.

The factor $\frac{d}{dt}\Omega(t, x, \lambda)$, provided $a > x-\frac{1}{2}$, is not greater in absolute value than $B\exp\beta(t-x)$ where $B$ is also finite. Hence this double integral is not greater than

$$2\pi AB\int_a^x \exp\beta(t-x)\,dt \quad \text{or} \quad 2\pi AB\{1-\exp\beta(a-x)\}/\beta,$$

or than

$$2\pi AB/\beta.$$

It therefore tends to the limit zero.

When $a < x-\frac{1}{2}$ the only difference is that for values of $t$ between $a$ and $x-\frac{1}{2}$ the term multiplied by $L$ becomes the most important in the numerator of $d\Omega/dt$, unless $L = 0$. This does not affect the result: we have seen that, for such values of $t$, $|d\Omega/dt|$ has a superior limit of the form $B\exp\beta(x-t-1)$, and the integral of this between $a$ and $x-\frac{1}{2}$ is negligible.

The other double integral tends to zero similarly. Hence the whole contribution of the major arc tends to the limit zero.

19. On the minor arc the range of values of $\theta$ is infinitesimal, and hence it is only necessary to prove that the factors of the form $d\Omega/dt$ are finite in order to show that the contribution of the minor arc is infinitesimal. Clearly the arc must be so drawn as not to pass through any point such as $\lambda_n$.

Now on the minor arc $\beta < \frac{1}{3}\log\alpha$ numerically, and therefore the numerator and denominator of $\frac{d}{dt}\Omega(t, x, \lambda)$ differ by infinitesimals from

$$E\lambda^{-\frac{1}{2}}\cos t\sqrt{\lambda}\sin(x-1)\sqrt{\lambda}-G\sin t\sqrt{\lambda}\sin(x-1)\sqrt{\lambda}$$
$$-H\cos t\sqrt{\lambda}\cos(x-1)\sqrt{\lambda}+K\sqrt{\lambda}\sin t\sqrt{\lambda}\cos(x-1)\sqrt{\lambda}+L\cos(x-t)\sqrt{\lambda},$$

and $$E\lambda^{-\frac{1}{2}}\sin\sqrt{\lambda}+(G+H)\cos\sqrt{\lambda}-K\sqrt{\lambda}\sin\sqrt{\lambda}-2L.\text{*}$$

Suppose first that $K$ does not vanish; then the term containing it is the chief term in the denominator, the rest diminishing without limit in comparison, so long as $\sin\sqrt{\lambda}$ is not small.

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* This approximation shows, if any proof is needed, that the series $\lambda_1, \lambda_2, \ldots$ is an infinite one, since by substituting successive even or odd multiples of $\frac{1}{2}\pi$ for $\sqrt{\lambda}$ we find that the expression changes sign an infinite number of times. The even multiples of $\frac{1}{2}\pi$ serve when $K = 0$, $G+H \neq 0$, the odd multiples in other cases.

Now $|\sin\sqrt{\lambda}\,|^2 = |\sin^2(\alpha+\iota\beta)\,| = \cosh^2\beta-\cos^2\alpha = \cosh^2\beta$,

if we take $\alpha =$ an odd multiple of $\frac{1}{2}\pi$. This implies that $\alpha$ is constant along the minor arc, which is therefore not circular, but parabolic; along it $\beta$ varies from $-\frac{1}{3}\log a$ to $\frac{1}{3}\log a$.

Similarly, $\quad |\sin^2 t\sqrt{\lambda}\,| = \cosh^2 t\beta-\cos^2 t\alpha \ngtr \cosh^2 t\beta$,

and $\quad |\cos^2 t\sqrt{\lambda}\,| = \cosh^2 t\beta-\sin^2 t\alpha \ngtr \cosh^2 t\beta$.

Hence $|K\sqrt{\lambda}\sin t\sqrt{\lambda}\cos(x-1)\sqrt{\lambda}\,| \ngtr K\sqrt{l}\cosh t\beta\cosh(x-1)\beta$,

which is less than $K\sqrt{l}\cosh\beta$ when $t<x<1$; similarly

$$|L\cos(x-t)\sqrt{\lambda}\,| \ngtr L\cosh\beta,$$

and so for the other terms of the numerator. Hence the contribution of the minor arc tends to zero when $K$ does not vanish.

Suppose next that $K=0$: then $GH \nless L^2$; so that $G+H$ cannot vanish unless $G$, $H$, $L$ are all zero. Take $G+H$ not to vanish. The same kind of proof applies, but, since

$$|\cos\sqrt{\lambda}\,|^2 = \cosh^2\beta-\sin^2\alpha,$$

we take $\alpha$ to be a multiple of $\pi$; so that $|\cos\sqrt{\lambda}\,| = \cosh\beta$.

If $G$, $H$, $K$, $L$ all vanish, the proof is the same as when $K$ is not zero.

Similarly for $\dfrac{d}{dt}\Omega(x, t, \lambda)$ when $x<t<1$.

20. Hence we have

$$F(x) = \lim_{n=\infty}\sum_{r=1}^{r=n}\int_a^b \Phi(x, t, \lambda_r)F(t)\,dt,$$

when $a<x<b$.

If $x$ were $<a$, we should have

$$2\iota\pi\sum_{r=1}^{r=n}\int_a^b \Phi(x, t, \lambda_r)\,F(t)\,dt = \iint_a^b \Omega(x, t, \lambda)\,F(t)\,dt\,d\lambda,$$

and this would be zero in the limit. Similarly, if $x>b$.

Thus $F(x)$ is expanded between $a$ and $b$ in a series of multiples of the functions $\phi(x, \lambda_n)$ and $\psi(x, \lambda_n)$; the sum of the series is zero when $0<x<a$ or $b<x<1$; the coefficients of $\phi(x, \lambda_n)$, $\psi(x, \lambda_n)$ are the values when $\lambda=\lambda_n$ of

$$\int_a^b \{(Eg+Hk)\phi+(Gg+Kk-L)\psi\}\,F(x)\,dx$$
$$\div\int_0^1 (E\phi\chi+G\psi\chi-H\phi\omega-K\psi\omega)\,dx$$

and $\int_a^b \{(Gg+Kk-L)\phi-(Ge+Kh)\psi\}\, F(x)\,dx$

$$\div \int_0^1 (E\phi\chi+G\psi\chi-H\phi\omega-K\psi\omega)\,dx$$

respectively; $\lambda_1, \lambda_2, \ldots$ are the values of $\lambda$ for which

$$Ee+Gg+Hh+Kk = 2L,$$

and, if two of these coincide, the coefficients just written become indeterminate and are to be replaced by such quantities as are indicated by (7), § 8.

It is easy to pass to a similar series which shall express different analytical functions in different parts of the range between 0 and 1. The functions need not in fact be analytical, since only the first derivative of $F(x)$ was used in the proof; it is necessary to the proof that this derivative should exist, but the whole series of derivatives is not needed.

21. When $GH = EK+L^2$* the numerator of $\Phi(x, t, \lambda)$ breaks up into factors, one a function of $x$, and the other the same function of $t$. This happens in the best known special cases. For instance, when $G$, $H$, $K$, $L$ all vanish the equation determining $\lambda_1, \lambda_2, \ldots$ is $e = 0$, and we have expansions in series of harmonic functions which all vanish at the two ends of the range of validity, since when $e = 0$ the functions $\phi$, $\chi$ only differ by a constant factor.

Again, when $E = K = 0$, $G = H = L = 1$, each term of the expansion has the same value and the same derivative at the two ends of the range; thus, if $X$ had 1 as a period, the expansion would be in periodic functions; in our notation the type of such a function would be $\chi-\phi$ or $\omega-\psi$ indifferently.

22. Again, taking $X = 0$, we have trigonometrical expansions, and, if $G = K = L = 0$, the terms are of the form $A \sin ax$, the values of $a$ being such that

$$\tan a = -Ha/E.$$

These are the expansions used by Fourier in his treatment of the problem

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* Note that $GH-EK-L^2$ is an algebraic invariant of the two forms

$$gh-ek-1, \qquad Ee+Gg+Hh+Kk-2L$$

in the variables $e$, $g$, $h$, $k$. If this invariant is positive, and $e$, $g$, $h$, $k$ are taken as coordinates of a point in space of four dimensions, the two loci

$$Ee+Gg+Hh+Kk-2L = 0, \qquad gh-ek-1 = 0$$

cut one another in a conicoid whose generators are imaginary, and, if the invariant vanishes, in a cone.

of the conduction of heat in a solid sphere (see Burkhardt's *Report*, p. 420; Fourier's *Works*, Vol. I., p. 312). A variety of other forms of trigonometrical expansion could be derived by giving different values to $E$, $G$, $H$, $K$, $L$.

23. The question whether the expansion of $F(x)$ is valid when $x = a$ or $b$ has not yet been answered, but it is readily seen as in § 17 that the expansion as a rule is only valid when $x = a$ if $F(a) = 0$, and when $x = b$ if $F(b) = 0$; more generally, the expansion of a discontinuous function does not hold good at the values where the discontinuities occur.

The cases $x = a = 0$ and $x = b = 1$ deserve special notice. Take first $x = a = 0$. We have

$$2\iota\pi \sum_{r=1}^{n} \int_0^b \Phi(0, t, \lambda_r) F(t)\, dt$$
$$= \iint_0^b \frac{d}{dt}\Omega(0, t, \lambda)\, \frac{d}{dt}\, \frac{F(t)}{T+\lambda}\, dt\, d\lambda - \int \Big/_{t=0}^{t=b} \left\{ \frac{d}{dt}\Omega(0, t, \lambda)\, \frac{F(t)}{T+\lambda} \right\} d\lambda.$$

Now $$\frac{d}{dt}\Omega(0, t, \lambda) = \frac{G\chi' t - K\omega' t - L\phi' t}{Ee+Gg+Hh+Kk-2L},$$

which is of the order of magnitude of $\exp(-\beta t)$, or possibly $\exp\beta(t-1)$ when $t > \frac{1}{2}$ and $L \neq 0$. Thus the double integral tends to zero as before. In the single integral the part given by substituting $b$ for $t$ tends to zero unless $b = 1$; taking $b = 1$ and writing $X_0$, $X_1$ for the values of $X$ when $x = 0, 1$, we have for the single integral

$$\int \left\{ \frac{Gg+Kk-L}{Ee+Gg+Hh+Kk-2L}\, \frac{F(0)}{X_0+\lambda} - \frac{G-Lh}{Ee+Gg+Hh+Kk-2L}\, \frac{F(1)}{X_1+\lambda} \right\} d\lambda.$$

The value of this is $2\iota\pi F(0)$ if $X_0 = X_1$, $E = 0$, and

$$H : L :: L : G :: F(1) : F(0);$$

or if $F(1) = 0 = F(0)$; or if $E, G, H, L$ all vanish and $X_0 \neq X_1$. It tends to the limit $2\iota\pi F(0)$ if $F(0) = 0$ and $K$ does not vanish, or if $F(0) = 0$ and $K$, $L$ vanish.

Similarly, when $x = b = 1$, we have

$$\frac{d}{dt}\Omega(t, 1, \lambda) = (-H\phi' t - K\psi' t + L\chi' t)/(Ee+Gg+Hh+Kk-2L),$$

and the single integral in this case is, if $a = 0$,

$$\int \left\{ (-H+Lg)\frac{F(0)}{X_0+\lambda} - (-Hh-Kk+L)\, \frac{F(1)}{X_1+\lambda} \right\} \frac{d\lambda}{Ee+Gg+Hh+Kk-2L}.$$

The value of this is $2\iota\pi F(1)$, if $X_0 = X_1$, $E = 0$, and

$$H : L :: L : G :: F(1) : F(0);$$

or if $F(1) = 0 = F(0)$; or if $E$, $G$, $H$, $L$ all vanish and $X_0 \neq X_1$.

Hence the expansion is valid for the extreme values 0 and 1 in three cases, at least,

(1) When $F(1) = F(0) = 0$.

(2) When $E$, $G$, $H$, $L$ vanish, so that the equation whose roots are $\lambda_1, \lambda_2, \ldots$ is $k = 0$.

(3) When $X_0 = X_1$, $E = 0$, $L = mG$, $H = m^2 G$, $F(1) = mF(0)$, $m$ being some real constant, finite or infinite.

There are other cases in which the expansion is valid for one extreme value or the other.

It is easily seen that these conclusions are not affected by discontinuities in $F(x)$ at intermediate values of $x$.

24. It has been assumed that none of the critical points of the differential equation fall between the values 0 and 1 of the independent variable or at either of those values. If 0 or 1 were a critical point, the expansion would not necessarily fail, as is shewn by the special case of Legendre's functions in which there is a critical point at each end of the range of validity. If the method of this paper were used, the initial conditions by which $\phi$, $\chi$, $\psi$, $\omega$ are defined would have to be modified.

## NOTES ADDED IN PROOF.

I. To the results in §§ 6, 7 the following may be added. If $Y$ stands for $Ee+Gg+Hh+Kk-2L$, $Y = 0$ is an equation for $\lambda$ whose roots are separated by those of $Eg+Hk = 0$, also by those of $Ge+Kh = 0$, and further by those of $Z+\kappa Y = 0$, where

$$Z = \alpha^2(Eg+Hk)+\beta^2(Eh+Gk)-\gamma^2(He+Kg)-\delta^2(Ge+Kh)$$
$$-2(\alpha\delta+\beta\gamma)M,$$

$\alpha$, $\beta$, $\gamma$, $\delta$, $\kappa$ being any real constants, and $M$ a real constant such that

$$M^2 \not> GH-EK-L^2.$$

It follows, in fact, from the formulæ of the paper that

$$Z\dot{Y}-Y\dot{Z}=\int_0^1\big[\{\alpha(E\chi-H\omega+L\psi)-\delta M\phi\}^2+\{\beta(E\phi+G\psi-L\omega)-\gamma M\chi\}^2$$
$$+\{\gamma(H\phi+K\psi-L\chi)+\beta M\omega\}^2+\{\delta(G\chi-K\omega-L\phi)-\alpha M\psi\}^2$$
$$+(\alpha^2\psi^2+\beta^2\omega^2+\gamma^2\chi^2+\delta^2\phi^2)(GH-EK-L^2-M^2)\big]\,dx.$$

This being never negative, the roots of $Y=0$ and $Z+\kappa Y=0$ separate each other as a rule; it is, however, possible for $Y$, $Z$ to vanish together when $GH-EK=L^2+M^2$. If $\lambda$, $\lambda_1$ were conjugate complex quantities, we should have $(ZY_1-YZ_1)/(\lambda_1-\lambda)$ essentially positive.

II. The results of §§ 11–13 may be proved, perhaps more simply, by the following method. Suppose, in § 15, that $\beta$ is not restricted, and that $\kappa$ is a superior limit to

$$|\sqrt{\lambda}\,\phi-\sin x\sqrt{\lambda}\,|/\cosh\beta x.$$

Then $$|\phi\sqrt{\lambda}\,|\not>(1+\kappa)\cosh\beta x;$$

thus $$|\phi\sqrt{\lambda}-\sin x\sqrt{\lambda}\,|\not> Bl^{-\frac{1}{2}}(1+\kappa)\int_0^x\cosh\beta t\cosh\beta(x-t)\,dt$$
$$\not> Bl^{-\frac{1}{2}}(1+\kappa)\left(\frac{x}{2}\cosh\beta x+\frac{1}{2\beta}\sinh\beta x\right);$$

so that $\kappa\not> Bl^{-\frac{1}{2}}(1+\kappa)$, since $\tanh\beta x<\beta x$, $x<1$.

Hence we may take $\kappa=B/(l^{\frac{1}{2}}-B)$, a small quantity, and the ratio of $\phi\sqrt{\lambda}$ to $\sin x\sqrt{\lambda}$ tends to unity when $\beta$ is increased indefinitely, since $|\sin x\sqrt{\lambda}\,|$ approaches equality with $\cosh\beta x$, and $\phi\sqrt{\lambda}-\sin x\sqrt{\lambda}$ is small compared with $\cosh\beta x$. In the same way $\phi'$ and $\cos x\sqrt{\lambda}$, $\psi$ and $\cos x\sqrt{\lambda}$, $\psi'$ and $-\sqrt{\lambda}\sin x\sqrt{\lambda}$, $\chi\sqrt{\lambda}$ and $\sin(x-1)\sqrt{\lambda}$, $\chi'$ and $\cos(x-1)\sqrt{\lambda}$, $\omega$ and $\cos(x-1)\sqrt{\lambda}$, $\omega'$ and $\sqrt{\lambda}\sin(1-x)\sqrt{\lambda}$ tend respectively to equality when $\beta$ is increased indefinitely.

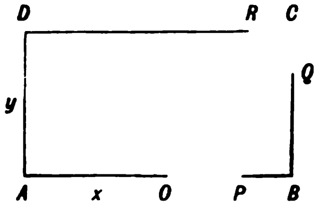
# CURRENT FLOW IN RECTANGULAR CONDUCTORS

*By* H. FLETCHER MOULTON.

[Received January 3rd, 1905.—Read January 12th, 1905.]

THE simplest case of current flow in two dimensions is that of electricity flowing across a rectangular plate the two opposite sides of which are the electrodes. I am, however, unacquainted with any formula dealing with the cases in which the electrodes do not extend over the whole of these sides, and it is difficult to see how any approximation based on *a priori* reasoning can be made for these cases.

By the use of Schwarz's transformation, however, the results come out in a very simple manner.



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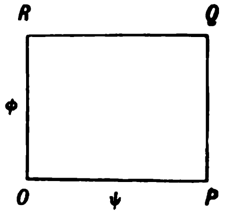

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Let (1) be the $x$, $y$ diagram, the electrodes being $OP$ and $QR$. Let (2) be the $\phi$, $\psi$ diagram. Writing, as usual, $z = x+\iota y$, $w = \psi+\iota\phi$, and for simplicity taking $\sigma$, the specific resistance, as unity, we put

$$dz = \frac{dt}{\sqrt{\{(t-\alpha)(t-\beta)(t-\gamma)(t-\delta)\}}},$$

$\alpha$, $\beta$, $\gamma$, $\delta$ being arbitrary quantities subject to the one condition that $(\beta-\gamma)(\alpha-\delta)/(\alpha-\gamma)(\beta-\delta) = \kappa^2$, where $\kappa$ is chosen so that the ratio of the periods $K'/K = AD/AB$. Then $\kappa$ is immediately found from Legendre's first table. Again, put

$$dw = \frac{dt'}{\sqrt{\{(t'-\alpha')(t'-\beta')(t'-\gamma')(t'-\delta')\}}},$$

where $(\beta'-\gamma')(\alpha'-\delta')/(\alpha'-\gamma')(\beta'-\delta') = l^2$, the modulus $l^2$ being such that on the $\psi$, $\phi$ diagram the ratio of corresponding periods, viz., $L'/L = OR/OP$, *i.e.*, $L'/L =$ equivalent resistance of plate.

The quantity $l$, of course, is at present undetermined, and it is our purpose to determine it. We have

$$t = \frac{\alpha(\beta-\delta)-\beta(\alpha-\delta)\operatorname{sn}^2 mz \pmod{\kappa}}{\beta-\delta-(\alpha-\delta)\operatorname{sn}^2 mz \pmod{\kappa}}$$

and

$$t' = \frac{\alpha'(\beta'-\delta')-\beta'(\alpha'-\delta')\operatorname{sn}^2 m'w \pmod{l}}{\beta'-\delta'-(\alpha'-\delta')\operatorname{sn}^2 m'w \pmod{l}},$$

where $m = \frac{1}{2}\sqrt{\{(\alpha-\gamma)(\beta-\delta)\}}$, $m' = \frac{1}{2}\sqrt{\{(\alpha'-\gamma')(\beta'-\delta')\}}$.

Since we have only one relation between $\alpha$, $\beta$, $\gamma$, and $\delta$, and between $\alpha'$, $\beta'$, $\gamma'$, and $\delta'$, we can take $m$ and $m'$ to be unity.

We can also simplify the problem without loss of generality by assuming the units such that $AB = K$, $OP = L$, and therefore $AD = K'$, $OR = L'$.

Now there is one and only one solution of the equation $\nabla^2 w = 0$ over the area $ABCD$ with given boundary conditions.

By the transformation adopted the boundary in the $z$ plane becomes the real axis in the $t$ plane, and the area $ABCD$ becomes the part of the $t$ plane above the real axis. So also the boundary in the $w$ plane becomes the real axis in the $t'$ plane, and the area $OPQR$ becomes the part of the $t'$ plane above the real axis.

If, therefore, we put $t = t'$, and so choose the constants that each point of the real axis represents corresponding points on the $z$ and $w$ planes, we shall arrive at the required relation between $z$ and $w$.

To do this we must determine $\alpha'$, $\beta'$, $\gamma'$, and $\delta'$, and we do so by putting $t = t'$ at $O$, $P$, $Q$, and $R$. Let the co-ordinates of $O$, $P$, $Q$, and $R$ in the $z$ plane be $z_1$, $z_2$, $z_3$, $z_4$. In the $w$ plane they are $O$, $K$, $K+\iota K'$, $\iota K'$.

Then

$$\alpha' = \frac{\alpha(\beta-\delta)-\beta(\alpha-\delta)\operatorname{sn}^2 z_1}{\beta-\delta-(\alpha-\delta)\operatorname{sn}^2 z_1},$$

$$\delta' = \frac{\alpha(\beta-\delta)-\beta(\alpha-\delta)\operatorname{sn}^2 z_2}{\beta-\delta-(\alpha-\delta)\operatorname{sn}^2 z_2},$$

$$\gamma' = \frac{\alpha(\beta-\delta)-\beta(\alpha-\delta)\operatorname{sn}^2 z_3}{(\beta-\delta)-(\alpha-\delta)\operatorname{sn}^2 z_3},$$

$$\beta' = \frac{\alpha(\beta-\delta)-\beta(\alpha-\delta)\operatorname{sn}^2 z_4}{(\beta-\delta)-(\alpha-\delta)\operatorname{sn}^2 z_4};$$

therefore

$$l^2 = \frac{(\beta'-\gamma')(\alpha'-\delta')}{(\alpha'-\gamma')(\beta'-\delta')} = \frac{(\operatorname{sn}^2 z_4-\operatorname{sn}^2 z_3)(\operatorname{sn}^2 z_1-\operatorname{sn}^2 z_2)}{(\operatorname{sn}^2 z_1-\operatorname{sn}^2 z_3)(\operatorname{sn}^2 z_4-\operatorname{sn}^2 z_2)}.$$

If the electrodes both commence at the same end, $l^2 = \operatorname{sn}^2 z_2/\operatorname{sn}^2 z_3$. If, in addition, they are equal, then $\operatorname{sn}^2 z_3 = 1/\kappa^2 \operatorname{sn}^2 z_2$ and $l^2 = \kappa^2 \operatorname{sn}^4 z_2$.

In the latter case, however, we get a simpler transformation by putting

$$AD/AB = 2K'/K, \qquad OR/OP = 2L'/L,$$

and putting $$\mathrm{dn}\,(z, \kappa) = t = \mathrm{dn}\,(w, l).$$

It is at once seen that $t$ is real round the boundaries, and that, when $z = 0$, $t = 1$ and $w = 0$, and, when $z = 2\iota K'$, $t = -1$ and $w = 2\iota L'$. If now $\kappa^2\,\mathrm{sn}^2 z_2 = l^2$, then $\mathrm{dn}\,(z, \kappa) = t = \mathrm{dn}\,(w, l)$ at all points.

Curves of current flow for $z_2 = \cdot 1K$ have been drawn from this formula in the case where $AD = 2AB$.

In the general case the formula giving the equipotential lines and the lines of current flow is

$$\mathrm{sn}^2\,(w, l) = \frac{\mathrm{sn}^2 z - \mathrm{sn}^2 z_1}{\mathrm{sn}^2 z - \mathrm{sn}^2 z_4}\,\frac{\mathrm{sn}^2 z_4 - \mathrm{sn}^2 z_2}{\mathrm{sn}^2 z_1 - \mathrm{sn}^2 z_2}.$$

The calculation of the lines of current flow in the above-named case where $AD = 2AB$ and $z_2 = 2K$ showed a very great increase of current density at the outer ends of the electrode. Theoretically the density would become infinite, but, of course, in practice the very great increase of current flow in this part of the electrode would cause a drop of potential.

The following examples of the equivalent resistances of a square conductor of unit specific resistance according to the position of the electrode may also prove interesting. They show that the increase of resistance due to the crowding together of the lines of current flow on their approach to a small electrode is by no means unimportant.

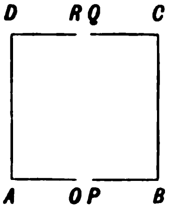
In each case the total length of an electrode is one-fifth of the side of the square.

(1) Electrodes opposite to each other and in the middle. Resistance = 1·745.

(2) Electrodes opposite to each other, but at one end. Resistance = 2·408.

(3) Electrodes on opposite sides, but at opposite corners. Resistance = 2·589.

(4) Electrodes bent equally round opposite corners. Resistance = 3·027.



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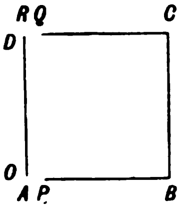

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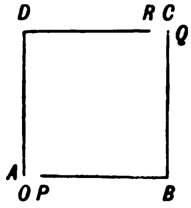

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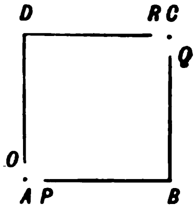

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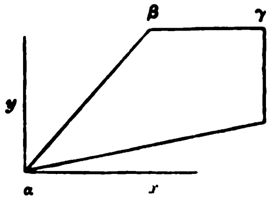
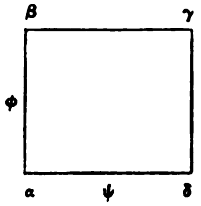
Schwarz's transformation also enables us to find the resistance of a plane rectilinear conductor one of whose angles is a right angle, the electrodes extending over the whole of opposite sides.





Let $\alpha\beta\gamma\delta$ be the conductor, $\beta\gamma\delta$ being a right angle, and construct the $x$, $y$ and $\psi$, $\phi$ diagrams as before.

Let $$dz = \frac{c_1\,dt}{\sqrt{\{(t-\alpha)(t-\beta)(t-\gamma)(t-\delta)\}}\,(t-\alpha)^p(t-\beta)^q(t-\delta)^r},$$
where
$$p+\tfrac{1}{2} = 1-\frac{\angle\delta\alpha\beta}{\pi}, \qquad q+\tfrac{1}{2} = 1-\frac{\angle\alpha\beta\gamma}{\pi}. \qquad r+\tfrac{1}{2} = 1-\frac{\angle\alpha\delta\gamma}{\pi}.$$

Let $$dw = \frac{c_2\,dt}{\sqrt{\{(t-\alpha)(t-\beta)(t-\gamma)(t-\delta)\}}}.$$

Then $$dz = \frac{c\,dw}{(t-\alpha)^p(t-\beta)^q(t-\delta)^r}, \quad \text{where } c = \frac{c_1}{c_2};$$

but $$t = \frac{\alpha(\beta-\delta)-\beta(\alpha-\delta)\,\mathrm{sn}^2 mw}{(\beta-\delta)-(\alpha-\delta)\,\mathrm{sn}^2 mw}\left[\mathrm{mod}\sqrt{\left\{\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)}\right\}}\right],$$

where $m = \frac{1}{2}\sqrt{\{(\alpha-\gamma)(\beta-\delta)\}}$ and can be put equal to unity without loss of generality.

We can also choose co-ordinates so that in the $\phi$, $\psi$ diagram $\alpha\delta = K$, $\alpha\beta = K'$, the modulus being $\sqrt{\left[\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)}\right]}$, and also make $c_1 = 1$, $c_2 = 1$. Then
$$t-\alpha = \frac{(\alpha-\beta)(\alpha-\delta)\,\mathrm{sn}^2 w}{(\beta-\delta)-(\alpha-\delta)\,\mathrm{sn}^2 w}, \qquad t-\beta = \frac{(\alpha-\beta)(\beta-\delta)}{(\alpha-\delta)-(\beta-\delta)\,\mathrm{sn}^2 w},$$
$$t-\delta = \frac{(\alpha-\delta)(\beta-\delta)\,\mathrm{cn}^2 w}{(\alpha-\delta)-(\beta-\delta)\,\mathrm{sn}^2 w};$$

so that $$dz = \frac{c\,dw\left[(\beta-\delta)-(\alpha-\delta)\,\mathrm{sn}^2 w\right]^{p+q+r}}{(\alpha-\beta)^{p+q}(\alpha-\delta)^{p+r}(\beta-\delta)^{q+r}\,\mathrm{sn}^{2p} w\,\mathrm{cn}^{2r} w},$$

or, since $p+q+r = 0$, we have
$$dz = c'dw\,\mathrm{sn}^{-2p} w\,\mathrm{cn}^{-2r} w.$$

If now the co-ordinates of $\beta$ and $\delta$ in the $x$, $y$ plane are $a+\iota b$ and $c+\iota d$, we get

$$\frac{a+\iota b}{c+\iota d} = \frac{\int_0^{\iota K'} dw \operatorname{sn}^{-2p} w \operatorname{cn}^{-2r} w}{\int_0^{K} dw \operatorname{sn}^{-2p} w \operatorname{cn}^{-2r} w} = \frac{\iota^{1-2p}\int_0^{K'} du \operatorname{sn}^{-2p}(u,\kappa') \operatorname{cn}^{2(p+r)}(u,\kappa')}{\int_0^{K} dw \operatorname{sn}^{-2p} w \operatorname{cn}^{-2r} w},$$

which can easily be proved to be the same as

$$\iota^{1-2p}\frac{\Gamma(\frac{1}{2}+p+r)\,\Gamma(1-p-r)\,F(\frac{1}{2},\,\frac{1}{2}-p,\,1+r,\,\kappa'^2)}{\Gamma(1+r)\,\Gamma(\frac{1}{2}-r)\,F(\frac{1}{2},\,\frac{1}{2}-p,\,1-p-r,\,\kappa^2)},$$

$F(\;)$ denoting the hypergeometric series.

From this we can determine $\kappa^2$ by approximation and consequently the resistance of the conductor. This approximation can be facilitated when $\kappa$ is either very small or nearly unity by observing that

$$F(\tfrac{1}{2},\,\tfrac{1}{2}-p,\,1-p-r,\,x)$$

and

$$F(\tfrac{1}{2},\,\tfrac{1}{2}-p,\,1+r,\,1-x)$$

are both solutions of the equation

$$y''+\frac{1-p-r-(2-p)x}{x(1-x)}\,y'-\frac{\frac{1}{2}(\frac{1}{2}-p)}{x(1-x)}\,y=0.$$

Call these two particular solutions $y_1$ and $y_2$.

Another solution in powers of $x$ is

$$y_3 = x^{p+r}F(\tfrac{1}{2}+p+r,\,\tfrac{1}{2}+r,\,1+p+r,\,x),$$

and another in powers of $(1-x)$ is

$$y_4 = (1-x)^{-r}F(\tfrac{1}{2}-p-r,\,\tfrac{1}{2}-r,\,1-r,\,1-x).$$

Since the above differential equation is of the second degree, we have

$$y_1 = Ay_2+By_4, \qquad y_2 = Cy_1+Dy_3,$$

where $A$, $B$, $C$, and $D$ are independent of $\kappa$ and are the same for all quadrilaterals with the same angles.

In the case where $r=0$, *i.e.*, where two adjacent angles are right angles, we find that $y_4$ degenerates into $y_2$, but in this case another solution in powers of $(1-x)$ is

$$y_4 = -\log(1-x)\,F(\tfrac{1}{2},\,\tfrac{1}{2}-p,\,1,\,1-x)$$
$$+\Sigma\,\frac{\frac{1}{2}\cdot\frac{3}{2}\,(n-\frac{1}{2})(\frac{1}{2}-p)(\frac{3}{2}-p)\ldots(n-\frac{1}{2}-p)}{(n\,!)^2}\,c_n(1-x)^n$$

where $$c_n = 2\sum_1^n \frac{1}{s} - \sum_1^n \frac{1}{s-\frac{1}{2}} - \sum_1^n \frac{1}{s-\frac{1}{2}-p}.$$

Now, since this equation is linear and of the second degree, we have

$$y_2 = Ay_1 + By_3, \qquad y_1 = Cy_2 + Dy_4,$$

where $A$, $B$, $C$, $D$ are constants for a given value of $p$.

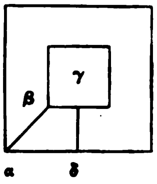
In employing this method when $\kappa$ is small, *i.e.*, the resistance is large, we approximate rapidly by means of the equation

$$\frac{a}{c} = \cos a \frac{\Gamma(\frac{1}{2}+p)\,\Gamma(1-p)}{\Gamma(\frac{1}{2})} \left(A + B\frac{y_3}{y_1}\right);$$

and, when $\kappa$ is nearly unity and the resistance is small,

$$\frac{c}{a} = \sec a \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+p)\,\Gamma(1-p)} \left(C + D\frac{y_4}{y_2}\right).$$

The case of $r = 0$ and $p = \frac{1}{4}$ gives the resistance of a plate when the electrodes are of the form which is occasionally used, in which the current flows from one square to a square symmetrically placed round the first. It is clear from symmetry that the lines bisecting the sides at right angles and the diagonals are lines of flow. The resistance therefore is $\frac{1}{8}$ of that of the quadrilateral $\alpha\beta\gamma\delta$.



If now $\alpha\delta$ is $a$ and $\alpha\beta$ is $c+\iota c$, we get

$$\frac{c}{a} = A' + B'\frac{y_3}{y_1} \quad \text{or} \quad \frac{a}{c} = C' + D'\frac{y_4}{y_2}.$$

The values to three places of $A'$, $B'$, $C'$, $D'$ are

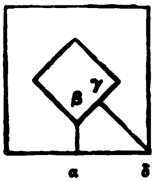
$$A' = 1, \quad B' = -{\cdot}457, \quad C' = 1{\cdot}605, \quad D' = -{\cdot}318.$$

If $k$ is small, it may be outside the limits of Legendre's tables, but we can approximate conveniently to the resistance $K'/K$ by means of the formula

$$K' = PK + Q\left\{K\log\kappa^2 + \frac{\pi}{2}\sum_1^\infty \kappa^{2p}(\tfrac{1}{2}\ \ldots\ n-\tfrac{1}{2})\sum_1^n \frac{1}{n(n-\frac{1}{2})}\right\},$$

where $P = {\cdot}883$, $Q = -{\cdot}318 = D'$: and, since $K = \frac{1}{2}\pi$ nearly and $\log\kappa^2$ is large, we get $R = P + Q\log\kappa^2$. But, if $\kappa$ is very small, $B'\kappa^{\frac{1}{2}} = c/a - 1$, and to a first approximation $R = -1{\cdot}28\log\frac{a-c}{a}$. If, however, $\kappa'$ is small, we have $\log\kappa'^2 = D'^{-1}a/c$ approximately, and therefore $1/R = a/c$, which could, of course, have been foreseen.

If the inner square is placed diamond-wise, then $\alpha\beta$ and $\gamma\delta$ are lines of flow, and, if the co-ordinates of $\beta$ are 0, $a$ and those of $\delta$ are $b$, 0,



$$\frac{a}{b}=\frac{F(\frac{1}{2},\frac{1}{2},\frac{5}{4},\kappa'^2)}{F(\frac{1}{2},\frac{1}{2},\frac{3}{4},\kappa^2)}\,\frac{\{\Gamma(\frac{3}{4})\}^2}{\Gamma(\frac{1}{4})\,\Gamma(\frac{5}{4})}.$$

Having determined $\kappa$ in any of the above cases, the resistance is given by $K'/K$, and the differential equations for the lines of flow and equipotential lines are

$$dx+\iota dy = c'(d\psi+\iota d\phi)\,[\mathrm{sn}(\psi+\iota\phi,\kappa)]^{-\frac{1}{2}}.$$

In the case of the two squares treated above the current density on the outer electrode is given by

$$\frac{\partial\psi}{\partial x}=\mathrm{sn}^{\frac{1}{2}}\psi,$$

and on the inner by

$$\frac{\partial\psi}{\partial x}=1/\kappa^{\frac{1}{2}}\,\mathrm{sn}^{\frac{1}{2}}\psi.$$

# A GENERALISATION OF THE LEGENDRE POLYNOMIAL

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[Received January 1st, 1905.—Read January 12th, 1905.]

## 1. *Source and Definition of the Function.*

The general homogeneous polynomial of degree $n$ in $x_1$, $x_2$, $x_3$, $x_4$ contains $\frac{1}{6}(n+1)(n+2)(n+3)$ arbitrary constants; if, however, it satisfies the differential equation

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}=0,$$

there will be $\frac{1}{6}(n-1)n(n+1)$ relations between these constants. The polynomial can therefore be expressed in terms of $(n+1)^2$ independent solutions.

To obtain a symmetrical representation of these independent solutions we make the substitution

$$x_1 = r\cos\theta\cos\phi, \qquad x_2 = r\cos\theta\sin\phi,$$
$$x_3 = r\sin\theta\cos\chi, \qquad x_4 = r\sin\theta\sin\chi;$$

the differential equation then becomes

$$\frac{\partial^2 V}{\partial r^2}+\frac{3}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r^2\cos^2\theta}\frac{\partial^2 V}{\partial \phi^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \chi^2}+\frac{\cot\theta-\tan\theta}{r^2}\frac{\partial V}{\partial \theta}=0.$$

If we assume as a trial solution $V = r^n e^{im\phi+ik\chi}\Theta$, we find that $\Theta$ must satisfy the equation

$$\frac{\partial^2\Theta}{\partial\theta^2}+(\cot\theta-\tan\theta)\frac{\partial\Theta}{\partial\theta}+\left\{n(n+2)-\frac{m^2}{\cos^2\theta}-\frac{k^2}{\sin^2\theta}\right\}\Theta=0.$$

Writing $\cos 2\theta = \mu$, the equation is transformed into

$$(1) \qquad \frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\}+\Theta\left\{\frac{n}{2}\left(\frac{n}{2}+1\right)-\frac{m^2}{2(1+\mu)}-\frac{k^2}{2(1-\mu)}\right\}=0.$$

Putting $\Theta = (1+\mu)^{\frac{1}{2}m}(1-\mu)^{\frac{1}{2}k}V$, we have

$$(2) \qquad (1-\mu^2)\frac{\partial^2 V}{\partial\mu^2}+[(m+1)(1-\mu)-(k+1)(1+\mu)]\frac{dV}{d\mu}$$
$$+\tfrac{1}{4}V[(n+1)^2-(k+m+1)^2]=0,$$

or, if $1-\mu = 2t$ and $1+\mu = 2(1-t)$,

(3)

$$(1-t)\frac{d^2V}{dt^2}+[k+1-(k+m+2)t]\frac{dV}{dt}+\tfrac{1}{4}(n-k-m)(n+k+m+2)V = 0.$$

A solution of this equation is given by

$$V = F\left(\frac{n+m+k+2}{2},\ -\frac{n-k-m}{2},\ k+1,\ t\right);$$

we may therefore define the function $\Theta_n^{m,k}(\mu)$ by the equation

$$\Theta_n^{m,k}(\mu) = \frac{(1+\mu)^{\frac{1}{2}m}(1-\mu)^{\frac{1}{2}k}}{2^{\frac{1}{2}(m+k)}\Pi(k)}\frac{\Pi\left(\dfrac{n+m+k}{2}\right)}{\Pi\left(\dfrac{n-m-k}{2}\right)}$$

$$\times F\left(\frac{n+m+k+2}{2},\ -\frac{n-k-m}{2},\ k+1,\ \frac{1-\mu}{2}\right).$$

When $n-m-k$ is an even positive integer the series terminates so that $\Theta_n^{m,k}(\mu)$ is then a polynomial in $\sin\theta$ and $\cos\theta$.

A polynomial of degree $n$ in $x_1$, $x_2$, $x_3$, and $x_4$ will contain terms of the form $r^n e^{im\phi+ik\chi}\Theta$, where $n-m-k$ is an even integer and $m\sim k$ and $m+k$ are numerically less than $n$. The different values of $r^n e^{im\phi+ik\chi}\Theta$ subject to these restrictions will give the $(n+1)^2$ independent solutions of degree $n$. The expression $r^n e^{\pm im\phi\pm ik\chi}\Theta_n^{m,k}(\mu)$ is clearly a solution of the equation whether $n$ is an integer or not.

## 2. *Connection with the Associated Legendre Function.*

If in equation (1) we put $k = m$, it becomes

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\}+\Theta\left\{\frac{n}{2}\left(\frac{n}{2}+1\right)-\frac{m^2}{1-\mu^2}\right\} = 0;$$

but this is the differential equation for the associated Legendre function.

Now when $m$ is an integer and $\theta$ is small $P_{\frac{1}{2}n}^m(\cos 2\theta)$ can be replaced by $\dfrac{\Pi(\frac{1}{2}n+m)}{\Pi(\frac{1}{2}n-m)2^m m!}\sin^m 2\theta$; therefore

(5)
$$\Theta_n^{m,m}(\cos\theta) = P_{\frac{1}{2}n}^m(\cos 2\theta).$$

Again, it can easily be verified that, if $F(x, y, z)$ is a solution of Laplace's equation, then

$$V = (x_1+ix_2)^{-k}F\{\sqrt{(x_1^2+x_2^2)},\ x_3,\ x_4\}$$

is a solution of the equation

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}=0.$$

Putting $\sqrt{(x_1^2+x_2^2)} = r_1 \cos\theta_1 = r\cos\theta,$

$$x_3 = r_1 \sin\theta_1 \cos\phi_1 = r\sin\theta\cos\chi, \qquad x_4 = r_1\sin\theta_1\sin\phi_1 = r\sin\theta\sin\chi,$$

the standard solution $r_1^n P_n^k(\cos\theta_1)e^{ik\phi_1}$ gives us

$$V = r^{n-\frac{1}{2}} P_n^k(\cos\theta)\, e^{-\frac{1}{2}i\phi+ik\chi} \cos^{-\frac{1}{2}}\theta.$$

Comparing this with the solution

$$r^{n-\frac{1}{2}}\Theta_{n-\frac{1}{2}}^{\pm\frac{1}{2},\pm k}(\cos 2\theta)\, e^{-\frac{1}{2}i\phi+ik\chi},$$

we see that there is a relation between $P_n^k(\cos\theta)$ and the quantities

$$\sqrt{(\cos\theta)}\,\Theta_{n-\frac{1}{2}}^{\pm\frac{1}{2},\pm k}(\cos 2\theta).$$

When $n$ and $k$ are integers $\Theta_{n-\frac{1}{2}}^{-\frac{1}{2},k}(\cos 2\theta)$ is of the form $(\cos\theta)^{-\frac{1}{2}}\sin^k\theta$ (polynomial in $\cos\theta$) when $n-k$ is even, and $\Theta_{n-\frac{1}{2}}^{\frac{1}{2},k}(\cos 2\theta)$ is of this form when $n-k$ is odd; accordingly

$$\sqrt{(\cos\theta)}\,\Theta_{n-\frac{1}{2}}^{-\frac{1}{2},k}(\cos 2\theta) = \frac{2^k\,\Pi(n-k)\,\Pi\left(\frac{n+k-1}{2}\right)}{\Pi\left(\frac{n-k}{2}\right)\Pi(n+k)}\,P_n^k(\cos\theta)$$

when $n-k$ is even;

$$\sqrt{(\cos\theta)}\,\Theta_{n-\frac{1}{2}}^{\frac{1}{2},k}(\cos 2\theta) = \frac{2^k\,\Pi(n-k)\,\Pi\left(\frac{n+k}{2}\right)}{\Pi(n+k)\,\Pi\left(\frac{n-k-1}{2}\right)}\,P_n^k(\cos\theta)$$

when $n-k$ is odd.

### 3. *Properties of the Function.*

If we put $\rho = r\cos\theta$, $\varpi = r\sin\theta$, our partial differential equation becomes

$$\frac{\partial^2 V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial V}{\partial\varpi}+\frac{1}{\varpi^2}\frac{\partial^2 V}{\partial\chi^2}=0,$$

from which it is easily seen that $J_m(\rho)J_k(i\varpi)e^{im\phi+ik\chi}$ is a solution; also, if this be expanded in powers of $r$, the coefficients will be of a polynomial character and must therefore be constant multiples of $\Theta_n^{m,k}(\mu)$. This is the case whether $m$ and $k$ are integers or not. Accordingly we may write

$$J_m(r\cos\theta)\,J_k(ir\sin\theta) = \sum_{s=0}^{\infty} A_n r^n \Theta_n^{m,k}(\cos\theta) \quad (n = m+k+2s).$$

To determine the coefficients we make $\theta$ very small and replace $\Theta_n^{m,k}(\cos 2\theta)$ and $J_k(ir\sin\theta)$ by $\dfrac{\Pi\left(\frac{n+m+k}{2}\right)}{\Pi\left(\frac{n-m-k}{2}\right)}\dfrac{\sin^k\theta}{\Pi(k)}$ and $\dfrac{i^k\sin^k\theta}{\Pi(k)}(\tfrac{1}{2}r)^k$ respectively.

Equating coefficients of $r^n$, we find that

$$A_n = (\tfrac{1}{2})^n i^{n-m}\frac{1}{\Pi\left(\frac{n+m+k}{2}\right)\Pi\left(\frac{n+m-k}{2}\right)},$$

giving the expansion

$$(8)\quad J_m(r\cos\theta)\,J_k(ir\sin\theta) = \sum_{s=0}^{\infty}(\tfrac{1}{2}r)^n i^{n-m}\frac{\Theta_n^{m,k}(\cos 2\theta)}{\Pi\left(\frac{n+m+k}{2}\right)\Pi\left(\frac{n+m-k}{2}\right)}$$

$$(n = m+k+2s).$$

If in this result we write $ir$ in place of $r$, $\frac{1}{2}\pi-\theta$ in place of $\theta$, we find that

$$J_m(ir\sin\theta)\,J_k(-r\cos\theta) = \sum_{0}^{\infty}(\tfrac{1}{2}r)^n i^{2n-m}\frac{\Theta_n^{m,k}(-\cos 2\theta)}{\Pi\left(\frac{n+m+k}{2}\right)\Pi\left(\frac{n+m-k}{2}\right)}.$$

Now $J_k(-r\cos\theta) = (-1)^k J_k(r\cos\theta)$; therefore

$$(9)\qquad \Theta_n^{m,k}(-\cos 2\theta) = (-1)^{\frac{1}{2}(n-m-k)}\frac{\Pi\left(\frac{n+m-k}{2}\right)}{\Pi\left(\frac{n+k-m}{2}\right)}\Theta_n^{k,m}(\cos 2\theta),$$

$n-m-k$ being an even integer.

Another expansion can be obtained from the equation

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}+V = 0.$$

Making the same substitution as before, we arrive at the solution

$$\frac{J_{n+1}(r)}{r}\,\Theta_n^{m,k}(\cos 2\theta)e^{im\phi+ik\chi}.$$

Again, it can be easily verified that the equation is satisfied by

$$J_m(r\cos\theta\cos\omega)\,J_k(r\sin\theta\sin\omega)e^{im\phi+ik\chi};$$

accordingly we may assume an expansion of the form

$$J_m(r\cos\theta\cos\omega)\,J_k(r\sin\theta\sin\theta) = \sum_{s=0}^{\infty} r^{-1}J_{n+1}(r)\,A_n\Theta_n^{m,k}(\cos 2\theta)\,;$$

but, since the expression on the left is symmetrical in $\theta$ and $\omega$, it is evident that $A_n$ must be of the form $B_n\Theta_n^{m,k}(\cos 2\omega)$. To determine the coefficients we put $\omega = \frac{1}{2}\pi - a$; the expansion then becomes

$$J_m(r\cos\theta\sin a)\,J_k(r\sin\theta\cos a)$$
$$= \sum_{s=0}^{\infty} B_n(-1)^{\frac{1}{2}(n-m-k)}\frac{J_{n+1}(r)}{r}\,\frac{\Pi\left(\frac{n+m-k}{2}\right)}{\Pi\left(\frac{n+k-m}{2}\right)}\,\Theta_n^{m,k}(\cos 2\theta)\,\Theta_n^{k,m}(\cos 2a).$$

Now make $a$ and $\theta$ very small: we find at once that

$$(\tfrac{1}{2}r)^{m+k} = \sum_{0}^{\infty}(-1)^{\frac{1}{2}(n-m-k)}B_n\frac{J_{n+1}(r)}{r}\,\frac{\Pi\left(\frac{n+m-k}{2}\right)}{\Pi\left(\frac{n+k-m}{2}\right)}\left\{\frac{\Pi\left(\frac{n+m+k}{2}\right)}{\Pi\left(\frac{n-m-k}{2}\right)}\right\}^2$$
$$(n = m+k+2s).$$

Comparing this with the well known expansion

$$(\tfrac{1}{2}r)^{m+k+1} = \sum_{s=0}^{\infty}(n+1)\,\frac{\Pi\left(\frac{n+m+k}{2}\right)}{\Pi\left(\frac{n-m-k}{2}\right)}\,J_{n+1}(r) \qquad (n = m+k+2s),$$

we see that

$$B_n = 2(-1)^{\frac{1}{2}(n-m-k)}(n+1)\,\frac{\Pi\left(\frac{n+k-m}{2}\right)\Pi\left(\frac{n-k-m}{2}\right)}{\Pi\left(\frac{n+m-k}{2}\right)\Pi\left(\frac{n+k+m}{2}\right)};$$

therefore

(10) $J_m(r\cos\theta\cos\omega)\,J_k(r\sin\theta\sin\omega)$

$$= \sum_{s=0}^{\infty}(-1)^{\frac{1}{2}(n-m-k)}(2n+2)\,r^{-1}J^{n+1}(r)\,\frac{\Pi\left(\frac{n+k-m}{2}\right)\Pi\left(\frac{n-k-m}{2}\right)}{\Pi\left(\frac{n+m-k}{2}\right)\Pi\left(\frac{n+m+k}{2}\right)}$$
$$\times\,\Theta_n^{m,k}(\cos 2\theta)\,\Theta_n^{m,k}(\cos 2\omega) \qquad (n = m+k+2s).$$

The other form of the expansion is

(11) $J_m(r\cos\theta\sin a)\,J_k(r\sin\theta\cos a)$

$$= 2r^{-1}\sum_{0}^{\infty}(n+1)\,\frac{\Pi\left(\frac{n-k-m}{2}\right)}{\Pi\left(\frac{n+k+m}{2}\right)}\,J_{n+1}(r)\,\Theta_n^{m,k}(\cos 2\theta)\,\Theta_n^{k,m}(\cos 2a).$$

If in this we make $\theta$ very small, we find that

$$(12)\qquad (\tfrac{1}{2}r)^{k+1}\cos k\, a\,.\,J_m(r\sin a) = \sum_{s=0}^{\infty}(n+1)\,J_{n+1}(r)\,\Theta_n^{k,\,m}(\cos 2a),$$

or, again, putting $\theta = 0$, $\omega = 0$ in (10), we get

$$(13)\quad \Pi(k)\,(\tfrac{1}{2}r)^{k+1}J_m(r)$$

$$= \sum_{s=0}^{\infty}(m+k+2s+1)(-1)^s\,\frac{\Pi(m+k+s)\,\Pi(k+s)}{\Pi(s)\,\Pi(m+s)}\,J^{m+k+2s+1}(r).$$

When $m = k$ the expansions take the form

$$(14)\quad J_m(r\cos\theta\cos\omega)\,J_m(r\sin\theta\sin\omega)$$

$$= \sum_{n=m}^{\infty}(-1)^{n-m}(2n+1)\,2r^{-1}J_{2n+1}(r)\,\frac{\Pi(n-m)}{\Pi(n+m)}\,P_n^m(\cos 2\theta)\,P_n^m(\cos 2\omega),$$

$$(15)\qquad (\tfrac{1}{2}r)^{m+1}(\cos^m a\,.\,J_m(r\sin a) = \sum_{n=m}^{\infty}(2n+1)\,J_{2n+1}(r)\,P_n^m(\cos 2a).$$

### 4. *Relations between Functions of Different Magnitudes.*

From equation (4) we see that $\Theta_n^{m,\,k}(\mu) = 0$ when $n-m-k$ is an even negative integer. When $k$ is a negative integer the first $k$ terms of the expansion disappear and the remainder can easily be identified with $(-1)^k\Theta_n^{m,\,-k}(\mu)$. Accordingly, we have the relation

$$(16)\qquad \Theta_n^{m,\,-k}(\mu) = (-1)^k\,\Theta_n^{m,\,k}(\mu)\quad (k \text{ being an integer}).$$

Another relation may be obtained from the equation

$$(17)\qquad (\tfrac{1}{2}r)^{m+1}\cos^m a\,.\,J_k(r\sin a) = \Sigma(n+1)\,J_{n+1}(r)\,\Theta_n^{m,\,k}(\cos 2a)$$

by making use of the addition formula

$$J_{k-1}(z)+J_{k+1}(z) = 2kz^{-1}J_k(z)\,;$$

therefore

$$(\tfrac{1}{2}r)^{m+1}\cos^m a\,[J_{k-1}(r\sin a)+J_{k+1}(r\sin a)]$$

$$= \Sigma(n+1)\,J_{n+1}(r)\,[\Theta^{m,\,k-1}+\Theta^{m,\,k+1}]\,;$$

therefore

$$(\tfrac{1}{2}r)^m\, k\, \frac{\cos^m a}{\sin a}\, J_k(r \sin a) = \Sigma\,(n+1)\, J_{n+1}(r)\, [\Theta_n^{m,\,k-1}+\Theta_n^{m,\,k+1}].$$

Comparing this with equation (17), we have

$$(18) \qquad \Theta_n^{m,\,k-1}(\cos 2a)+\Theta_n^{m,\,k+1}(\cos 2a) = k \cot a\, \Theta_n^{m-1,\,k}(\cos 2a).$$

Again, substituting

$$(n+1)\, J_{n+1}(r) = \tfrac{1}{2}r\,[J_n(r)+J_{n+2}(r)],$$

we find that

$$(19) \qquad \Theta_n^{m,\,k}(\cos 2a)+\Theta_{n-2}^{m,\,k}(\cos 2a) = n \cos a\, \Theta_{n-1}^{m-1,\,k}(\cos 2a).$$

Changing $a$ into $\frac{1}{2}\pi-\theta$, and making use of equation (9), we get

$$(20) \qquad k \tan\theta\, \Theta_n^{k,\,m-1}+\Theta_n^{k+1,\,m} = \tfrac{1}{4}(n+k-m+1)(n-k+m+1)\,\Theta_n^{k-1,\,m},$$

(21)

$$(n+m-k)\Theta_n^{k,\,m}(\cos 2\theta)+(n+k-m)\,\Theta_{n-2}^{k,\,m}(\cos 2\theta) = n \sin\theta\, \Theta_{n-1}^{k,\,m-1}(\cos 2\theta).$$

Interchanging $a$ and $\theta$, $k$ and $m$, so as to keep the formulæ uniform, we have the equations

$$(20)' \quad m \tan a\, \Theta_n^{m,\,k-1}(\cos 2a)+\Theta_n^{m+1,\,k}(\cos 2a)$$
$$= \tfrac{1}{4}(n+m-k+1)(n-m+k+1)\Theta_n^{m-1,\,k}(\cos 2a),$$

$$(21)' \quad (n+k-m)\, \Theta_n^{m,\,k}(\cos 2a)-(n+m-k)\, \Theta_{n-2}^{m,\,k}(\cos 2a)$$
$$= 2n \sin a\, \Theta_{n-1}^{m,\,k-1}(\cos 2a).$$

Multiplying equation (19) by $k$ and subtracting (21)′, we have

$$(22) \quad (n+m+k)\, \Theta_{n-2}^{m,\,k}(\cos 2a)-(n-m-k)\Theta_n^{m,\,k}(\cos 2a)$$
$$= 2n \sin a\, \Theta_{n-1}^{m,\,k+1}(\cos 2a).$$

The last two equations give us the addition formula

$$(23) \quad (n-m-k)(n+k-m)(n-2)\Theta_n^{m,\,k}(\mu)$$
$$+n(n+m+k-2)(n+m-k-2)\Theta_{n-4}^{m,\,k}(\mu)$$
$$= 2(n-1)\,[k^2-m^2+n(n-2)\mu]\,\Theta_{n-2}^{m,\,k}(\mu)\,;$$

when $m = k$ this reduces to the well known addition formula for the associated Legendre function.

Other formulæ may be obtained by differentiating, for example,

(24)

$$\sqrt{(1-\mu^2)}\frac{d}{d\mu}[\Theta_n^{m,k}(\mu)] = \tfrac{1}{4}(n+m-k+1)(n-m+k+1)\Theta_n^{m-1,k+1}-\Theta_n^{m+1,k-1},$$

$$(\mu^2-1)\frac{d\Theta_n}{d\mu} = \frac{1}{2n}\{n^2\mu+k^2-m^2\}\Theta_n-\frac{(n+m+k)(n+m-k)}{2n}\Theta_{n-2}$$

$$= \frac{(n-m-k+2)(n+k-m+2)}{2(n+2)}\Theta_{n+2}$$

$$-\frac{1}{2(n+2)}\{(n+2)^2\mu+k^2-m^2\}\Theta_n.$$

I have only proved these formulæ for the case when $\Theta_n$ is a polynomial, but it seems probable that they will be true for unrestricted values of $n$, $m$, and $k$.

## 5. *The Integral Formulæ.*

Let $\Theta_n^{m,k}(\mu)$ and $\Theta_p^{m,k}(\mu)$ be two polynomials of different orders. $n-p$ being an even integer, we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta_n}{d\mu}\right\}+\Theta_n\left\{\frac{n}{2}\left(\frac{n}{2}+1\right)-\frac{m^2}{2(1+\mu)}-\frac{k^2}{2(1-\mu)}\right\} = 0,$$

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta_p}{d\mu}\right\}+\Theta_p\left\{\frac{p}{2}\left(\frac{p}{2}+1\right)-\frac{m^2}{2(1+\mu)}-\frac{k^2}{2(1-\mu)}\right\} = 0.$$

Multiplying the first equation by $\Theta_p$, the second by $\Theta_q$, and subtracting, we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\left(\Theta_p\frac{d\Theta_n}{d\mu}-\Theta_n\frac{d\Theta_p}{d\mu}\right)\right\} = \Theta_n\Theta_p\left[\frac{n}{2}\left(\frac{n}{2}+1\right)-\frac{p}{2}\left(\frac{p}{2}+1\right)\right];$$

so that, if $n \neq p$, and $m$ and $k$ are both $> 0$, we have

$$\int_{-1}^{+1}\Theta_n(\mu)\Theta_p(\mu)\,d\mu = 0.$$

We must now evaluate this integral for $n = p$. Putting $\theta = \omega$ in the expansion (10), we have

$$J_m\left[\frac{r}{2}(1+\mu)\right]J_k\left[\frac{r}{2}(1-\mu)\right]$$

$$= \frac{2}{r}\sum_{s=0}^{\infty}(-1)^{\frac{1}{2}(n-m-k)}(n+1)J_{n+1}(r)\frac{\Pi\left(\frac{n+k-m}{2}\right)\Pi\left(\frac{n-k-m}{2}\right)}{\Pi\left(\frac{n+m-k}{2}\right)\Pi\left(\frac{n+m+k}{2}\right)}[\Theta_n^{m,k}(\mu)]^2.$$

Let $$F_{m,k} = \tfrac{1}{2}\int_{-1}^{+1} J_m\left[\frac{r}{2}(1+\mu)\right] J_k\left[\frac{r}{2}(1-\mu)\right] d\mu.$$

Writing $1-\mu = 2x$, it becomes

$$F_{m,k} = \int_0^1 J_m[r(1-x)]\, J_k(rx)\, dx.$$

Now $$\frac{dJ_k(z)}{dz} = \tfrac{1}{2}\{J_{k-1}(z) - J_{k+1}(z)\};$$

so that

$$\begin{aligned}\tfrac{1}{2}\{F_{m,k-1} - F_{m,k+1}\} &= \int_0^1 J_m[r(1-x)]\, J_k'(rx)\, dx \\ &= \int_0^1 J_m'[r(1-x)]\, J_k(rx)\, dx + \left[J_m[r(1-x)]\, J_k(rx)\,\frac{1}{r}\right]_0^1 \\ &= \tfrac{1}{2}\{F_{m-1,k} - F_{m+1,k}\};\end{aligned}$$

therefore $$\begin{aligned}F_{m,k-1} - F_{m-1,k} &= -[F_{m+1,k} - F_{m,k+1}] \\ &= [F_{m+2,k+1} - F_{m+1,k+2}],\end{aligned}$$

and so on; but $F_{m,k} = 0$ when $m$ and $k$ are very large. Accordingly we have

$$F_{m,k-1} = F_{m-1,k}.$$

This shows that the integral $F_{m,k}$ depends only on $m+k$. When $m$ and $k$ are integers and $m = k$ the integral can at once be evaluated, for then

$$\begin{aligned}&J_m\left[\frac{r}{2}(1+\mu)\right] J_m\left[\frac{r}{2}(1-\mu)\right] \\ &\qquad = \frac{2}{r}\sum_{n=m}^{\infty} (-1)^{n-m}(2n+1)\, J_{2n+1}(r)\,\frac{(n-m)!}{(n+m)!}\,[P_n^m(\mu)]^2,\end{aligned}$$

and so

$$\tfrac{1}{2}\int_{-1}^{+1} J_m\left[\frac{r}{2}(1+\mu)\right] J_m\left[\frac{r}{2}(1-\mu)\right] d\mu = \frac{2}{r}\sum_{n=m}^{\infty} (-1)^{n-m} J_{2n+1}(r),$$

*i.e.*, $$\int_0^1 J_m[r(1-x)]\, J_m[rx]\, dx = \frac{2}{r}\sum_{n=m}^{\infty} (-1)^{n-m} J_{2n+1}(r),$$

and $$\begin{aligned}\int_0^1 J_m[r(1-x)]\, J_{m+2}[rx]\, dx &= \int_0^1 J_{m+1}[r(1-x)]\, J_{m+1}[rx]\, dx \\ &= \frac{2}{r}\sum_{n=m+1}^{\infty} (-1)^{n-m-1} J_{2n+1}(r),\end{aligned}$$

also $$J_m[rx] + J_{m+2}[rx] = \frac{2(m+1)}{rx} J_{m+1}(rx);$$

therefore $$\int_0^1 J_m[r(1-x)]\,J_{m+1}[rx]\,\frac{dx}{x}=\frac{1}{m+1}J_{2m+1}(r),$$

$m$ being an integer.

We shall now prove that

$$G_{m,k}=\int_0^1 J_m[r(1-x)]\,J_k[rx]\,\frac{dx}{x}=\frac{1}{k}J_{k+m}(r)$$

for all values of $m$ and $k$ greater than zero.

For, if the integral on the left-hand side be expanded in powers of $r$, the coefficient of $(\frac{1}{2}r)^{m+k+2s}$ is

$$\sum_{\rho+\sigma=s}(-1)^{\rho+\sigma}\frac{1}{\Pi(m+\rho)\,\rho!}\,\frac{1}{\Pi(k+\sigma)\,\sigma!}\int_0^1(1-x)^{m+2\rho}x^{k+2\sigma-1}$$

$$=\sum_{\rho+\sigma=s}(-1)^{\rho+\sigma}\frac{\Pi(m+2\rho)\,\Pi(k+2\sigma-1)}{\Pi(m+\rho)\,\rho!\,\Pi(k+\sigma)\,\sigma!\,\Pi(m+k+2\rho+2\sigma)}.$$

We have got to show therefore that

$$\text{(A)}\quad\sum_{\rho+\sigma=s}\frac{\Pi(m+2\rho)\,\Pi(k+2\sigma-1)}{\Pi(m+\rho)\,\rho!\,\Pi(k+\sigma)\sigma!\,\Pi(m+k+2\rho+2\sigma)}=\frac{1}{k\,s!\,\Pi(m+k+s)}.$$

Now $G_{m,k}=\frac{1}{2}rk^{-1}[F_{m,k-1}+F_{m,k+1}]$ on account of the relation

$$J_{k-1}(rx)+J_{k+1}(rx)=(2k/rx)J_k(rx);$$

accordingly $2kG_{m,k}$ depends on $m+k$ only. If, then, the equation (A) be cleared of fractions, it may be considered as an equation of the $s$-th degree in $(m+k)$; but we know that the equation is satisfied whenever $m=k=$ an integer; it is therefore satisfied for more than $s$ values of $m+k$, and so must be an identity. Thus, if $m$ and $k$ are any positive quantities, we have the identity

$$\sum_{\rho+\sigma=s}\frac{\Pi(m+2\rho)\,\Pi(k+2\sigma-1)}{\Pi(m+\rho)\,\rho!\,\Pi(k+\sigma)\,\sigma!}=\frac{\Pi(m+k+2s)}{ks!\,\Pi(m+k+s)};$$

hence the equation

$$\int_0^1 J_m[r(1-x)]\,J_k[rx]\,\frac{dx}{x}=\frac{1}{k}J_{m+k}(r)$$

holds for all values of $m$ and $k$ greater than zero. This gives us the further equation

$$F_{m,k}=2r^{-1}[(k+1)G_{m,k+1}-(k+3)G_{m,k+3}+\ldots]$$

$$=2r^{-1}[J_{m+k+1}(r)-J_{m+k+3}(r)+\ldots].$$

We can now evaluate the integral $\int_{-1}^{+1}[\Theta_n(\mu)]^2\,d\mu$ for

$$2r^{-1}\sum_{n=m+k}^{\infty}(-1)^{\frac{1}{2}(n-m-k)}(n+1)\,J_{n+1}(r)\,\frac{\Pi\left(\frac{n+k-m}{2}\right)\Pi\left(\frac{n-k-m}{2}\right)}{\Pi\left(\frac{n+m-k}{2}\right)\Pi\left(\frac{n+m+k}{2}\right)}$$

$$\times\int_{-1}^{+1}[\Theta_n^{m,\,k}(\mu)]^2\,d\mu$$

$$=\int_{-1}^{+1}J_m\left[\frac{r}{2}(1+\mu)\right]J_k\left[\frac{r}{2}(1-\mu)\right]d\mu = 2F_{m,\,k}$$

$$=4r^{-1}\sum_{n=m+k}^{\infty}(-1)^{\frac{1}{2}(n-m-k)}J_{n+1}(r).$$

Comparing the coefficients of $J_{n+1}(r)$, we have the formula

$$\int_{-1}^{+1}[\Theta_n^{m,\,k}(\mu)]^2\,d\mu = \frac{2}{n+1}\,\frac{\Pi\left(\frac{n+m-k}{2}\right)\Pi\left(\frac{n+m+k}{2}\right)}{\Pi\left(\frac{n+k-m}{2}\right)\Pi\left(\frac{n-k-m}{2}\right)},$$

$m$ and $k$ being any quantities greater than 0, and such that $n-m-k$ is an even positive integer.

If we assume a convergent expansion of the form

$$f(\mu)=\sum_{s=0}^{\infty}A_n\Theta_n^{m,\,k}(\mu)\quad(n=m+k+2s),$$

the coefficients will be determined by the formula

$$\int_{-1}^{+1}f(\mu)\,\Theta_n^{m,\,k}(\mu)\,d\mu = \frac{2A_n}{n+1}\,\frac{\Pi\left(\frac{n+m-k}{2}\right)\Pi\left(\frac{n+m+k}{2}\right)}{\Pi\left(\frac{n+k-m}{2}\right)\Pi\left(\frac{n-k-m}{2}\right)}.$$

## 6. *Various Formulæ.*

If we equate coefficients of $(\frac{1}{2}r)^n$ in the expansion

$$J_m(r\cos\theta)\,J_k(ir\sin\theta)=\sum_{s=0}^{\infty}(\tfrac{1}{2}r)^n\,i^{n-m}\,\frac{\Theta_n^{m,\,k}(\cos 2\theta)}{\Pi\left(\frac{n+m+k}{2}\right)\Pi\left(\frac{n+m-k}{2}\right)},$$

we find that

$$\Theta_n^{m,k}(\cos 2\theta) = \Pi\left(\frac{n+m+k}{2}\right)\Pi\left(\frac{n+m-k}{2}\right)\sum_{\rho=0}^{\frac{1}{2}(n-m-k)}(-1)^{\frac{1}{2}(n-m-k-2\rho)}$$

$$\times\frac{\cos^{m+2\rho}\theta\,\sin^{n-m-2\rho}\theta}{\Pi(m+\rho)\,\Pi(\rho)\,\Pi\left(\frac{n-m-k-2\rho}{2}\right)\Pi\left(\frac{n-m+k-2\rho}{2}\right)}.$$

This shows that when $n-m-k$ is an even integer $\Theta_n^{m,k}(\mu)$ can be written in the form

$$\Theta_n^{m,k}(\mu) = \frac{\Pi\left(\frac{n+m+k}{2}\right)(-1)^{\frac{1}{2}(n-m-k)}}{2^n\,\Pi\left(\frac{n-m-k}{2}\right)\Pi\left(\frac{n-m+k}{2}\right)}(1-\mu)^{-\frac{1}{2}k}(1+\mu)^{-\frac{1}{2}m}$$

$$\times\frac{d^{\frac{1}{2}(n-m-k)}}{d\mu^{\frac{1}{2}(n-m-k)}}\left[(1-\mu)^{\frac{1}{2}(n-m+k)}(1+\mu)^{\frac{1}{2}(n+m-k)}\right].$$

From this we may deduce the definite integral

$$\Theta_n^{m,k}(\mu) = \frac{\Pi\left(\frac{n+m+k}{2}\right)}{2^n\,\Pi\left(\frac{n-m+k}{2}\right)}\;\frac{(-1)^{\frac{1}{2}(n-m-k)}}{2\pi i}$$

$$\times\int_C(1-\mu)^{-\frac{1}{2}k}(1+\mu)^{-\frac{1}{2}m}(1-t)^{\frac{1}{2}(n-m+k)}(1+t)^{\frac{1}{2}(n+m-k)}(t-\mu)^{-\frac{1}{2}(n-m-k+2)}\,dt,$$

taken round a simple contour enclosing the point $t=\mu$.

Another definite integral may be deduced from the formula*

$$V=\int_0^{2\pi}f(x_1\cos t+x_2\sin t+ix_3,\; x_1\sin t-x_2\cos t+ix_4,\; t)\,dt$$

for a solution of $\dfrac{\partial^2V}{\partial x_1^2}+\dfrac{\partial^2V}{\partial x_2^2}+\dfrac{\partial^2V}{\partial x_3^2}+\dfrac{\partial^2V}{\partial x_4^2}=0;$

the result is

$$i^n\Theta_n^{m,k}(\cos 2\theta) = \frac{\Pi(m)\,\Pi\left(\frac{n-m+k}{2}\right)}{2\pi\,\Pi(k)\,\Pi\left(\frac{n-m-k}{2}\right)}$$

$$\times\int_0^{2\pi}(\cos\theta\, e^{it}+i\sin\theta)^{\frac{1}{2}(n-m-k)}(\cos\theta\, e^{-it}+i\sin\theta)^{\frac{1}{2}(n+m+k)}e^{imt}\,dt.$$

If we put $r\cos\theta=\rho\cos\alpha\cos\omega$, $ir\sin\theta=\rho\sin\alpha\sin\omega$, so that

$$r^2=\tfrac{1}{2}\rho^2(\cos 2\alpha+\cos 2\omega)=\tfrac{1}{2}\rho^2(\mu+\mu'),$$

$$r^2\cos 2\theta=\tfrac{1}{2}\rho^2(1+\cos 2\alpha\cos 2\omega)=\tfrac{1}{2}\rho^2(1+\mu\mu'),$$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 457.

we have

$$\sum_{s=0}^{\infty}\left(\frac{r}{2}\right)^{n} i^{n-m}\frac{\Theta_n^{m,\,k}\left(\dfrac{1+\mu\mu'}{\mu+\mu'}\right)}{\Pi\left(\dfrac{n+m+k}{2}\right)\Pi\left(\dfrac{n+m-k}{2}\right)}$$

$$= J_m(r\cos\theta)\,J_k(ir\sin\theta) = J_m(\rho\cos\alpha\cos\omega)\,J_k(\rho\sin\alpha\sin\omega)$$

$$= \sum_{s=0}^{\infty}(-1)^s\frac{2}{\rho}(m+k+2s+1)\,J_{m+k+2s+1}(\rho)\frac{\Pi\left(\dfrac{n+k-m}{2}\right)\Pi\left(\dfrac{n-k-m}{2}\right)}{\Pi\left(\dfrac{n+m-k}{2}\right)\Pi\left(\dfrac{n+m+k}{2}\right)}$$

$$\times\,\Theta_n^{m,\,k}(\mu)\,\Theta_n^{m,\,k}(\mu') \quad (n=m+k+2s).$$

Comparing the coefficients of $(\tfrac{1}{2}\rho)^{m+k+2p}$ in the two expansions, we find that

$$i^k\left(\frac{\mu+\mu'}{2}\right)^{\frac{1}{2}(m+k+2p)}\Theta_{m+k+2p}^{m,\,k}\left(\frac{1+\mu\mu'}{\mu+\mu'}\right)$$

$$= \sum_{s=0}^{p}(m+k+2s+1)\frac{\Pi(k+s)\,\Pi(s)\,\Pi(m+p)\,\Pi(m+k+p)}{\Pi(m+s)\,\Pi(m+k+s)\,\Pi(p-s)\,\Pi(m+k+s+p+1)}$$

$$\times\,\Theta_n(\mu)\,\Theta_n(\mu') \quad (n=m+k+2s).$$

When $m=k=0$, $2p=2n$, $2s=2r$, this formula becomes

$$\left(\frac{\mu+\mu'}{2}\right)^{n}P_n\left(\frac{1+\mu\mu'}{\mu+\mu'}\right) = \sum_{r=0}^{n}(2r+1)\frac{\Pi(n)\,\Pi(n)}{\Pi(n-r)\,\Pi(n+r+1)}P_r(\mu)\,P_r(\mu').$$

# THE KINEMATICS AND DYNAMICS OF A GRANULAR MEDIUM IN NORMAL PILING

*By* J. H. JEANS.

[Received and Read January 12th, 1905.]

## *Introduction.*

1. The conception of a medium which consists of a number of small hard spherical grains in "normal piling," *i.e.*, holding the same geometrical relations to one another as spherical shot piled in the usual way, has been brought into prominence by the recent memoir of Prof. Osborne Reynolds.* According to Prof. Reynolds a system of this kind forms the "one, and only one, purely mechanical system capable of accounting for all the physical evidence, as we know it, in the Universe." It is therefore of interest to examine any exact evidence which can be obtained as to the ability of this ideal mechanical system to account for natural phenomena. And whether or not the mechanical system is ultimately found to be able to substantiate the high claims put forward on its behalf by Prof. Reynolds—and the results of the present investigation give no countenance at all to this hope—it is a system which possesses great interest in itself, being sufficiently closely related to our conceptions of an elastic solid or a dense gas, to be worthy of examination for its own sake. It is, indeed, one of the very few instances of a medium, whose ultimate mechanical specification is fully given, of which the properties can be worked out in full.

The analytical methods to be employed in the present investigation consist largely of combinations and modifications of those employed in Elastic Solid Theory and the Dynamical Theory of Gases. Since, however, the physical conditions in the present problem are very different from those either of an elastic solid or of a gas, I have on the whole been reluctant to quote known theorems from these subjects, and have tried to develop the dynamics of the present system almost from first principles.

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* *The Sub-Mechanics of the Universe*, Cambridge University Press (for the Royal Society), 1903.

This seemed to be the only way by which it was possible to feel sure that the physical assumptions or hypotheses appropriate to the discussion of an elastic solid or a gas were not introduced in disguise into the present discussion. The medium we are now examining being fully specified at the outset, it is unnecessary, and would therefore be illegitimate, to help the development of the problem by introducing extraneous facts from dynamics or physics. The same remark applies to the introduction of the "assumptions," true or untrue, of the so-called statistical method.

The geometrical meaning of normal piling has been fully explained in the last papers of Prof. J. D. Everett.* It will not be necessary, then, to discuss the geometry of normal piling here.

In the problem before us, we are given a space of unlimited, or practically unlimited, extent, and in this space a number of equal and similar spherical grains. Given the diameter $\sigma$ of each grain, there is a superior limit to the number of grains which can be put into any space, subject to the condition that no two grains may overlap. This limit is reached when the number of grains is so great that the average space for each grain is $\sigma^3/\sqrt{2}$. When this limit is reached, the grains are in closest piling. This piling is characterised by the feature that every sphere touches twelve others, the centres of any four which mutually touch one another forming a regular tetrahedron. The piling may, however, be either the normal piling of Prof. Reynolds, or the antinormal piling described by Prof. Everett, or a combination of the two. In Fig. 1 we have a tier of spheres arranged in closest order. In stacking a tier above this, we can either place the centres of spheres over the points indicated

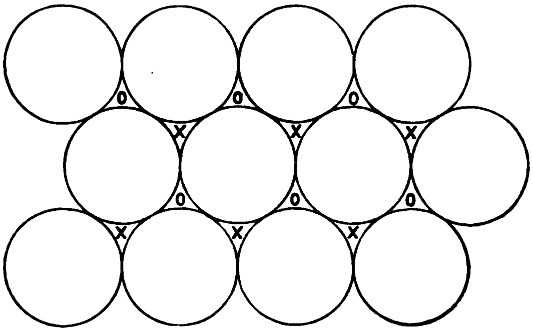

FIG. 1.

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* "On a Calculus of Point Assemblages," *Proc. Lond. Math. Soc.* [2], Vol. 1, p. 437: "On Normal Piling, as connected with Osborne Reynolds' Theory of the Universe," *Proc. Phys. Soc.*, Vol. XIX., p. 223; *Phil. Mag.* [6], Vol. 8, p. 30.

by $X$'s or over the points indicated by $O$'s. The choice is a matter of indifference; for one arrangement changes into the other on turning through 180° about the centre of any sphere of the original tier. Let us then decide on putting spheres above the $X$'s. Let us next extend the piling by putting a layer *below* the original layer. A similar choice is again before us, but it is now no longer a matter of indifference which course we take. If we decide to place the sphere centres below the $O$'s, the piling is *normal;* if we place them below the $X$'s, the piling is *antinormal.* Thus, after two tiers are in position there are two courses open as regards the arrangement of the third tier. In piling a great number of tiers the course adopted might be indicated by a sequence of letters, such as

$$O,\ O,\ A,\ N,\ N,\ A,\ N,\ \ldots,$$

the meaning being that, after tiers 1 and 2 are in position, 3 is piled so that 1, 2, 3 are in antinormal piling; 4 so that 2, 3, 4 are in normal piling; 5 so that 3, 4, 5 are in normal piling; and so on. The order

$$O,\ O,\ N,\ N,\ N,\ \ldots$$

is the normal piling of Osborne Reynolds.

It may be that the order of piling is determined by boundary conditions. For instance, in piling shot on a triangular base it will be found that the arrangement adopted, if the shot are to form a regular tetrahedral figure, must be wholly normal ($O, O, N, N, N, \ldots$). In packing into a square box, on the other hand, it must be wholly antinormal ($O, O, A, A, A, \ldots$). And in packing into a vessel of irregular shape the arrangement which is to take least space will in general be neither wholly normal nor wholly antinormal; the sequence of arrangements will be determined by the shape of the vessel.

We see, then, that, if space is infinite, there will be an infinite number of separate and distinct arrangements of closest packing. If space were finite, the arrangement might be determined by the shape of the boundary, and, in particular, if space were non-Euclidean and finite, it is at least conceivable that it might be necessary for the packing to be either wholly normal or wholly antinormal.

The theory of Osborne Reynolds is concerned only with piling which is wholly normal, and the theory does not explain why antinormal piling is excluded.* In wholly normal piling, the centres of the grains form a regular three-dimensional network, of which every alternate "mesh" will be a regular tetrahedron of edge $\sigma$. The total number of grains is just less than the maximum number which could be packed into the space available; so that the geometry of the system is what may be described as "loose normal piling." There are small interstices between the grains, and at any instant the centres of four adjacent spheres form a tetrahedron which in general is not quite regular, the sides being each somewhat greater than $\sigma$ and not necessarily equal. We suppose the number of grains to be such that the mean space available for each, instead of being $\sigma^3/\sqrt{2}$, is $\tau^3/\sqrt{2}$, $\tau$ being slightly greater than $\sigma$. The grains are free to move, and in the problem before us there is no limitation on their motion except that no two spheres may overlap.

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* A suggestion of Prof. Everett on this point will be dealt with later.

The motion of the grains is governed by certain laws, and these are to be the same as if the grains were hard and perfectly elastic spheres, of the kind with which the theory of gases has made us familiar, If the model is to be a true model of the ultimate structure of the Universe, the laws ought, perhaps, to be stated kinematically rather than dynamically, but this does not affect the actual problem before us, which is simply that of determining the motion. The problem may be regarded as a dynamical problem, and the motion as that of elastic spheres impinging on one another.

### REPRESENTATION OF THE SYSTEM IN A GENERALISED SPACE.

2. We are not in any way concerned with the rotations or orientations of the grains, so that the configuration of the system may be supposed to be completely specified when for each grain we know the values of $x$, $y$, $z$, the co-ordinates in space of its centre, and of $u$, $v$, $w$, the components of velocity of its centre in directions parallel to the axes of $x$, $y$, $z$. Let us suppose there are altogether $N$ grains, distinguished by the letters $a$, $b$, $c$, ..., and let us denote the co-ordinates and velocity components of grain $a$ by $x_a$, $y_a$, $z_a$, $u_a$, $v_a$, $w_a$. The configuration of the system will then be specified when we know the values of $6N$ variables,

$$x_a,\ y_a,\ z_a,\ u_a,\ v_a,\ w_a,\quad x_b,\ y_b,\ z_b,\ u_b,\ v_b,\ w_b,\quad x_c,\ \ldots. \tag{1}$$

All configurations of this system can be represented in a generalised space of $6N$ dimensions, the axes of co-ordinates corresponding to the $6N$ variables of the system. Every possible configuration of the system can be represented by a point in this space, but there will also be points in the $6N$-dimensional space which do not represent possible configurations, for no account has yet been taken of the condition that two spheres may not overlap.

The equation $$(x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2 = \sigma^2 \tag{2}$$

represents a tubular locus of $6N-1$ dimensions in the generalised space, and this locus separates points which represent configurations in which the spheres $a$, $b$ overlap from those in which they do not overlap. All configurations represented by points inside this locus may, then, be excluded as impossible. We can conveniently refer to the locus given by equation (2) as the tube $ab$. There are similar tubes corresponding to all other pairs of grains, $ac$, $bc$, ..., $\frac{1}{2}N(N-1)$ in all. The necessary and sufficient condition that a configuration of the system shall be a possible one is that the corresponding point in the generalised space shall be outside *all* these tubes.

3. The regions of the space which are outside all the tubes—and with which alone we are concerned—may obviously be broken up into any number of disconnected regions by the tubes. The various regions, whether disconnected or not, will each be repeated $N!$ times in exact similarity, each region occurring once for each of the $N!$ ways in which it is possible to allot the *rôles* of the spheres $a$, $b$, $c$, ... to the different grains.

4. The tubes specified by equations (2) are the same for all values of $u_a, v_a, w_a, u_b, v_b, w_b, u_c, \ldots$. It will be convenient for the moment to disregard possible variations of these velocity co-ordinates; or, what comes to the same thing, to discuss only the intersection of the tubes with the locus $u_a = v_a = w_a = u_b = \ldots = 0$, this intersection being a space of $3N$ dimensions. This amounts virtually to considering only the system at rest, and representing the possible values of $x_a, y_a, z_a, x_b, y_b, z_b, x_c, \ldots$, in space of $3N$ dimensions.

5. When we take into account the boundary of the real space, this imaginary space is, of course, of finite extension in every direction. If $f(x, y, z) = 0$ is the equation of the boundary of the real space in which the grains are piled, then the imaginary $3N$-dimensional space is bounded by the system of loci

$$f(x_a, y_a, z_a) = 0, \qquad f(x_b, y_b, z_b) = 0, \qquad \ldots.$$

Since no part of a grain and the boundary may overlap, we must further exclude all parts of the space which correspond to configurations in which any one of the points $x_a, y_a, z_a, x_b, y_b, z_b, \ldots$ are within a normal distance $\frac{1}{2}\sigma$ of the boundary. Thus, if $\psi(x, y, z)$ is the shortest normal distance from the point $x, y, z$ to the boundary in the real space, those parts of the imaginary space are to be excluded for which

$$\psi(x_a, y_a, z_a) < \tfrac{1}{2}\sigma, \qquad \psi(x_b, y_b, z_b) < \tfrac{1}{2}\sigma, \qquad \ldots .$$

Thus the non-excluded space must be supposed bounded by the system of loci

$$\left.\begin{array}{c}(x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2 = \sigma^2 \ldots (ab,\ bc,\ \ldots) \\ \psi(x_a, y_a, z_a) = \tfrac{1}{2}\sigma \ldots (a,\ b,\ c,\ \ldots)\end{array}\right\} . \tag{3}$$

## *Systems in Closest Piling.*

6. A system of grains in closest piling of any kind, not necessarily normal or antinormal, will be represented by a single point in this space, such that it is impossible to move in any direction from this point without passing into parts of the space which are excluded by the loci (3). Hence at a point which represents closest piling at least $3N$ of the loci (3) must intersect, and, except in special cases, the number of intersections will not be greater than $3N$. The interpretation of a point being on one of the loci (3) is that there is contact between either two grains or a grain and the boundary. Hence in closest piling of any kind there will in general be $3N$ contacts. Each contact between a pair of grains affects two grains; each contact with the boundary affects only one. When the space is of very great extent, the number of boundary contacts may be disregarded in comparison with the contacts between pairs of grains. We may say, then, that the $N$ grains experience $6N$ contacts, or an average of six contacts per grain. In special cases the number may be greater than this. For instance, when the boundary is such as to admit of exact normal piling we have seen that the number is twelve.

7. In the actual problem before us the grains of diameter $\sigma$ are not in closest piling, but would be if $\sigma$ were increased to $\tau$. Thus, if we replace $\sigma$ by $\tau$ in equations (3), the corresponding loci determine isolated points. These, as explained in § 3, must be $N!$ in number.

Let us now suppose the parameter $\tau$, which measures the diameter of the grains, gradually to decrease. By the time that $\tau$ has decreased to $\sigma$, the actual partition of space appropriate to the problem before us is reached.

When the diminution of $\tau$, *i.e.*, the contraction of the tubes, begins, the first change which will occur will be that each of the $N!$ points spreads out into a space of finite extension. As the shrinking proceeds, two new features may occur. In the first place, connections may be formed between the spaces already in existence, and, in the second place, new spaces may be formed. A helpful analogy will be found in the retirement of a flood from a mountainous country. As the level of the water falls mountain passes may become uncovered, and so connect peaks previously isolated, and also new peaks may appear above the surface of the water. Indeed, the analogy may be turned into an exact statement of what happens, as follows :—

Let us represent the possible configurations in a space of $3N+1$ dimensions corresponding to the variables $x_a$, $y_a$, $z_a$, $x_b$, $y_b$, $z_b$, $x_c$, ..., $\theta$, where $\theta$ is the parameter which measures the diameters of the grains. The loci

$$(x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2=\theta^2, \;\ldots, \quad \psi(x_a, y_a, z_a)=\tfrac{1}{2}\theta, \;\ldots \qquad (4)$$

will, as before, determine which regions must be excluded from the space, and the section of the space by $\theta=\sigma$, whatever $\sigma$ may be, will represent the partition of the original space when the value of the parameter is $\sigma$. Or, in other words, this section will represent all configurations which are possible in the real space for $N$ grains of diameter $\sigma$. The surface which separates the space excluded by equations (4) from the non-excluded space is an exact model of a mountainous country, the axis of $\theta$ being vertical, and the horizontal consisting of $3N$ dimensions instead of two. The highest peaks are $N!$ in number, and are each of height $\tau$. By the time we reach $\theta=0$, the whole section by a horizontal plane has become connected—for when the diameter of the grains has shrunk to zero the non-excluded $3N$-dimensional space is obviously all connected. Thus

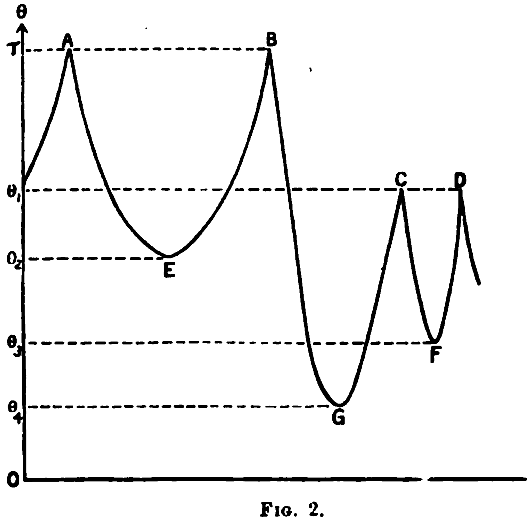

FIG. 2.

the lowest points on the mountainous surface are of altitude not less than $\theta=0$. Graphically, this can be represented as in Fig. 2. Here $A$, $B$

represent two highest peaks, of height $\theta = \tau$; $C$, $D$, two lower peaks, of height $\theta = \theta_1$, these representing new arrangements of closest piling; and $E$, $F$, $G$, the mountain passes, of heights $\theta_2$, $\theta_3$, $\theta_4$ respectively connecting the different peaks as shewn in the figure.

To represent the configurations possible for grains of diameter $\sigma$, we examine the section of the mountain range by the plane $\theta = \sigma$. The contour line, or series of contour lines, of height $\sigma$ represents the loci (3); the spaces enclosed by these contour lines represent configurations possible for the grains.

### *General Features of the Motion.*

8. We turn now to the kinematics of the system of grains. The $3N$-dimensional space enclosed by any contour $\theta = \sigma$ must be extended in the directions $u_a$, $v_a$, $w_a$, $u_b$, $v_b$, $w_b$, $u_c$, ... from $-\infty$ to $+\infty$ in each direction. In this way we obtain the original $6N$-dimensional space in which a point represents a possible arrangement of grains, both as regards position and velocity. To study the kinematics of the system of grains we fill this space with homogeneous fluid, which moves so that every possible motion of the grains is represented by the motion of a point in this fluid. Actually, the motion of the grains is determined by conditions which are exactly the same as would determine the motion if the spheres were hard rigid bodies obeying the dynamical law of energy and momentum, and exerting forces on one another only at contact.

The motion of the fluid in the imaginary space is therefore the same as I have described elsewhere[*] in connection with the Theory of Gases. It can be proved—

> (i.)[†] That the initial homogeneity of the fluid in the $6N$-dimensional space remains unaffected by the progress of the motion.
>
> (ii.)[‡] That, unless we are given the initial positions and velocities of every sphere, and calculate the motion by continued application of the dynamics of collisions (a possibility which is obviously excluded entirely from the present problem), then, if the non-excluded space is divided into equal infinitesimal cells, every cell has an equal chance of containing the point which represents the system at any instant, provided only that the motion has been in progress for a sufficiently great time. If the motion has originated from a fortuitous concourse of grains, the result is true without the necessity of making the last provision. This result, however, is not true if we have a knowledge of the values of any or all of the quantities which remain constant

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[*] *The Dynamical Theory of Gases*, Cambridge, 1904, p. 35, *et seq.*

[†] *L.c.*, § 37.

[‡] *L.c.*, § 58.

throughout the motion. An appeal of this kind to the theory of probability implies that a certain amount of knowledge is given, while some is withheld. What is given in this case is the specification of the dynamical system: what is withheld is a knowledge of the values of the co-ordinates of the system.

(iii.)* That, if the energy, but nothing else, is given, so that the system is represented through all time by points on a locus

$$u_a^2+v_a^2+w_a^2+u_b^2+\ldots = \text{constant},$$

then all equal small elements of this locus have an equal chance of representing the motion, from which it follows that the law of distribution of velocity components is Maxwell's law of trial and error

$$\sqrt{\frac{h^3m^3}{\pi^3}}\, e^{-hm(u^2+v^2+w^2)}\,du\,dv\,dw.$$

(iv.) That, as regards the *position* of the grains at any instant, if we divide up the $3N$-dimensional space of § 7 into infinitesimal equal cells, every cell has an equal chance of containing the point which represents the configuration at a given instant. This follows from the fact that the selection of velocity co-ordinates and the selection of positional co-ordinates in the $6N$-dimensional space are absolutely independent events. This proposition is not subject to any reservation of the kind which had to be applied to proposition (ii.), for clearly there can be no functions of the positional co-ordinates only, which remain constant throughout every possible motion.

It will be found that the proofs of these propositions rest on no assumption, except the assumption that the grains behave like hard elastic spheres, subject to no forces except when in contact. In particular, it is not necessary to make any use of the debateable assumption which Burbury has designated "assumption A."

9. When the $3N$-dimensional space consists of isolated portions, it is obvious that the representative point has no possibility of passing from one of the isolated regions to another. It will, therefore, be sufficient to consider the motion of the fluid inside *one* of these isolated regions. Excluding the infinitely improbable case of the representative point describing a periodic orbit in the generalised space—*i.e.*, excluding the discussion of a universe in which time would appear to be re-entrant—the representative point will wander throughout the whole of the available space, and the time spent in any portion will be exactly proportional to the volume $(\iiiint \ldots dx_a\,dy_a\,dz_a\,dx_b \ldots)$ of that portion.

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* *L.c.*, §§ 56, 57.

### THE PERMANENCE OR NON-PERMANENCE OF PECULIARITIES OF PILING.

10. We are now in a position to discuss the wandering of the representative point in any section of the surface which is represented symbolically in Fig. 2.

The different arrangements of closest packing are represented by the highest points of the different peaks, *i.e.*, points at which an increase of $\theta$ involves the violation of the condition of non-overlapping of grains.

Let us call these various peaks

$$\alpha, \alpha', \alpha'', \ldots, \beta, \beta', \beta'', \ldots.$$

It has been seen that the peaks occur in sets of $N!$. Each of a set of peaks may be indicated by the same letter, and the particular peak identified by the number of accents. The different *types* of closest packing are now represented by the peaks $\alpha, \beta, \ldots$. We may refer to these types of packing as $\alpha$-packing, $\beta$-packing, &c. These packings are not necessarily homogeneous. For instance, one peak might represent an arrangement in which part of the grains were in strained normal piling, and the remainder in strained cubical piling. And, as has already been said, even when the piling is homogeneous, it will not in general be unstrained; it is only for very special forms of boundary that it will be possible to arrange the grains so that closest packing is also homogeneous unstrained piling. The highest peaks will obviously represent the various types of normal and antinormal pilings, generally strained, so that each grain will on the average be touched by six, and not twelve, neighbours (cf. § 6).

The whole system of arrangements represented by the mountain under peak $\alpha$ may be referred to as loose inexact $\alpha$ piling. The looseness increases as we descend vertically from the peak, and the inexactness increases as we move away horizontally from the vertical through the peak. Obviously, when we are vertically under the peak, the measure of inexactness may be taken to be zero, for the centres of the grains are each in precisely the same positions as in the configuration, represented by the peak, which defines the type of packing in question. At the ridge which connects peak $\alpha$ with peak $\beta$ the inexactness is so great that it is impossible to determine whether the packing is of type $\alpha$ or $\beta$.

We shall now suppose that a single closed contour line encloses peaks $\alpha, \beta, \gamma, \ldots$, and that the system of grains each of diameter represented by the height of this contour is moving in such a way that the representative point is inside the contour. We shall shew that at any given instant it is infinitely probable that the piling is, with certain limitations, of one definite kind.

11. Let us suppose that the whole section of the surface in the $(3N+1)$-dimensional space, which is made by the horizontal $\theta = \sigma$, consists of $s$ closed curves, so that there are $s$ isolated contour lines at height $\sigma$. Let us suppose, first, that the section cuts a *finite* number of types of peaks, these types being $\phi, \chi, \psi, \ldots$. Then there are $N!$ peaks of each of these types to be placed within the $s$ separate contour lines, so that from the exact symmetry *inter se* of the $N!$ peaks of any given type, it follows that there must be $N!/s$ peaks of each of the types $\phi, \chi, \psi, \ldots$ inside each contour line.

The probability of the piling being of any given kind at a given instant is measured simply by the area of that part of the section which represents piling of the given kind, this area being a volume in the $3N$-dimensional space of § 7; hence it follows that the ratio of the probabilities of the piling being of types $\phi, \chi, \psi, \ldots$ is the ratio of the cross sections of single peaks of types $\phi, \chi, \psi, \ldots$ by the horizontal $\theta = \sigma$.

12. Let us first consider the simple case in which each peak is a right circular cone of semi-vertical angle $a$, the heights of cones of types $\phi, \chi, \psi$ above the section $\theta = \sigma$ being $h_\phi, h_\chi, h_\psi, \ldots$. The section of the cone $\phi$ is a "circle" in $3N$ dimensions of radius $h_\phi \tan a$; its area is accordingly

$$[\pi^{\frac{3}{2}N}/\Gamma(1+\tfrac{3}{2}N)]\,(h_\phi \tan a)^{3N}; \tag{5}$$

so that the ratio of the areas of the cross sections of cones of different types is

$$h_\phi^{3N} : h_\chi^{3N} : h_\psi^{3N} : \ldots. \tag{6}$$

If $\phi$ is the highest type of peak, so that $h_\phi$ is the greatest of the quantities $h_\phi, h_\chi, h_\psi, \ldots$, this ratio may be written

$$1 : \left(\frac{h_\chi}{h_\phi}\right)^{3N} : \left(\frac{h_\psi}{h_\phi}\right)^{3N} : \ldots,$$

which, on putting $N = \infty$, becomes

$$1 : 0 : 0 : \ldots, \tag{7}$$

except when the two highest $h$'s differ only by terms in $1/N$. Except in this special case, one cross section is infinitely greater than any of the others, and the order of this infinity is that of $e^N$.

13. When we abandon the simplification of regarding the peaks as right circular cones, the method fails. We can, however, say that each expression of the type (5) must be replaced by the product of $3N$ linear factors; so that each ratio of the type $h_\phi^{3N} : h_\chi^{3N}$ must be replaced by the product of $3N$ numbers. There is no reason why any of these numbers

should be equal to unity; so that it is infinitely probable that, in the limit when $N$ is made infinite, their product will be either infinite or zero, to the order of $e^{\pm N}$. Thus it is infinitely probable that in the corrected form of the ratio (6) one term will be infinite in comparison with all the others, the order of this infinity being $e^N$. Thus the series of probability ratios is again of form

$$1:0:0:0:0\ldots,$$

the order of the 0's being that of $e^{-N}$.

14. It may be useful to discuss this same question somewhat more closely, and from a different point of view. We could compare two $s$-dimensional cross sections in the following way. First measure the greatest intercepts of each on a line parallel to any one of the axes of co-ordinates, say $\xi_1$, and let these be $\eta_1$, $\eta_1'$. We are going to regard the ratio of the areas as a product of $s$ factors, and of these the first will be taken to be $\eta_1/\eta_1'$. Next we measure the greatest section by planes parallel to $\xi_1\xi_2$. These may be taken to be $\eta_1\eta_2$, $\eta_1'\eta_2'$, where $\eta_2$, $\eta_2'$ are the greatest intercepts on lines parallel to the axis of $\xi_2$, each multiplied by a correcting factor. The second term in our product of $s$ factors may be taken to be $\eta_2/\eta_2'$. We can continue the process indefinitely, and finally obtain as the value of $\rho$ the ratio required:—

$$\rho = \frac{\eta_1}{\eta_1'}\,\frac{\eta_2}{\eta_2'}\,\frac{\eta_3}{\eta_3'}\,\ldots.$$

Let us first suppose that nothing at all is known about the $s$-dimensional cross sections. Then $\eta_1/\eta_1'$ is just as likely to be above unity as below, and the chances of two values $\eta_1/\eta_1' = q$, $\eta_1/\eta_1' = q^{-1}$ are exactly equal. The same is true for $\eta_2/\eta_2'$, and so on. Thus

$$\log\rho = \log\left(\frac{\eta_1}{\eta_1'}\right) + \log\left(\frac{\eta_2}{\eta_2'}\right) + \ldots \tag{8}$$

is a series of terms such that equal positive and negative values are equally likely for each term. The expectation of $\log\rho$ is therefore zero, as it obviously ought to be, since we do not know whether $\rho$ is the ratio of $A$ to $B$ or of $B$ to $A$. If, however, we square equation (8), we obtain

$$(\log\rho)^2 = \left[\log\left(\frac{\eta_1}{\eta_1'}\right)\right]^2 + \left[\log\left(\frac{\eta_2}{\eta_2'}\right)\right]^2 + \ldots + 2\log\left(\frac{\eta_1}{\eta_1'}\right)\log\left(\frac{\eta_2}{\eta_2'}\right) + \ldots. \tag{9}$$

Of the terms which form the right-hand member the squared terms are necessarily positive; for the remainder equal positive and negative values are equally likely. Thus the expectation of $(\log\rho)^2$

$$= \sum_{r=1}^{r=s}\left\{\text{expectation of }\left[\log\left(\frac{\eta_1}{\eta_1'}\right)\right]^2\right\} = s\chi,$$

where $\chi$ is the average expectation of the squared logarithms. Thus, when $s$ is finite, this expectation of $(\log \rho)^2$ is finite; when $s$ is infinite, this expectation is an infinity of the order of $s$; so that the expectation is that $\rho$ will be infinite or zero to the order of $e^{\pm\sqrt{s}}$.

On the other hand, if, instead of nothing at all being known about the cross sections, we know that they are cross sections of peaks of heights $h$, $h'$, then there is a probability that $\log(\eta_1/\eta'_1)$, $\log(\eta_2/\eta'_2)$, ... will all be of the same sign as $\log(h/h')$; so that the result of equating the expectations of the two sides of equation (9) will be more nearly represented by

$$(\log \rho)^2 = s^2 \left(\log \frac{h}{h'}\right)^2,$$

giving $\rho = (h/h')^s$, as in § 12.

15. This treatment, then, brings us to the same conclusion as that of § 12, namely, that it is infinitely probable that the ratio of probabilities will be found to be

$$1 : 0 : 0 : 0,$$

the first term corresponding to the highest peak. If, as we have already supposed, the number of types of peak is finite, the sum of the series $1+0+0+0+\ldots$ is unity; so that it is infinitely probable that the representative point at any instant will be found under the highest peak. In particular, if the contour includes any of the peaks which represent homogeneous normal and anti-normal piling, it is infinitely probable that the whole system at any instant will be found to be in normal and anti-normal piling. The condition that the contour shall include peaks of normal and anti-normal piling is simply that it should be possible, if we were able to handle the individual grains, to arrange them in such piling without taking them outside the boundary of the real space in which they exist. If this condition is satisfied, it is infinitely probable that any divergences from this piling, if of the kind represented by separate peaks, would in a very short space of time disappear. We may perhaps summarise the question by saying that the "positional entropy"[*] of the system is a maximum when the system is in normal and anti-normal piling.

16. In Prof. Reynolds's theory, the representative point obviously has access to the peaks of normal piling. The diameter of a grain is $5{\cdot}5 \times 10^{-18}$; the mean path is $8{\cdot}6 \times 10^{-28}$. In a linear centimetre there are about $2 \times 10^{17}$ grains; so that by closing up the spaces between these we save a linear space of about $16 \times 10^{-11}$ cms.—space enough in which to

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[*] Cf. *The Dynamical Theory of Gases*, § 194.

manipulate millions of grains at once, and so establish normal or anti-normal piling.

In this theory, matter, electricity, and permanent magnetism are supposed to be represented by certain definite kinds of deviations from homogeneous normal piling—by certain peculiarities, we may say, in the ether structure. If these peculiarities are such as are represented in the peaks of normal piling, then clearly, as the representative point moves about under these peaks, the peculiarities will continually disappear and reappear, and, if we do not know the velocities of every grain of the ether, these appearances and disappearances will appear to be the result of pure chance. If, on the other hand, these peculiarities are represented by separate peaks, then the representative point, as we have just seen, will hasten from any one of these separate peaks to the peaks of normal and anti-normal piling, and at any instant it is infinitely probable that it will be found under these latter peaks. Thus matter, electricity, and magnetism would have no permanence, and their appearance would be transitory and infinitely improbable phenomena. We can form some estimate of what would be the rate of disappearance of these phenomena if Prof. Reynolds's theory were true. The motion of ordinary matter is, according to this theory, a progressive rearrangement of the piling of the ether. The average velocity of a molecule in a gas at ordinary temperatures is about $5\times10^4$ cms. per second; so that the time of passage over the diameter of a grain is about $10^{-22}$ seconds. This, then, would be the time of rearrangement of one layer of grains. The time of rearrangement of the whole universe, *i.e.*, of the disappearance of matter, electricity, and magnetism, would be such that the matter, &c., would be reduced to $e^{-1}$ of their initial amount in a time comparable with this, just as in the theory of gases deviations from maximum entropy are reduced to $e^{-1}$ in a time comparable with the description of a single free path.

17. An alternative possibility, although excluded from Prof. Reynolds's theory by the numerical values given in his memoir, is of sufficient interest to be worth mentioning. Contours can be found enclosing a series of peaks which do not include any peaks of normal piling. In this case, one type of peak, say $\phi$, will have the greatest cross section; so that at any instant it is infinitely probable that the piling will be of type $\phi$. By hypothesis, this type is not normal piling; so that the universe will tend to assume some state other than that represented by the complete dissolution of all matter, electricity, and magnetism.

There is a simple criterion which enables us to determine whether a configuration is under a peak of normal piling or a peak of some other kind. If we imagine all the grains to swell uniformly in size, the representative point moves vertically upwards. When, on account of the swelling, a grain touches either another grain or the boundary of the real space, the representative point has reached the surface of the mountain through which it is moving. The motion is now up the side of the mountain; at first we can suppose it to be up a line of greatest slope, but when a second contact occurs between two grains or a grain and the boundary the representative point finds itself in the angle of a gully on the mountain-side, and the motion has to continue up this gully. And so we proceed indefinitely, until it is impossible for the swelling to proceed further: the grains are jammed together. At this point the peak is reached.

Now according to Prof. Reynolds the existence of each atom of matter, each charge of electricity, &c., is characterised by a surface of misfit, and there is no gearing between the grains inside this surface and those outside. An *increase* in the size of the grains cannot cause this want of gearing to disappear, so that at the peak we must have the surfaces of misfit still existing in, or near, their original positions. Thus all points under one peak represent the same atoms of matter, and the same charges of electricity and magnetism, all in approximately the same positions. If, then, this peak is isolated from all other peaks by the contour line of height $\sigma$, the matter, electricity, and magnetism in the universe will be fixed both in quantity and position. If, on the other hand, the contour of height $\sigma$ includes a number of such peaks, but no peaks of normal piling, the matter, &c., will in general be variable in quantity and also in position. The motion of the universe will tend to a certain definite arangement of matter, &c.—that of the peak above the greatest cross section made by the contour. This arrangement might, of course, represent the cold motionless universe, with the matter all clustered together, with which we are familiar. But in the motion towards this configuration we should in general find no continuity, either in quantity or position, of matter, electricity, and magnetism. The motion, as far as one can see, would probably consist of the dissolution of matter at some points and the appearance of matter at and near the centre of the universe. The time taken for the universe to cover $e^{-1}$ of its path towards the final stage would be about $10^{-22}$ seconds, as in § 16. This possibility however, as we have already said, is excluded by the figures given by Prof. Reynolds.

18. We have so far only discussed the case in which the contour includes only a finite number of types of peaks. If this number becomes

infinite with $N$, two new difficulties occur. In the first place, it may be that the heights of the peaks decrease by steps which are of the order only of $1/N$ or even smaller, in which case the ratio of the different cross sections can no longer be taken to be $1:0:0:0\ldots.$ And, secondly, the sum of the terms of the probability ratios may be a divergent series.

To obviate these difficulties we may agree to divide up the different peaks only into a finite number of classes, each class presenting one definite peculiarity. Then in calculating the ratio of the probabilities of the representative point being under peaks of two different kinds, we are again calculating the product of an infinite number of factors, so that, as in § 15, it is infinitely probable that this product will be infinite or zero. Thus we may assume a probability ratio of the form $1:0:0:0\ldots$ consisting of a finite number of terms.

Let us agree that all arrangements of normal and antinormal piling are together to form one class.

There is no obvious justification for identifying the first term of the probability ratio with this class. There will be a final state towards which the universe will tend, but this has not been proved to be a state in which ether alone exists. At the same time it is almost obvious, in spite of the absence of definite proof, that such must be the case. And, whether this is so or not, with the figures suggested by Prof. Reynolds, the motion of the universe towards its ultimate state would still be at the rapid rate which has been mentioned, and the peculiarities in the ether would still not possess permanence either of quantity or position.

19. The question of the ratio of the probabilities, as between peaks of homogeneous normal piling, homogeneous antinormal piling, and combinations of the two kinds of piling is one of importance. Prof. Reynolds states that shot shaken together tend to arrange themselves in homogeneous normal piling, while Prof. Everett has suggested that this may be because the six close lines of normal piling act in effect as battering-rams, breaking up weaker structures which are provided with fewer lines of battering-rams; so that homogeneous normal piling ultimately becomes established by an evolutionary process.

We see at once that Prof. Everett's suggestion is not borne out by the analysis of the present paper. As regards the other question at issue, we may notice that, except for the influence of the shape of the boundary (an influence which becomes infinitesimal when the number of grains is very great), all arrangements of normal and antinormal pilings are equally probable, these arrangements being represented by peaks of the same height and shape. If there are $N$ tiers, the probability is $2^{N-2}-1$ to

1 against homogeneous normal piling. We have, however, seen (§ 16) that the height of contour proposed by Prof. Reynolds is far below the level of the passes connecting the various types of normal and antinormal pilings; so that the piling ultimately established would be very "inexact" (in the sense of § 10).

## THE PERMANENCE OR NON-PERMANENCE OF MATERIAL MOMENTUM.

20. Just as the conservation of mass requires the continued existence of a peculiarity in the piling of the medium, so the conservation of momentum requires the continued existence of a peculiarity in the distribution of the velocities of the grains, *i.e.*, of a deviation from Maxwell's Law. The average rate of subsidence of such a deviation can be calculated just as in the Theory of Gases. The velocities of the grains, independently of the arrangement of the piling, tend to arrange themselves according to Maxwell's Law. As regards the rate at which this arrangement takes place, we require to distinguish between three kinds of deviation from Maxwell's Law :—

> (i.) Deviations in which the mean values of $u$, $v$, $u^2-v^2$, $uv$, ... vanish at the point in question, while the mean value of $u^2+v^2+w^2$ at this point is different from the value at points in the neighbourhood. Such a deviation becomes dissipated by conduction and wave-motion; in each case the deviation spreads out in all directions at a rate which depends only on the structure of the medium.
>
> (ii.) Deviations in which the mean values of $u^2-v^2$, $uv$, ... vanish at the point, while the mean values of $u$, $v$, and $w$ do not vanish. Such a deviation becomes dissipated by viscosity and wave-motion, the features of this dissipation being the same as before.
>
> (iii.) Deviations in which the mean values of $u^2-v^2$, $uv$, and other functions of degree higher than the first, which would vanish if Maxwell's Law were obeyed, do not vanish. Such a deviation vanishes without propagation of any kind. It falls to $e^{-1}$ of its value in a time which Maxwell calls the "time of relaxation," this time being comparable with the time of description of a free path. In Prof. Reynolds' theory this is comparable with $10^{-29}$ seconds.

In so far as I understand the theory of Prof. Reynolds, momentum is a deviation of the third type; so that the momentum of the universe would be reduced to $e^{-1}$ of its value in about $10^{-29}$ seconds. The alternative would be that momentum would spread itself out in all directions through the ether, travelling with the velocity of light.

## DYNAMICS OF THE ETHER.

21. We now turn to a discussion of the phenomena which might be expected in free ether, all peculiarities of piling being absent. It will not be necessary to introduce any of the figures suggested by Prof. Osborne Reynolds as suitable for the actual ether: we shall simply discuss the abstract question of the motion of a granular medium.

### *Specification of the Strains of the Medium.*

22. We suppose the state of the medium to be specified in terms of its divergences from a state of loose uniform normal piling. This state, as we shall see, may be regarded as an equilibrium position for the medium. The whole space is supposed to be of such extent, and the boundary to be of such a shape, that, if the diameter of each grain were $\tau$ instead of $\sigma$, the grains could just be accommodated in close and exact normal piling. Each sphere in this case is touched by twelve others, and the points of contact lie on six diameters. By drawing all these diameters for each grain, we arrive at a network of straight lines. These lines lie in six directions in space, these directions being parallel to the six edges of a regular tetrahedron. It will be convenient to speak of these lines as the *frame-lines* of the piling. The frame-lines intersect in sixes, and the points so determined are the centres of grains of diameter $\tau$ in close normal piling.

If we now let the diameter shrink from $\tau$ to $\sigma$, the shrinkage being so small that $\tau-\sigma$ vanishes in comparison with $\tau$, and if at the same time we endow each grain with motion so that its centre describes small rectilinear paths about its former position, we have a medium in loose normal piling, the piling being determined by the same frame-lines as before.

This conception explains the configuration of the medium in terms of the frame-lines. In general, however, the problem will be to determine the frame-lines from the observed configuration of the medium. If the medium is strained, the frame-lines are displaced and distorted; as the strain relaxes, the frame-lines return to their old position. What we require to study is the *motion* of the frame-lines; so that it would be objectionable to define the position of these frame-lines in any manner which involves averaging over an interval of time. We, therefore, define the position of the frame-lines at any instant as follows:—

Let any $M$ grains be taken, these grains being contiguous grains on the same frame-line in the undisturbed state of the medium. The centres of these grains will not, in general, be collinear, but it will always be possible to find the line of closest fit (*i.e.*, the line such that the sum of the squares of the distances of the $M$ centres from this line is a minimum). Let us do this, and then take the next $M$ grains and again find their line of closest fit, and so on indefinitely. If the strain of the medium may be regarded as homogeneous over a length which contains a great number of grains (say $M'$), we shall be able to choose $M$ so as to be very great compared with unity, but very small compared with $M'$. If this is done, the lines of closest fit, each of length $M\sigma$, may be regarded as elements of arc of a continuous curve, and, moreover, the position of this curve will not be influenced by the to-and-fro motion of the grains, but only by the motion of the medium as a whole. It is clear that these curves will be the same no matter what value of $M$ is chosen, so long as it satisfies the conditions mentioned above. We *define* the curves obtained in this way to be the frame-lines of the strained medium.

The frame-lines will still intersect in sixes, at any rate except for small distances of a high order of infinitesimals. We *define* these intersections to be the *mean-centres* of the corresponding grains. If the strained medium is at rest, the mean-centres as just defined become identical with the mean position of the centres averaged through a sufficient time.

The vector distance between the mean-centre of a grain in the standard (normal-piling) configuration and the mean-centre of the same grain in the medium when strained we define to be the *displacement* of the medium at the point in question.

23. As regards analysis which is not fine-grained enough to separate out the individual grains of the medium, the displacement of the medium, as just defined, will be a continuous function of position in the medium. We may, therefore, introduce the strain components $e, f, g, a, b, c$, and the rotations $\varpi_1, \varpi_2, \varpi_3$, defined as usual by

$$e=\frac{\partial u}{\partial x}, \quad \dots, \quad a=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}, \quad \dots, \quad 2\varpi_1=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \quad \dots,$$

where $u, v, w$ are the components of displacement at $x, y, z$.

A short line passing through the point $x, y, z$ and having direction-cosines $l, m, n$ is extended by the strain by an amount*

$$(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c)(l, m, n)^2$$

of its original length.

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* Cf. Love, *Elasticity*, I., § 9.

24. When a strain is homogeneous, the frame-lines are straight lines in the strained state, and alternate meshes of the lattice formed by them are similar and equal tetrahedra. When the strain is not homogeneous, but satisfies the condition of varying only over lengths great compared with the diameter of a grain, we may regard the alternate meshes in the neighbourhood of any point as consisting of equal and similar tetrahedra. And, if the strain is small, these tetrahedra differ only slightly from the regular tetrahedra of edge $\tau$ which occur in the unstrained state. Let the edges of the tetrahedra in the neighbourhood of any point be $a_1, a_2, a_3, a_4, a_5, a_6$, and their direction-cosines in the unstrained position be $l_1, m_1, n_1, \ldots$. Then the extension $a_1-\tau$ of the first edge is given by

$$\frac{a_1-\tau}{\tau} = el_1^2+fm_1^2+gn_1^2+am_1n_1+bn_1l_1+cl_1m_1,$$

and there are five similar equations for the other edges. If $a_1, a_2, \ldots, a_6$ are given, these six equations may be regarded as equations to determine $e, f, g, a, b, c$, and as such they are independent. Thus, instead of being given the strain components $e, f, g, a, b, c$, the strain is sufficiently specified by a knowledge of the edges of the tetrahedra at any point, and it will be this specification of strain which we shall find most useful.

By adding the six above equations we obtain

$$\frac{1}{\tau}\Sigma(a-\tau) = 2(e+f+g);$$

so that the cubical dilation is three times the mean linear expansion of the six frame-lines through any point.

25. We cannot, of course, assign arbitrary values to the edges of the tetrahedra at the different points of the medium: the condition must be satisfied that they fit together and fill all space. The obvious fact that the edges of the tetrahedra are not arbitrary is simply Saint-Venant's theorem that the strains in an elastic solid cannot be given arbitrarily.* And the condition that the tetrahedra shall fit together is equivalent to the condition that any journey along the frame-lines, which is such that the algebraic total of the "steps" (in Prof. Everett's sense) taken parallel to each frame-line is zero, shall bring us back to the starting point. This brings us at once to the conditions that a given system of strains may be possible (cf. Love, p. 121).

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* Love, *Elasticity*, I., p. 121.

### *Stresses in the Medium.*

26. The stresses are the result of the transfers of momentum which take place at the different collisions between the grains. At a collision the components of momentum in the tangent plane remain unaltered, but the two components along the line of centres are interchanged. There is therefore a transfer of momentum across the tangent plane equal to the difference of the original components of momentum, and this may be interpreted as an impulsive stress acting across the tangent plane. Thus the system of stresses in the medium is the resultant of a system of forces which have the frame-lines of the medium for lines of action. Our first task will be to determine the forces along the frame-lines.

This obviously will involve a determination of the number of collisions of any given type, and the ultimate basis for any calculation of this kind must be a generalised space of the kind described in § 2. We must not use the whole of this generalised space, but only those parts which represent configurations having certain specified frame-lines in the neighbourhood of the point $P$ at which we are calculating the pressures. The calculation of the pressures may be supposed to proceed as follows.

27. When a collision occurs between two grains moving with relative normal velocity $V$ there is a transfer of momentum $mV$ across the common tangent plane, and this is produced by the action of an impulsive action and reaction $mV$ along the line of centres. We may replace these impulses by ordinary forces of repulsion $mV^2/\epsilon$ acting for a time $\epsilon/V$, where $\epsilon$ is to be made to vanish in the limit. Let us agree that the time $\epsilon/V$ is to be taken as beginning $\epsilon/V$ before the collision, and terminating at the collision; then $\epsilon$ will be the distance apart of the two grains when these forces first come into play. Let us select a point at random from those parts of the generalised space which represent configurations having the given frame-lines at and near the point $P$. On examining the motion and configuration represented by the selected point in the generalised space, we shall find that certain pairs of grains will undergo collision within a time $\epsilon/V$. Between these pairs of grains, then, we must suppose forces to act. On compounding all the forces which act upon the grains intersected by any plane, we obtain the system of forces (pressures and tractions) acting across that plane.

These forces will have different resultants for the different points in the generalised space. As in the Theory of Gases,* we shall find that the resultant is the same for all except an infinitesimal fraction of the points in the generalised space, and that, in any case, the *observed* system of forces (*i.e.*, the actual system averaged through a sufficient time) is the average system when the average is taken through all those parts of the generalised space with which we are concerned, equal elements of volume receiving equal weight in taking the average. In taking this average, it will be convenient to use the language of the theory of probability, the statement "the probability of the occurrence of $x$ is $p$" meaning that $x$ occurs in a fraction $p$ of that part of the volume of the generalised space through which the average is being taken. In taking this average, we have to average with respect to the $6N$ variables (1), subject only to the limitation that the values must give a point in the non-excluded part of the generalised space.

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* See the author's *Dynamical Theory of Gases*, § 120.

28. We can perform the averaging with respect to the velocity coordinates at once. The law of distribution of velocities is, as we have seen, that of Maxwell; so that the probability that two neighbouring grains have velocities between $u$ and $u+du$, $u'$ and $u'+du'$ respectively along the line of centres is

$$\frac{hm}{\pi} e^{-hm(u^2+u'^2)} du\, du'.$$

Let us put $u-u' = V$, $u+u' = 2W$, so that $V$ is the relative velocity, and $W$ is the velocity along the line of centres, of the centre of gravity. Then the foregoing expression may be written as

$$\frac{hm}{\pi} e^{-\frac{1}{2}hmV^2} e^{-2hmW^2} dV\, dW,$$

and on integrating from $W = -\infty$ to $W = +\infty$ we obtain, as the probability of a relative velocity between $V$ and $V+dV$,

$$\sqrt{\frac{hm}{2\pi}}\, e^{-\frac{1}{2}hmV^2} dV.$$

Let $P$ be the probability, obtained by averaging the positional coordinates, that two specified grains are within a distance $\epsilon$ of one another. If this condition is satisfied, and if $V$ is positive, a collision will occur within time $\epsilon/V$; so that we must suppose a repulsion $mV^2/\epsilon$ to act between the grains. Hence, integrating over all values of $V$, we obtain, as the expectation of force between the two grains in question,

$$P\int_0^\infty \sqrt{\frac{hm}{2\pi}}\, e^{-\frac{1}{2}hmV^2} \frac{mV^2}{\epsilon} dV = \frac{P}{2\epsilon h}.$$

The system of pressures is accordingly the resultant of a force $P/2\epsilon h$ along each frame-line, $P$ being, of course, different for the different frame-lines.

### *Stress-Strain Relations.*

29. We can now show that the stresses and strains are connected by linear relations, and can determine the equations expressing these relations. The problem is, of course, the same as that of determining $P$ in terms of the edges of the strained fundamental tetrahedron. This involves averaging in the generalised space with respect to the $3N$ variables

$$x_a, y_a, z_a, \quad x_b, y_b, z_b, \quad \ldots.$$

We begin by giving fixed values to $x_b, y_b, z_b, x_c, \ldots$, and averaging with respect to $x_a, y_a, z_a$. The values given to $x_b, y_b, z_b, x_c, \ldots$ are to be such that those of the grains $B, C, \ldots$ which are near to $A$ are at their mean-centres, the medium being strained so that the edges of the fundamental tetrahedron are $a_1, a_2, \ldots, a_6$. Let $B, C, D, \ldots, M$ be the twelve grains surrounding $A$; then all relations of the type

$$(x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2 > \sigma^2 \quad (ab, bc, ca, \ldots)$$

are already satisfied by the values assigned to the variables, except for the twelve relations

$$ab, ac, ad, \ldots, am.$$

We have, therefore, to average over all values of $x_a, y_a, z_a$ which are consistent with these twelve relations, assigning equal weight to equal values of $dx_a dy_a dz_a$. This, however, is exactly the same thing as averaging for all positions of the centre of $A$ which are possible in the real space, subject to this centre not approaching to within a distance $\frac{1}{2}\sigma$ of any one of the twelve grains $b, c, \ldots, m$. The space available for the centre of $A$ is accordingly bounded by twelve spheres, each of radius $\sigma$, having their centres at the mean-centres $B, C, \ldots, M$. The linear dimensions of this space are small compared with the radii of the spheres, so that the bounding surfaces may be regarded as plane. The space available for the centre of $A$ is therefore a dodecahedron. Each face is a quadrilateral, and in the unstrained medium these quadrilaterals become equal rhombuses (cf. Everett's paper, § 17). In the strained state, let $S_B, S_C, \ldots$ be the areas of these faces, $S_B$ being the face which lies in the tangent plane to $B$. This tangent plane will be perpendicular to $BA$, where $A$ is the mean-centre for the grain $A$, and its distance from $A$ will be $AB-\sigma$. Hence, if we regard the space available for the centre of $A$ as made up of twelve pyramids on $S_B, S_C, \ldots, S_M$ as bases, we arrive at once at the expression

$$\tfrac{1}{3}[(AB-\sigma)S_B+(AC-\sigma)S_C+\ldots+(AM-\sigma)S_M]$$

for the volume of this space. The probability that $A$ shall be within a distance $\epsilon$ of $B$ is the ratio of a small volume of base $S_B$ and height $\epsilon$ to the above expression, say $P'$, where

$$P' = \frac{8\epsilon S_B}{(AB-\sigma)S_B + \ldots + (AM-\sigma)S_M}. \tag{10}$$

To find $P$ we have to calculate this expression for other possible values of $x_b$, $y_b$, $z_b$, $x_c$, ..., and average it over all these values. We first calculate $P'$, as found above.

30. In the figure, let $A$, $B$, ... be the mean-centres of $A$ and the twelve surrounding grains, $C, D, E, F$ being the four which are contiguous both to $A$ and $B$. $ABCD$ may be regarded as the fundamental tetrahedron; its edges being $a_1$, $a_2$, ..., $a_6$, as marked. The whole figure is,

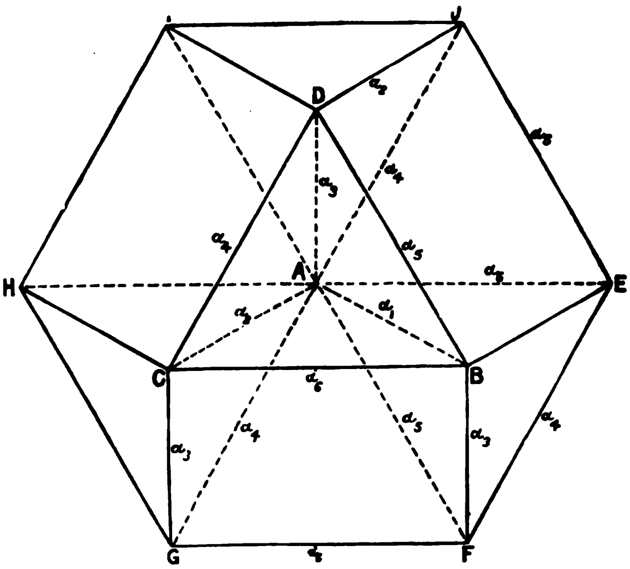

FIG. 3.

except for the strain, what Prof. Everett has called the "normal pile solid" (see his paper, Fig. 6); $B$, $C$, $D$ form an equilateral triangle; $EFGHIJ$ is a regular hexagon parallel to $BCD$, having $A$ as its centre; and $K$, $L$, $M$ form a second equilateral triangle, its plane being parallel to $BCD$ and equidistant from $O$, its projection on the plane $BCD$ coinciding with the triangle $BCD$ if turned through 180°. (The triangle $KLM$ is omitted from the figure for clearness.) The lengths of other lines in the figure can be obtained by noticing that all lines joining mean-centres of contiguous spheres are elements of one or other of the six frame-lines.

Clearly $CD$ and $EF$, being similar elements of parallel frame-lines, are equal and parallel. Thus $C, D, E, F$ are coplanar, and form a parallelogram. Similarly, $A, C, B, E$ form a parallelogram; so that $CE$ bisects $AB$. Again, $D, A, F, B$ form a parallelogram; so that $DF$ bisects $AB$.

Thus the intersection of the diagonals of the original parallelogram $CDEF$ is also the middle point of $AB$. The line $AB$ and the parallelogram $CDEF$ are reproduced in Fig. 4.

To find $S_B$ we draw a plane perpendicular to $BA$ at a distance $\sigma$ from $B$, and then planes perpendicular to $CA$, $DA$, $EA$, $FA$, at distances $\sigma$ from $C, D, E, F$ respectively.

The quadrilateral traced out on the first plane by the last four is the required quadrilateral of which the area is $S_B$.

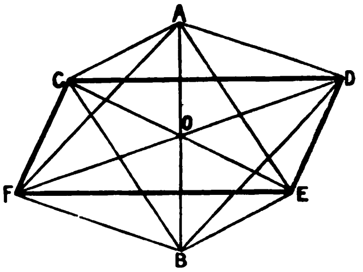

Fig. 4.

31. In the unstrained medium, in which $a_1 = a_2 = \ldots = \tau$, let the value of $S_B$ be $S_0$. Then in the strained medium, in which $a_1, a_2, \ldots$ each differ slightly from $\tau$, we may write

$$S_B = S_0 + (a_1 - \tau)\frac{\partial S_B}{\partial a_1} + \ldots + (a_6 - \tau)\frac{\partial S_B}{\partial a_6}$$

where the differential coefficients are calculated in the unstrained state. Now in this unstrained state $OC = OF$; so that $CDEF$ is a rectangle, and $AC, AD, AE, AF$ are each equal to $\tau$; so that $AO$ is perpendicular to the plane of this rectangle. Thus, in the unstrained state $AC, AD, AE, AF$ are symmetrically placed in Fig. 4. The lengths $AC, AD, AE, AF$, however, are simply the edges $a_2, a_3, a_6, a_5$ of the fundamental tetrahedron. Hence it follows that in the unstrained state

$$\frac{\partial S_B}{\partial a_2} = \frac{\partial S_B}{\partial a_3} = \frac{\partial S_B}{\partial a_5} = \frac{\partial S_B}{\partial a_6};$$

so that the value of $S_B$ may be written

$$S_B = S_0 + p(a_1 - \tau) + q(a_2 + a_3 + a_5 + a_6 - 4\tau) + r(a_4 - \tau) \tag{11}$$

where $p, q, r$ do not depend on $a_1, a_2, \ldots, a_6$.

To determine $p, q, r$ we consider the case in which $a_2 = a_3 = a_5 = a_6$. In Fig. 4 $AC, AD, AE, AF$ are now equal; so that $CDEF$ is a rectangle, and $AO$ is perpendicular to its plane.

Take $A$ as origin, and axes of $x, y, z$ parallel to $AO, DC, FC$. Then the co-ordinates of $C, D, E, F$ are $\frac{1}{2}a_1$, $\pm\frac{1}{2}a_4$, $\pm\frac{1}{2}\theta$, where $\theta = CF$; so that

$$\tfrac{1}{4}(a_1^2 + a_4^2 + \theta^2) = a_2^2 = a_3^2 = a_5^2 = a_6^2 = \kappa^2, \text{ say.}$$

The plane perpendicular to $AC$ at a distance $\sigma$ from $C$ is

$$a_1x+a_4y+\theta z = 2\kappa(\kappa-\sigma),$$

that perpendicular to $AB$ at a distance $\sigma$ from $B$ is $x = a_1-\sigma$, and the trace of the latter on the former is the line

$$a_4y+\theta z = 2\kappa(\kappa-\sigma)-a_1(a_1-\sigma).$$

This makes, with the axes of $y$ and $z$, a triangle of area

$$\tfrac{1}{2}(\theta a_4)^{-1}[2\kappa(\kappa-\sigma)-a_1(a_1-\sigma)]^2,$$

and this is $\frac{1}{4}S_B$, since the quadrilateral now becomes a rhombus of which the triangle just found is a quarter. Thus

$$S_B = 2(\theta a_4)^{-1}[2\kappa(\kappa-\sigma)-a_1(a_1-\sigma)]^2.$$

On differentiation, we obtain

$$dS_B = \frac{2\sqrt{2}}{\tau}(\tau-\sigma)(2\tau-\sigma)(2d\kappa-da_1)+\frac{\sqrt{2}}{\tau}(\tau-\sigma)^2(da_1-da_4-4d\kappa)$$

in the standard configuration. And, by differentiation of equation (11),

$$dS_B = p\,da_1+4q\,d\kappa+r\,da_4,$$

from which $p$, $q$, $r$ are found by comparison. If we neglect $\tau-\sigma$ in comparison with $\tau$, these values become

$$p = -2\sqrt{2}(\tau-\sigma), \qquad q = \sqrt{2}(\tau-\sigma), \qquad r = 0.$$

Writing $\qquad a_1-\tau = a_1, \qquad \ldots, \qquad \tau-\sigma = \varpi,$

the value of $S_B$ is found to be

$$S_B = \sqrt{2}[\varpi^2+\varpi(a_2+a_3+a_5+a_6-2a_1)].$$

Also

$$(AB-\sigma)S_B+\ldots+(AM-\sigma)S_M$$

$$= 2\Sigma(\varpi+a_1)\sqrt{2}\varpi^2\left(1+\frac{a_2+a_3+a_5+a_6-2a_1}{\varpi}\right) = 2\sqrt{2}\varpi^2(6\varpi+3\Sigma a_1)$$

as far as the first powers of $a_1$, $a_2$, …, $a_6$; so that

$$P' = \frac{3\epsilon(\varpi+a_2+a_3+a_5+a_6-2a_1)}{2\varpi(6\varpi+3\Sigma a_1)}$$

$$= \frac{\epsilon}{4\varpi}\left[1+\frac{1}{2\varpi}(a_2+a_3-a_4+a_5+a_6-5a_1)\right].$$

32. We now proceed to average $P'$ for all other positions of the grains. The value of $P'$ has depended only on the positions of the twelve grains contiguous to $A$; so that, if we keep these twelve fixed in the positions

already assigned to them, the process of averaging for the positions of the remaining grains produces no effect on $P'$.

The value of $S_B$ depended only on the positions of the four grains $BCDE$. We can therefore average for all positions of the remaining eight grains without affecting the value of $S_B$. The value of

$$(AB-\sigma)S_B+\ldots+(AM-\sigma)S_M$$

depends, however, on the position of these eight grains. Since the values for which this expression has been calculated corresponded to mean positions of these eight grains, and since we are neglecting squares of deviations from these positions, we may say that equal deviations from the mean positions are equally likely on either side, and hence we see that the averaged value of the expression in question is exactly that which has been calculated. Hence the average value of $P'$ also is exactly that which has been calculated.

We can apply a similar argument to the final process of averaging with respect to the four remaining grains $B, C, D, E$. We accordingly arrive at the result that $P$, the probability of which we are in search, is the same as the $P'$ which has been calculated.

33. We have found the cubical dilatation to be $\Sigma(a_1-\tau)/2\tau$, or, in our present notation, $\Sigma a_1/2\tau$. Let us denote this by $\Delta$. Then the value of $P'$ (or $P$) can be expressed in the form

$$P=\frac{\epsilon}{4\varpi}\left(1+\frac{1}{\varpi}(\Delta\tau-a_4-3a_1)\right).$$

The force along the frame-line corresponding to the edge $a_1$ of the fundamental tetrahedron has been found to be $P/2\epsilon h$. Calling this force $F_1$, its value is now seen to be

$$F_1=\frac{1}{8\varpi h}\left(1+\frac{1}{\varpi}(\Delta\tau-a_4-3a_1)\right).$$

Thus the stress-strain relations are completely known.

*Work Function.*

34. A system of parallel frame-lines in the unstressed configuration meet a perpendicular plane, to the number of unity per $\tau^2/\sqrt{2}$ of area. Hence in a unit volume of the medium there will be a length $\sqrt{2}/\tau^2$ of frame-lines of each system, containing $\sqrt{2}/\tau^3$ elements. It follows that

in changing the state of strain of the medium the work done, per unit volume of the unstrained medium, is

$$\sqrt{2}\tau^{-3}(F_1 da_1 + F_2 da_2 + \ldots + F_6 da_6).$$

This is seen to be a complete differential $dW$ where

$$W = \frac{\sqrt{2}}{8\varpi h\tau^3}\left[2\Delta\tau + \frac{1}{\varpi}\left\{\Delta^2\tau^2 - \tfrac{1}{2}\Sigma a_1^2 - 3(a_1 a_4 + a_2 a_5 + a_3 a_6)\right\}\right].$$

Unit volume in the unstrained state is, however, represented by a volume $(1+\Delta)$ in the strained state; so that the work function per unit volume in the strained state, say $W'$, is

$$W' = \frac{W}{1+\Delta} = \frac{\sqrt{2}}{8\varpi h\tau^3}\left[2\Delta\tau(1-\Delta) + \frac{1}{\varpi}(\Delta^2\tau^2 - \Theta)\right],$$

where
$$\Theta = \tfrac{1}{2}\Sigma a_1^2 - 3(a_1 a_4 + a_2 a_5 + a_3 a_6).$$

To reduce this to rectangular co-ordinates, we have to use the relations

$$a_1/\tau = el_1^2 + fm_1^2 + gn_1^2 + am_1 n_1 + bn_1 l_1 + cl_1 m_1, \ \ldots,$$

which have already been obtained, and from this transformed work function the stress-strain relations in orthogonal co-ordinates can be obtained in the usual way.

### *Introduction of "Ether Temperature."*

35. The reciprocal of the quantity $h$ measures the energy of agitation of the medium. In the Theory of Gases we put $\tfrac{1}{2}h^{-1} = RT$, where $R$ is a constant and $T$ is the temperature. In the present problem also we shall put $\tfrac{1}{2}h^{-1} = RT$, and shall define $T$ to be the *ether temperature*. In the final state $T$ is constant throughout the medium, but in discussing a strained medium we have already supposed the state to be other than this final state, and so are compelled also to consider the possibility of non-uniformity of $T$.

The work function we have obtained is the work function only for strains at constant ether temperature—the isothermal work function. In terms of this temperature, it becomes

$$W_{\text{isoth}} = \frac{\sqrt{2}\,T}{4\varpi\tau^3}\left(2\Delta\tau(1-\Delta) + \frac{1}{\varpi}(\Delta^2\tau^2 - \Theta)\right).$$

This work function contains implicitly the thermo-elastic properties of the medium. We see that each elastic constant is directly proportional to $T$, as in a perfect gas (Charles' Law).

### *The Adiabatic Work Function.*

36. This last circumstance, and the occurrence of $\Delta$ in its first power in the work function, remind us that the medium is in some ways more like a gas than a solid. There is a second work function to be considered, namely that appropriate to adiabatic motion, *i.e.*, motion which is so rapid that no energy escapes by conduction during the progress of the motion.

In unit volume, measured when unstrained, there are $\sqrt{2}/\tau^3$ grains. These each have a mean kinetic energy $3/4h$, so that the kinetic energy of agitation per unit volume of the unstrained medium is $3\sqrt{2}/4h\tau^3$ or $3\sqrt{2}\,T/2\tau^3$.

When strained, let $dQ$ be the quantity of energy per unit volume which escapes by conduction; then we have, as the energy equation of any element,

$$d\left(\frac{3\sqrt{2}\,T}{2\tau^3}\right)-dQ+dW_{\text{isoth}}=0,$$

and, as far as the first order of small quantities, we may take

$$dW_{\text{isoth}}=\frac{\sqrt{2}\,T}{2\varpi\tau^3}\,d\Delta.$$

Putting $dQ=0$, the energy equation becomes

$$\frac{dT}{T}+\frac{\tau}{3\varpi}\,d\Delta=0.$$

This is the adiabatic relation between $\tau$ and $\Delta$. If we replace $d\Delta$ by $dv/v$, where $v$ is the volume occupied by any part of the medium, we obtain the integral

$$Tv^{\tau/3\varpi}=\text{constant}.$$

Since $\tau/\varpi$ is very great, $T$ varies very rapidly in comparison with $v$. Thus the medium, when moving adiabatically, is almost incompressible, any excess of energy spent upon an element in the medium being used in increasing the internal agitation of the medium rather than in compressing the element. The medium then, behaves like a gas in which the ratio of the specific heats is nearly infinite.

If $T_0$ is the temperature of an element when unstrained, and if the motion of straining is adiabatic, the strained temperature will be

$$T_0\left(1-\frac{\tau}{3\varpi}\,\Delta\right).$$

To obtain the adiabatic work function, we must replace $T$ by this expression in the former (isothermal) work function. We thus obtain for the two work functions

$$W_{\text{isoth}} = \frac{T}{2\sqrt{2}\,\varpi\tau^3}\left(2\Delta\tau(1-\Delta)+\frac{1}{\varpi}(\Delta^2\tau^2-\Theta)\right),$$

$$W_{\text{adiab}} = \frac{T}{2\sqrt{2}\,\varpi\tau^3}\left(2\Delta\tau(1-\Delta)+\frac{1}{\varpi}(\tfrac{1}{3}\Delta^2\tau^2-\Theta)\right).$$

35. The meaning and method of calculation of $\Theta$ have already been explained. Taking the axes of $x$, $y$, $z$ to be respectively parallel to $a_4$, perpendicular to $a_4$ in the plane of $a_4\,a_5\,a_6$, and perpendicular to this plane, in the unstrained medium, we obtain

$$\begin{aligned}
a_1 &= \tfrac{1}{3}f+\tfrac{2}{3}g-\tfrac{1}{3}\sqrt{2}a,\\
\left.\begin{matrix}a_2\\ a_3\end{matrix}\right\} &= \tfrac{1}{4}e+\tfrac{1}{12}f+\tfrac{2}{3}g+\frac{1}{\sqrt{18}}a\mp\left(\frac{1}{\sqrt{6}}b+\frac{1}{\sqrt{48}}c\right),\\
a_4 &= e,\\
\left.\begin{matrix}a_5\\ a_6\end{matrix}\right\} &= \tfrac{1}{4}e+\tfrac{3}{4}f\pm\frac{3}{\sqrt{48}}c,
\end{aligned}$$

whence upon substitution

$$\begin{aligned}
\Theta &= \tfrac{1}{2}\Sigma a_1^2-3(a_1a_4+a_2a_5+a_3a_6)\\
&= -\tfrac{7}{8}e^2+\tfrac{1}{4}f^2+\tfrac{2}{3}g^2+\tfrac{1}{3}fg-\tfrac{8}{3}ge-\tfrac{11}{6}ef\\
&\qquad+\tfrac{1}{6}a^2+\tfrac{1}{6}b^2+\tfrac{7}{12}c^2+\frac{5}{3\sqrt{2}}(bc+ae)-\frac{5}{3\sqrt{2}}af.
\end{aligned}$$

It is at once clear that the medium is æolotropic. It is unnecessary to discuss the nature of the æolotropy from the algebraic form of the work function, since a sufficient idea of it can be obtained by considering the arrangement of the frame-lines. We may notice, however, that the number of elastic constants is 18. Writing the work function in the form

$$(c_{11},\ c_{22},\ \ldots,\ c_{66},\ c_{12},\ \ldots\big)\!\!\big(\,e,\ f,\ g,\ a,\ b,\ c)^2,$$

we have the relations

$$c_{14}=c_{56},\qquad c_{25}=c_{64},\qquad c_{36}=c_{45},$$

reducing the number of constants from 21 to 18.[*] These are three of

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[*] To prove the relations, we have

$$\begin{aligned}
c_{14}-c_{56} &= \sum_{14,\,25,\,36}(l_1^2m_4n_4+l_4^2m_1n_1-l_1m_1l_4n_4-l_1n_1l_4m_4)\\
&= \Sigma\,(l_1m_4-l_4m_1)(l_1n_4-l_4n_1)\\
&= \Sigma\,\cos(z\,.\,14)\cos(y\,.\,14),
\end{aligned}$$

where $(z\,.\,14)$ is the angle between the axis of $z$ and the common perpendicular to the opposite the relations by which Cauchy reduces the number of elastic constants of the general crystal from 21 to 15: the remaining three, namely,

$$c_{23} = c_{44}, \qquad c_{31} = c_{55}, \qquad c_{12} = c_{66},$$

are not true. The conditions which must be satisfied for Cauchy's relations to hold in an elastic solid are that the force between two point atoms should be along the line joining their centres, and should be a function of their distance apart. In the present problem, we may suppose the centres of the grains to be point atoms, and the force between two such point atoms to be proportional to their frequency of collision. This, as we have seen, does not depend solely on the distance apart of their mean-centres, but depends also on the other edges of the fundamental tetrahedron.

In cubical piling, on the other hand, the frequency of collisions along a frame-line depends solely on the extension or contraction of that frame-line, and it will be found that the six relations of Cauchy are all satisfied.

### *Conclusion.*

36. The æolotropy which we have found to be associated with a granular medium in normal piling seems to dispose sufficiently of any claims to represent the ether which such a medium might be thought to possess. At the same time it may be of interest to notice a few other characteristics in which such a medium differs from the ether as we know it.

I. In a gas the rate of propagation of a disturbance by wave motion is very great compared with the rate of dissipation of an excess of energy by conduction, the reason being that the disturbance is propagated at a rate comparable with the velocity of the molecules, while the excess heat is, so to speak, buffeted backwards and forwards almost at random by collisions between molecules, in its effort to escape by conduction. I have not worked out the problem of conduction in a granular medium, because it is pretty clear that considerations similar to those just mentioned for a gas must have at least equal weight here; so that we are led to suppose that the propagation of light would be adiabatic. On the other hand, the ratio of the electrostatic and electromagnetic units would obviously be the isothermal velocity of propagation. Thus this ratio would differ from

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edges 1, 4 of the tetrahedron; and, since the three shortest distances between opposite edges of a regular tetrahedron are mutually perpendicular, this expression

$$= \cos(y \, . \, z) = 0.$$

the velocity of light by an amount which, as is shown by an inspection of the two work functions, would be comparable with the whole value of either.

II. The phenomena of viscosity and conduction of heat resolve themselves, when we pass to the ultimate mechanism of the medium, into a redistribution of the energy of agitation of the different parts of the medium—in fact they simply represent the tendency for the velocities of the grains to assume Maxwell's law of distribution. Hence in any granular medium there must be a constant degradation of energy of wave motion or mass motion into energy of random agitation of the medium; so that energy of radiation and the motion of material masses would be "dissipated" in raising the ether temperature. According to Prof. Reynolds, the energy of the normal wave would be spent entirely in raising the ether temperature in the immediate neighbourhood of the origin of the wave. Incidentally, it may be remarked that the analogy of sound motion leaves little hope that any perceptible amount of light could reach us through a granular medium, from even the nearest of the fixed stars. In any case, however, the elastic constants, and therefore also the velocity of propagation of light, would be different in the neighbourhood of moving or radiating bodies.

A first consequence of this would be that as the material energy of the universe became dissipated into the ether the velocity of light would continually increase. This, however, would be unnoticeable, amounting only to a uniform shrinkage of the unit of time.

A second consequence would be that in the neighbourhood of moving or radiating bodies, where the ether temperature would be highest, the velocity of light would be greater than elsewhere (or, what comes to the same thing, the local unit of time would be less). Consequently the periods of vibration of atoms would be less near hot or moving bodies than elsewhere—the radiation from regions in which the ether temperature was higher than that near the Earth would shew a uniform displacement towards the violet when analysed in a terrestrial spectroscope, while that from regions of lower ether temperature would shew a similar displacement towards the red.

A third consequence would be that light passing from a higher to a lower ether temperature would undergo refraction. And, as the medium has already been seen to be æolotropic, there might be additional refraction consequent on the distortion of the frame-lines of the medium.

If a region of the ether could be kept permanently at a higher ether temperature than the remainder, a condition of hydrostatic equilibrium of the medium would be that the density of the ether should be less in this region than elsewhere. Thus there would be an outward *displacement* of ether from a region of high ether temperature, and the linear dimensions of matter, if any, in this region would increase.

Other differences between the behaviour of a granular medium and the observed behaviour of the ether will suggest themselves to the reader. I have only attempted to mention a few of the more obvious ones. As regards the ability of the granular medium to represent the structure of the universe, it is useless to mention differences of this kind at all if, as I have tried to shew to be the case, there is an infinite probability against the continued existence of those peculiarities in the ether which must be taken to represent matter, electricity and magnetism, and if the ether itself is crystalline.

# AN EXTENSION OF BOREL'S EXPONENTIAL METHOD OF SUMMATION OF DIVERGENT SERIES APPLIED TO LINEAR DIFFERENTIAL EQUATIONS

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[Received December 6th, 1904.—Read December 8th, 1904.]

THE object of the following discussion is to make somewhat more precise the connexion between Laplace's definite integral solution of linear differential equations with rational coefficients and of "rank"[*] unity in the neighbourhood of $z = \infty$ and the development of this solution in the form of a power series with an exponential factor. Poincaré has shewn[†] that these developments are "asymptotic expansions" of the integrals from which they are derived, and as such serve to calculate numerically the value of the integral for large values of the variable. But from the function theory point of view asymptotic expansions are of little value, inasmuch as they do not represent unique analytic functions. In fact, if we consider the integrals of a given differential equation, not only does an asymptotic expansion fail to represent a definite solution, but a definite solution has different expansions for different phases of the complex variable.

Now it seems clear that there should be some more definite relation between the divergent series and the integrals of the equation than the foregoing seems to suggest, and it appears that some approach to this connexion may be made on the lines of Borel's theory of *summable* (*sommable*) divergent series.[‡]

The divergent expansions we have to consider are of the form

$$e^{az}x^{-\rho}\left\{u_0+\frac{u_1}{x}+\frac{u_2}{x^2}+\dots\right\}.$$

Leaving out of account for the present the exponential factor, we shall extend Borel's theory to include series of the form

$$x^{-\rho}\left\{u_0+\frac{u_1}{x}+\dots\right\},$$

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[*] Forsyth, *Linear Differential Equations*, p. 271.

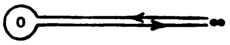
[†] *Acta Math.*, t. VIII., p. 296.

[‡] E. Borel, *Leçons sur les Séries Divergentes*, pp. 97 *ff.*

$\rho$ not being an integer. We require first a generalized form of the $\Gamma$ function. It is well known that for all values of $z$

$$\Gamma(z) = \frac{i}{2\sin\pi z}\int e^{-t}(-t)^{z-1}\,dt,^{*}$$

the integral being taken along a contour which goes from infinity along the axis of $t$ to the origin, makes a small positive circuit round the origin, and returns to infinity along the axis. Integration along such a path as this will be denoted by the suffix $A$. The path may be shown diagrammatically by the figure



Integrating with respect to $a$ along a like contour, we have

$$\begin{aligned}\int_A \frac{u_r a^{z+r}}{\Gamma(z+r+1)}\,e^{-a}\,da &= (-1)^{-(z+r)}\{-2\iota\sin\pi(z+r+1)\}\,u_r \\ &= e^{-\pi\iota(z+r+1)}\{e^{\pi\iota(z+r+1)}-e^{-\pi\iota(z+r+1)}\}\,u_r \\ &= u_r(1-e^{-2\pi\iota z}),\end{aligned}$$

or

$$u_r = (1-e^{-2\pi\iota z})^{-1}\int_A \frac{u_r a^{z+r}}{\Gamma(z+r+1)}\,e^{-a}\,da.$$

Consider now a series

$$u = u_0+u_1+u_2+\dots$$

which may be divergent; and let

$$u_z(a) = \frac{a^z u_0}{\Gamma(z+1)} + \frac{a^{z+1}u_1}{\Gamma(z+2)} + \dots + \frac{a^{z+r}u_r}{\Gamma(z+r+1)} + \dots.$$

Then we have the formal equation

$$\frac{1}{1-e^{-2\pi\iota z}}\int_A e^{-a}u_z(a)\,da = \left[\Sigma\int\frac{u_r a^{z+r}}{\Gamma(z+r+1)}\,e^{-a}\,da\right]\frac{1}{1-e^{-2\pi\iota z}} = \Sigma u_n.$$

Further

$$u_z\left(\frac{a}{x}\right) = \left(\frac{a}{x}\right)^z\frac{u_0}{\Gamma(z+1)}+\dots,$$

and

$$\frac{1}{1-e^{-2\pi\iota z}}\int_A e^{-a}u_z\left(\frac{a}{x}\right)da = \Sigma\frac{u_n}{x^{n+z}} = x^{-z}\sum_0^\infty u_n x^{-n},$$

the equation being once more merely formal so far.

We proceed to show that, if we agree to call the definite integral on the left-hand side the sum of the divergent series on the right, Borel's propositions regarding the sums of divergent series are equally true of the more general series and their sums so defined.

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[*] Whittaker's *Analysis*, p. 181.

We shall assume that $z$ is not an integer; for otherwise the expression on the left is illusory. If $z$ is a positive integer, Borel's method of summation is applicable without modification. If $z$ is a negative integer, the series begins with a finite number of positive integral powers of $x$ and the remainder of the series can be summed by Borel's method.

Assuming then that $z$ is not an integer, let

$$x^{-z}\sum_{0}^{\infty} u_n x^{-n}$$

be a series diverging for all values of $x$, however large.

Let $f(\xi)$ denote the associated series

$$\xi^{z}\sum_{0}^{\infty} u_n \xi^{n}/\Gamma(z+n+1),$$

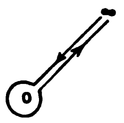
and let this series be convergent within a finite circle about $\xi = 0$; and, by continuation, define an analytic function, existing over the whole plane, isolated singular points excepted; and suppose that none of these singular points occur within a certain angle $\Theta$ bounded by two lines through $\xi = 0$ between which lies the real axis, nor within a circle of finite radius including $\xi = 0$. Further, suppose that a quantity $k$, real, finite, and positive, can be found such that

$$e^{-k\xi}\,|\,f^{(\lambda)}(\xi)\,| \;\ldots\; \lambda = 0, 1, \ldots$$

tends uniformly to zero as $\xi$ becomes large within this sector.

Now
$$\int e^{-a} f\left(\frac{a}{x}\right) da = \int_{B} e^{-bx} f(b)\,\frac{db}{x},$$

where on the left the integration is taken as before, and on the right a straight line inclined to the real axis takes the place of that axis in the path of integration. Such integrations will in future be expressed by the suffix $B$. The path may be shown diagrammatically by the figure



The integral on the right exists if the straight line in question lies within the angle $\Theta$; and, if this is so, that path may be taken along the real axis without changing the value of the integral. We must also have the real part of $x > k$. The existence of the former integral is contingent upon $(a/x)$ lying within the angle $\Theta$; $a$ being real and positive, $x$ must lie within an angle such that $x^{-1}$ lies within $\Theta$. Let the region which fulfils this condition, and also $R(x) > k$, be called $\bar{\Theta}$; in the future the variation of $x$ will be supposed restricted to the region $\bar{\Theta}$.

The series $x^{-z}\sum_{n=0}^{\infty} u_n x^{-n}$ will be called "absolutely summable" within the region $\bar{\Theta}$, and either of the above written integrals will be denoted by

$$(1-e^{-2\pi i z})\,u,$$

$u$ being called the "sum" of the above series.

It will now be shewn that the series

$$x^{-z}\sum_{n=1}^{\infty} u_n x^{-n},$$

obtained by omitting the first term of the previous series, is likewise absolutely summable within $\bar{\Theta}$, and has for sum $u-u_0x^{-z}$.

The associated series is here

$$\begin{aligned}\phi(a,x) &= x^{-z}\left[\frac{u_1}{x}\,\frac{a^z}{\Gamma(z+1)}+\frac{u_2}{x^2}\,\frac{a^{z+1}}{\Gamma(z+2)}+\dots\right]\\ &= \frac{\partial}{\partial a}\left(f\left\{\frac{a}{x}\right\}\right)-x^{-z}u_0\,\frac{a^{z-1}}{\Gamma(z)},\end{aligned}$$

the series $\dfrac{\partial}{\partial\xi}\,\{f(\xi)\}$ being convergent within a finite circle, and giving by continuation throughout the plane the analytic function $f'(\xi)$. Hence, since $e^{-k\xi}\,|f'(\xi)|$, $e^{-k\xi}\,|f''(\xi)|$, ... tend uniformly to zero within $\Theta$, the integral

$$\int_A e^{-a}\phi(a,x)\,da$$

exists, and the said series is absolutely summable within $\bar{\Theta}$.

Its sum

$$\begin{aligned} &= \frac{1}{1-e^{-2\pi i z}}\int_A e^{-a}\left[\frac{\partial}{\partial a}\left(f\left\{\frac{a}{x}\right\}\right)-x^{-z}u_0\frac{a^{z-1}}{\Gamma(z)}\right]da\\ &= -u_0x^{-z}+\frac{1}{1-e^{-2\pi i z}}\left[e^{-a}f\left(\frac{a}{x}\right)\right]_A+\int_A e^{-a}f\left(\frac{a}{x}\right)da\\ &= u-u_0x^{-z}.\end{aligned}$$

Conversely, if a series be absolutely summable within $\bar{\Theta}$, the series obtained by prefixing a term is so also; and its sum is the result of adding the term prefixed to the original sum. The proof of this is a natural extension of Borel's proof for the more limited definition, and need not be given in full.*

From these two propositions it follows immediately that the addition or subtraction of any finite number of terms at the beginning of a series does not affect its summability; and, therefore, that the interchange of any finite number of terms in a series affects neither its summability nor its sum. Next it is clear that two series of the same index $z$ or of indices differing only by integers may be added or subtracted term by term, the result being a series absolutely summable in the region common to the two series, its sum being the sum or difference of the sums of the two separately.

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* *Leçons sur les Séries Divergentes*, pp. 101-2.

We may similarly extend Borel's proposition as to the multiplication of two such series. Let

$$x^{-\rho}\left\{u_0+\frac{u_1}{x}+\ldots\right\},\qquad x^{-\sigma}\left\{v_0+\frac{v_1}{x}+\ldots\right\}$$

be two series having a common region within which they are absolutely summable. Then, if

$$w_n = u_0v_n+\ldots+u_nv_0,$$

the series

$$x^{-(\rho+\sigma)}\left\{w_0+\frac{w_1}{x}+\ldots\right\}$$

will be absolutely summable within that common region, and, if $u$, $v$, $w$ denote the sums of the three series respectively, $w = uv$.

Letting $u(a)$, $v(b)$ denote the associated series

$$\left(\frac{a}{x}\right)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}+\frac{u_1}{\Gamma(\rho+2)}\left(\frac{a}{x}\right)+\ldots\right\},$$

$$\left(\frac{b}{x}\right)^{\sigma}\left\{\frac{v_0}{\Gamma(\sigma+1)}+\frac{v_1}{\Gamma(\sigma+2)}\left(\frac{b}{x}\right)+\ldots\right\},$$

the product of the sums of the series is given by

$$(1-e^{-2\pi i\rho})(1-e^{-2\pi i\sigma})uv = \int_A e^{-a}u(a)da\int_A e^{-b}v(b)db.$$

Inasmuch as $e^{-a}u(a)$ and $e^{-b}v(b)$ tend uniformly to zero, as $a$ and $b$ become infinite along the given contours, this product may be written as a double integral, viz.,

$$\int_A\int_A e^{-(a+b)}u(a)\,v(b)\,da\,db.$$

Call this integral $W$, and let the variables be changed to

$$c = a+b,\qquad \gamma = a-b.$$

Then

$$W = \int e^{-c}\left[\int u\left(\frac{c+\gamma}{2}\right)v\left(\frac{c-\gamma}{2}\right)\frac{d\gamma}{2}\right]dc.$$

The range of integration with respect to $c$ and $\gamma$ will be specified a little later.

Let $\int u\left(\frac{c+\gamma}{2}\right) v\left(\frac{c-\gamma}{2}\right)\frac{d\gamma}{2}$ be denoted by $\bar{w}(c)$; so that

$$W = \int e^{-c}\bar{w}(c)\,dc.$$

This integral from its formation is known to be valid.

Within the common circle of convergence of $u(a)$, $v(b)$ these two series may be multiplied together, giving an absolutely converging double series of powers of $a$ and $b$. The typical term in the product is

$$\frac{u_r v_s a^{\rho+r} b^{\sigma+s}}{(x)^{\rho+r+\sigma+s}\Gamma(\rho+r+1)\Gamma(\sigma+s+1)}$$

or

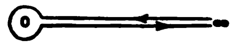
$$\frac{u_r v_s (c+\gamma)^{\rho+r}(c-\gamma)^{\sigma+s}}{(2x)^{\rho+r+\sigma+s}\Gamma(\rho+r+1)\Gamma(\sigma+s+1)}.$$

Call this term $w_{rs}$.

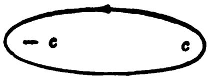
Now as to the paths to be described by the variables $c$, $\gamma$ during the integration :—

For a given value of $c$, $= a+b$, $\gamma$ describes a path from $-c$ to $+c$, a small positive circle round the latter point, returns along itself to $-c$, and closes the path by a small positive circle round $-c$.

The variation of $c$ is then represented by the diagram used above :—



The $\gamma$ contour may be represented diagrammatically by



and an integration along such a contour will be denoted by $\int_C$.

Supposing $|c|$ so small that the corresponding $\gamma$ contour lies wholly within the region for which $\frac{1}{2}(c+\gamma)$ and $\frac{1}{2}(c-\gamma)$ lie within the circles of convergence of $u(a)$, $v(b)$, the integral $w(c)$ can be integrated term by term. The typical term is

$$\tfrac{1}{2}\int_C w_{rs}\,d\gamma = \frac{\frac{1}{2}u_r v_s}{(2x)^{\rho+r+\sigma+s}}\,\frac{1}{\Gamma(\rho+r+1)\Gamma(\sigma+s+1)}\int_C (c+\gamma)^{\rho+r}(c-\gamma)^{\sigma+s}\,d\gamma.$$

Put now $\qquad c+\gamma = 2ct,$

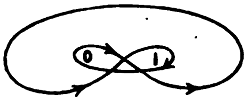
so that $\qquad c-\gamma = 2c(1-t).$

Then $$\int_C (c+\gamma)^{\rho+r}(c-\gamma)^{\sigma+s}\,d\gamma = \int_{C'} t^{\rho+r}(1-t)^{\sigma+s}\,dt\,(2c)^{\rho+r+\sigma+s+1},$$

where in $\int_{C'}$ the contour is of the same kind as in $\int_{C}$, but the points $-c$ and $c$ are replaced by 0 and 1. The integral last written is equal to

$$(2c)^{\rho+r+\sigma+s+1}\int\frac{t^{\rho+r}(1-t)^{\sigma+s}\,dt}{e^{2\pi\iota(\rho+r+\sigma+s)}-1}$$

taken along the contour represented diagrammatically by



and the integral in this expression is Pochhammer's generalization of the Eulerian function of the first kind.* It is equal to

$$-\frac{4\sin\{(\rho+r+1)\pi\}\sin\{(\sigma+s+1)\pi\}\,e^{\pi\iota(\rho+r+s+2)}\,\Gamma(\rho+r+1)\,\Gamma(\sigma+s+1)}{(e^{2\pi\iota(\rho+r+\sigma+s)}-1)\,\Gamma(\rho+r+\sigma+s+2)}$$

$$=\frac{(1-e^{-2\pi\iota\rho})(1-e^{-2\pi\iota\sigma})}{1-e^{-2\pi\iota(\rho+\sigma)}}\,\frac{\Gamma(\rho+r+1)\,\Gamma(\sigma+s+1)}{\Gamma(\rho+r+\sigma+s+2)}.$$

Thus $$\tfrac{1}{2}\int w_{rs}\,d\gamma = \frac{u_r v_s c^{\rho+r+\sigma+s+1}(1-e^{-2\pi\iota\rho})(1-e^{-2\pi\iota\sigma})}{x^{\rho+r+\sigma+s}\,\Gamma(\rho+r+\sigma+s+2)\,(1-e^{-2\pi\iota(\rho+\sigma)})}.$$

If now all the terms of $\bar{w}(c)$ which have the same value for $(r+s)$ be grouped together, we obtain

$$\bar{w}(c) = \frac{(1-e^{-2\pi\iota\rho})(1-e^{-2\pi\iota\sigma})}{1-e^{-2\pi\iota(\rho+\sigma)}}\sum_0^\infty \frac{w_n}{x^{\rho+\sigma+n}}\,\frac{c^{\rho+\sigma+n+1}}{\Gamma(\rho+\sigma+n+2)},$$

provided $c$ lies within a certain circle.

Since $u\{\tfrac{1}{2}(c+\gamma)\}$, $v\{\tfrac{1}{2}(c-\gamma)\}$ are analytic functions which exist for all values of $c$ and $\gamma$, so too is $\bar{w}(c)$; thus, without the circle of convergence of the series for $\bar{w}(c)$, the continuation of this series will give the value of

$$\tfrac{1}{2}\int_C u\{\tfrac{1}{2}(c+\gamma)\}\,v\{\tfrac{1}{2}(c-\gamma)\,d\gamma.$$

It is now necessary to see that $\bar{w}(c)$ satisfies the same conditions that $u(a)\,v(b)$ satisfy, viz., that $e^{-c}|\bar{w}^{(\lambda)}(c)|$ tends uniformly to zero as $c$ tends to infinity, when $x$ lies within the region we have called $\bar{\Theta}$.

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* *Math. Ann.*, t. XXXV.

Within this region we have

$$|u\{\tfrac{1}{2}(c+\gamma)\}| < |\alpha e^{\frac{1}{2}(c+\gamma)}|$$

and $$|v\{\tfrac{1}{2}(c-\gamma)\}| < |\beta e^{\frac{1}{2}(c-\gamma)}|,$$

$\alpha$, $\beta$ being assignable finite constants.

Hence $$|u\{\tfrac{1}{2}(c+\gamma)\}\, v\{\tfrac{1}{2}(c-\gamma)\}| < |e^{c}\alpha\beta|;$$

therefore $$|\bar{w}(c)| < 2\left|\tfrac{1}{2}\int_{-c}^{c} |\alpha\beta e^{c}|\, d\gamma\right| < 2\alpha\beta\, |ce^{c}|,$$

or, putting $c/x = \xi$, $$\left|\frac{\bar{w}(\xi x)}{x}\right| < 2\alpha\beta\, |\xi e^{\xi x}|.$$

The left-hand side is now independent of $x$, and this inequality holds for all points within the region $\bar{\Theta}$. Let $\kappa$ be the least value of $R(x)$ within this region. Then

$$e^{-c}|\bar{w}(c)|/x < 2\alpha\beta\, |\xi e^{-\xi(x-\kappa)}|.$$

Hence, provided $R(x) > \kappa$, $e^{-c}\bar{w}(c)$ tends to zero when $\xi$, and therefore $c$, tends to infinity. Similarly, we may show that, since $u'(a)$, $v'(b)$ also satisfy this condition at infinity, so also does $w'(c)$, and so on.

Consider now the integral

$$\int_{A} e^{-c}\bar{w}(c)\, dc = [e^{-c}\bar{w}(c)]_{A} + \int_{A} e^{-c}\frac{\partial}{\partial c}\bar{w}(c)\, dc = \int_{A} e^{-c}\frac{\partial}{\partial c}\{\bar{w}(c)\}\, dc,$$

what we have just proved being sufficient to ensure the existence of this integral. But

$$\frac{\partial}{\partial c}\{\bar{w}(c)\} = \Sigma\frac{w_n}{x^{\rho+\sigma+n}}\cdot\frac{c^{\rho+\sigma+n}}{\Gamma(\rho+\sigma+n+1)}\,\frac{(1-e^{-2\pi i\rho})(1-e^{-2\pi i\sigma})}{1-e^{-2\pi i(\rho+\sigma)}};$$

so that $$\{1-e^{-2\pi i(\rho+\sigma)}\}\, uv = \int_{A} e^{-c}\, dc\, w(c)$$

where $w(c)$ is the function defined by the series

$$\Sigma\frac{w^{n}}{x^{\rho+\sigma+n}}\,\frac{c^{\rho+\sigma+n}}{\Gamma(\rho+\sigma+n+1)}.$$

This series is exactly the series associated with the divergent series $\Sigma w_n/x^{\rho+\sigma+n}$, and, inasmuch as we have proved that

$$\underset{c=\infty}{\mathrm{Lt}}\,\{e^{-c}\,|w^{(\lambda)}(c)|\} = 0$$

within the region $\bar{\Theta}$, this series is absolutely summable, and we have shewn its sum to be $uv$.

Next consider the series formed by differentiating the series $x^{-\rho}\Sigma u_n x^n$ term by term.

For convenience put $x = 1/z$. Then, with the extended definition of absolute summability which has been adopted, we may apply all Borel's propositions as to the summability of the derived series to the series

$$z^{\rho}\{u_0+u_1 z+\dots\}.$$

The proofs are so identical with those which he gives that they need not be repeated here.*

Thus, if the last written series be denoted by $u(z)$ and its sum by $u$, the divergent series

$$u'(z) = u_0\rho z^{\rho-1}+u_1(\rho+1)z^{\rho}+\dots$$

is absolutely summable within the same region as $u(z)$ and has for sum $du/dz$. Reverting to the original series in powers of $1/x$,

$$\frac{d}{dx}\{u(x)\} = -x^{-2}u'(z).$$

But $x^2$ may be looked upon as the limiting form of a series absolutely summable within any assigned region, and $u'(z)$ has been shewn to be so within a certain region. Thus $u'(x)$ is absolutely summable within that region, and has for sum

$$-z^2\frac{du}{dz} = \frac{du}{dx}.$$

Thus the propositions as to differentiation are extended to series proceeding in descending powers of $x$.

We may therefore state the following proposition, which includes practically all that has been developed, and which is the generalization of Borel's theorem :—†

Let $u$, $u$, $w$, ... be series absolutely summable for $x = x_0$ and each of the typical form

$$x^{-\rho}\left\{u_0+\frac{u_1}{x}+\dots\right\},$$

and let

$$P(u,\ v,\ w,\ \dots,\ u^{(\lambda)},\ v^{(\lambda)},\ w^{\lambda},\ x)$$

be a polynomial in the series $u$, $v$, $w$ and their derivatives, the coefficients being developable for $|x| > |x_0|$ in the form $x^{-m}\left\{a_0 + \frac{a_1}{x} + \ldots\right\}$, $m$ being any index and the series being convergent.

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* Borel, *Leçons*, pp. 108–115; *Ann. de l'École Norm.*, 1899, pp. 94–5.

† Borel, *Leçons*, p. 114.

Then, if the polynomial $P$ is calculated as if the series were absolutely convergent, and the terms whose indices differ by integers are collected together, the result is an aggregate of series absolutely summable for $x = \theta x_0$, $\theta > 1$: and, if the sums of these series be substituted in their place, the result is what would be obtained by substituting in $P$ in the first instance the sums of the series $u, v, w, \ldots$.

Further, it is clear that the sum of an absolutely summable series is identically zero when, and only when, each coefficient vanishes separately. Hence, if the equation $P(u, v, \ldots, x) = 0$ is satisfied formally by the absolutely summable series $u, v, w, \ldots$, the analytic functions defined by these series also render the polynomial $P$ identically zero.

We now proceed to consider expressions of the form

$$e^{\alpha x}x^{-\rho}\left\{u_0 + \frac{u_1}{x} + \ldots\right\},$$

where $x^{-\rho}\{u_0 + \ldots\}$ is absolutely summable within a certain region, and is denoted by $u$, its sum being $\bar{u}$.

Two such expressions with the same exponential factor may clearly be added term by term to give a like expression. Two series $e^{\alpha x}.u$ and $e^{\beta x}.v$, if multiplied formally, give $e^{(\alpha+\beta)x}uv$, and the product $uv$ is absolutely summable, giving the product of the sums of $u$ and $v$, that is, $\bar{u}\bar{v}$.

Consider now the process of differentiation applied to such expressions. Differentiating formally, we have

$$\frac{d}{dx}\{e^{\alpha x}.u\} = e^{\alpha x}\{u' + \alpha u\}.$$

Now $u'$ is absolutely summable and may be added term by term to $\alpha u$; so that

$$\frac{d}{dx}\{e^{\alpha x}u\} = e^{\alpha x}u^{(1)}$$

where $u^{(1)}$ is absolutely summable. Further, the sum of

$$e^{\alpha x}u^{(1)} = e^{\alpha x}\{\bar{u}' + \alpha\bar{u}\} = \frac{d}{dx}\{e^{\alpha x}\bar{u}\}.$$

Thus the process of differentiation gives a series of like form whose sum is the derivative of the sum of the original expression.

It is now clear that the general proposition stated above may be extended to the case where $u$, $v$, $w$ represent absolutely summable series, each multiplied by a factor of the form $e^{\alpha x}$, and where $P$ may also contain explicitly expressions of the same form.

We may now bring what has been said above into line with the normal series satisfying linear differential equations with rational coefficients, of rank 1 at infinity, and of order $n$. Subject to the condition that a certain algebraic equation has its roots all different, we know that there are $n$ expressions of the form

$$e^{\alpha x}x^{-\rho}\left\{u_0+\frac{u_1}{x}+\ldots\right\}$$

which formally satisfy the equation.

Assuming that this series is absolutely summable for certain values of $x$, it follows that the sum, viz.,

$$\frac{1}{1-e^{-2\pi i\rho}}\int_A e^{\alpha x}e^{-a}\left(\frac{a}{x}\right)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}+\frac{u_1}{\Gamma(\rho+2)}\frac{a}{x}+\ldots\right\}da,$$

is an integral of the equation for those values of $x$.

Now this integral may easily be changed into Laplace's definite integral solution.

Put $a=-x(t-\alpha)$. Then the above sum becomes at once

$$\frac{1}{1-e^{-2\pi i\rho}}(-1)^{\rho+1}\int e^{tx}x(t-\alpha)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}-\frac{u_1}{\Gamma(\rho+2)}(t-\alpha)+\ldots\right\}dt.$$

The contour now consists of a line from infinity to the point $t=\alpha$, encircling that point and returning to infinity in the direction whence it came, namely, the direction such that $x(t-\alpha)$ is real and negative. Calling this path $B'$ and integrating by parts, the sum becomes

$$\frac{1}{1-e^{-2\pi i\rho}}(-1)^{\rho+1}\left[e^{tx}(t-\alpha)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}-\ldots\right\}\right]_{B'}$$
$$+\frac{1}{1-e^{-2\pi i\rho}}(-1)^{\rho}\int_{B'}e^{tx}(t-\alpha)^{\rho-1}\left\{\frac{u_0}{\Gamma(\rho)}-\frac{u_1}{\Gamma(\rho+1)}(t-\alpha)+\ldots\right\}dt.$$

The assumption as to the absolute summability of the series causes the quantity within brackets in the integrated part to vanish at the infinite limits. Thus we are led to the conclusion that, provided

$$e^{tx}\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[(t-\alpha)^{\rho-1}\left\{\frac{u_0}{\Gamma(\rho)}-\frac{u_1}{\Gamma(\rho+1)}(t-\alpha)+\ldots\right\}\right]$$

converges uniformly to zero when $t$ becomes infinite in a certain direction for all values of $x$ within a certain region $\bar{\Theta}$, the integral

$$\int_{B'} e^{tx}(t-a)^{\rho}\left\{\frac{u_0}{\Gamma(\rho)}-u_1\frac{t-a}{\Gamma(\rho+1)}+\dots\right\}dt$$

exists and is an integral of the given equation.

Now it has been shown directly from the recurrence formula for the coefficients $u_r$ that the series within the last written integral satisfies the equation known as Laplace's transformation of the given equation,* and hence that a finite number $\lambda$ exists, such that, as $t$ becomes infinite by real positive values,

$$\mathrm{Lt}_{t=\infty}\, e^{-t\mu}\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[(t-a)^{\rho-1}\left\{\frac{u_0}{\Gamma(\rho)}-\dots\right\}\right]=0$$

for all values of $\mu$ such that $R(\mu)>\lambda$.†

We have thus arrived directly from the normal series at an analytic function satisfying the equation within a given region of the $x$ plane, and therefore throughout its region of existence; that is to say, we have shown that a normal series formally satisfying an equation defines a unique integral of that equation by the method here developed, and that these series may be added, multiplied, and differentiated within a certain region as if they were absolutely convergent.

Poincaré's proposition that they asymptotically represent the definite integrals is included in the fact of their absolute summability. The proof is essentially the same as that which Poincaré gives.‡

As was stated at the outset, the object of this discussion has been rather to make closer the connection between the divergent normal series and the ordinary integrals of the differential equation than to obtain fresh knowledge of the integrals from the divergent series apart from the known integrals. One tangible result, at least, emerges from this reversal of the procedure which begins with the definite integral, viz.:—If two differential equations with rational coefficients, and each of rank 1, are satisfied formally by one normal series, these equations have a common integral, even if that series be divergent; and consequently, if one equation be irreducible, the second admits of all the integrals of that one as integrals of itself.

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* *V.* Schlesinger, t. I., § III.

† *V.* Forsyth, *Linear Differential Equations*, pp. 319-322.

‡ See also, Le Roy, "Mémoire sur les Séries div.," *Ann. de la Fac. des Sci. de Toul.*, 1900, p. 427.

Suppose two such equations to be denoted by $R$ and $S$, $S$ being irreducible, and $R$ containing all the integrals of $S$, and being therefore of higher order than $S$.

If the Laplace transformations of these two equations be called $U$, $V$ respectively, then at least one integral of $V$ must satisfy $U$; and, if $V$ is irreducible, all the integrals of $V$ must satisfy $U$. In this case, therefore, $U$ must be of higher order than $V$, unless the equations are identical.

But the order of the Laplace transformation is the degree of the first coefficient in the given equation. Hence, if a linear equation of rank 1 with rational coefficients is irreducible, and also its Laplace transformation, and a normal series exists satisfying this equation and another equation of rank 1 with rational coefficients, this other equation must not only be of higher order, but must have its coefficients of higher degree. In particular, it follows that no two equations of Laplace's type, *i.e.*, with linear coefficients, can be satisfied by a common normal series.

# ON THE GENERAL THEORY OF TRANSFINITE NUMBERS AND ORDER TYPES

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[Received January 26th, 1905.—Read February 9th, 1905.]

1. In accordance with Cantor's general theory of ordinal numbers and of aleph-numbers, there exist two aggregates

$$1,\ 2,\ 3,\ \ldots,\ n,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \Omega,\ \Omega+1,\ \ldots,\ \beta,\ \ldots,$$

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \ldots,\ \aleph_n,\ \ldots,\ \aleph_\omega,\ \aleph_{\omega+1},\ \ldots,\ \aleph_\Omega,\ \aleph_{\Omega+1},\ \ldots,\ \aleph_\beta,\ \ldots;$$

the first is the aggregate of all ordinal numbers, and the second is that of all $\aleph$ cardinal numbers. These aggregates are both normally ordered (*wohlgeordnet*) and are similar to one another; and they contain respectively every ordinal number, and every cardinal number which belongs to a normally ordered aggregate.

In accordance with the principle which is fundamental in the whole theory, that every normally ordered aggregate has a definite order type, or ordinal number, and also a definite cardinal number, it is seen that the above aggregates have an ordinal number $\gamma$ and a cardinal number $\aleph_\gamma$. The ordinal number $\gamma$ must itself occur in the first aggregate, and must therefore be the greatest ordinal number, *i.e.*, the last element of the aggregate; moreover $\aleph_\gamma$ must occur in the second aggregate, and must be the last element of that aggregate. There can, however, be no last ordinal number; for, on the assumption of the existence of $\gamma$, an aggregate of ordinal number $\gamma+1$ can be formed; for example, by placing the first element of either of the above aggregates, after $\gamma$ or $\aleph_\gamma$; it can then be shown that there can be no last aleph-number $\aleph_\gamma$. We have thus arrived at a contradiction.

Burali Forti, who first pointed out this contradiction,* accounted for it by denying the truth of the theorem, that any two distinct ordinal numbers $a_1$, $a_2$ must necessarily satisfy one of the relations $a_1 > a_2$, $a_1 < a_2$, in accordance with Cantor's definition of the meaning of these relations.

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* *Rend. del Circolo mat. di Palermo*, Vol. XI., 1897, "Una Questione sui Numeri Transfinite."

However, Cantor's proof[*] of this theorem does not appear to be capable of refutation; so that the origin of the contradiction cannot be explained in the manner indicated.

B. Russell has suggested that the aggregates of all ordinal numbers and of all aleph-numbers are not normally ordered, and therefore that these aggregates have no ordinal number, and that their cardinal number is consequently not necessarily an aleph-number. He admits, however, that the segments of either aggregate are normally ordered. This explanation is confuted by the argument due[†] to Jourdain, that, if the above aggregates are not normally ordered, then they must contain parts of type $^*\omega$; such a part would then be a part also of a segment of one of the aggregates, and such segment would not be normally ordered.

The contradiction has been explained by Jourdain, by means of the suggestion that there are ordered aggregates which have no order type and no cardinal number, and that the above aggregates belong to such class. To such aggregates he gives the name *inconsistent* aggregates, in virtue of the fact that of such an aggregate it is impossible to think, without contradiction, as a "collection by the mind of definite, distinct objects *to a whole.*" It appears from a statement made by Jourdain,[‡] that Cantor had himself, some years previously, arrived at the same conception and name. In accordance with this view of the matter there exists an ordered aggregate, that of all the ordinal numbers, every segment of which is normally ordered and has a cardinal number, and yet the aggregate itself being "inconsistent" cannot, without contradiction, be thought of as having an order type. This amounts to a denial of the universal validity of the fundamental principle that every ordered aggregate has a definite order type; and yet it is by means of this very principle that the existence of the successive ordinal numbers is regarded as having been established. Each successive ordinal number was defined to be the order type of the ordered aggregate of all the preceding ordinal numbers.

The doubt thus thrown upon the validity of the principle by means of which the existence of the complete series of ordinal numbers, and, simultaneously, that of the aleph-numbers, is established in Cantor's theory, naturally suggests that a further scrutiny of the foundations of that theory is needed. It is not clear, *a priori*, that an aggregate which is inconsistent, in the sense employed above, may not be reached at an earlier stage of the process of forming the successive classes of ordinal numbers, before the aggregate of all such numbers, in the sense of Cantor's theory, is reached. Moreover, it would seem reasonable to expect that so fundamental a distinction, as that involved in the notion of an inconsistent aggregate, should be indicated in the general definition of an ordered aggregate, or in close connection therewith. In any case, an explanation of the contradiction on these lines cannot be regarded as satisfactory, until criteria have been obtained which shall suffice to decide, in respect of any particular ordered aggregate, whether such aggregate has an order type and a cardinal number, or whether it is an inconsistent aggregate.

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[*] *Math. Annalen*, Vol. XLIX., p. 215.

[†] *Phil. Mag.*, 1904, "On the Transfinite Numbers of well-ordered Aggregates."

[‡] *Loc. cit.*, p. 67, note. See also Hilbert, *Jahresbericht der Deutsch. Math. Vereing.*, Vol. VIII. p. 184.

2. Before proceeding to attempt the consideration of the question how far Cantor's general theory of ordinal numbers and aleph-numbers can be accepted as sound, some remarks will be made as to the definition of an aggregate in general, with a view to discover whether it has, in the form given by Cantor, the requisite degree of precision. An attempt will then be made to decide what limitations or qualifications must be imposed upon the nature of an aggregate, so that in the development of the theory, the possibility of being confronted by such a contradiction as that which was pointed out by Burali Forti may be removed at its source.

Cantor has given the following definition of an aggregate (Menge) :— "Menge ist jede Zusammenfassung $M$ von bestimmten wohlunterschiedenen Objecten $m$ zu einem Ganzen, $M = \{m\}$." The term aggregate being thus taken as denoting a collection of distinct objects, in the most general sense, the difficult question arises as to when the elements of the aggregate can be regarded as adequately defined. In the case of a finite aggregate, the elements may be defined by means of individual specification, but this is not possible in the case of a transfinite aggregate. Individual specification must then, in the latter case, be replaced by a law or a set of laws, forming the *norm* by which the aggregate is defined. The most general definition which presents itself, as a *prima facie* definition of an aggregate, is that an aggregate consists of all objects each of which satisfies certain specified conditions. It is, however, convenient to admit the case of two or more alternative sets of conditions; thus an aggregate may contain all objects each of which satisfies either the conditions $A$ or else one of the sets of conditions $B, C, \ldots, K$. The conditions forming the norm by which the aggregate is defined must be of a sufficiently precise character to make it logically determinate as regards any particular object whatever, whether such object does, or does not, belong to the aggregate. In such a case as the aggregate of prime numbers, we may practically be unable to decide, as regards a particular number, whether that number is a prime or not, but we know that the question could be settled by means of a process containing a finite number of steps. On the other hand, if we consider the aggregate of all algebraical numbers, we may possess no present means of deciding, by means of a known process, whether a particular number is an algebraical number or not. It would, however, be an undue restriction of the conception of an aggregate, to insist upon the possibility of actual determination by means of a known finite process, whether a particular object satisfies the norm by which the aggregate is defined, or not. We therefore agree to fall back upon logical determinacy as sufficient; thus, for example, it is logically determinate as regards a number defined in any particular manner, whether that number is algebraical or transcendental, although we possess no perfectly general method of making the actual determination.

We shall accordingly define the meaning to be attached to the term aggregate as follows :—

*All objects which are such as to satisfy a prescribed norm are said to belong to an aggregate defined by that norm. The norm consists of a set of specified conditions, or of a specified set of alternative specified conditions, and must be sufficient to render it logically determinate, as regards any particular object whatever, whether that object belongs to the aggregate or not.*

In the case of a finite aggregate, the norm may take the form of individual specification of the objects which form the aggregate.

3. It is not clear that an aggregate, defined in the above sense, is necessarily capable of being ordered at all. For example, it is difficult to see that such an aggregate as that of "all propositions" could conceivably be ordered; it being assumed that the meaning of the word "proposition" is taken as so definite that this aggregate has a norm, in accordance with the definition above. Again, to take an example among aggregates of the kind usually considered in mathematical theory, we may consider the aggregate obtained by covering (*Belegung*) the aggregate of real numbers by itself; this aggregate, which, according to Cantor, has the cardinal number $f \equiv c^c$, is equivalent to the aggregate of all functions of a real variable: it is difficult, if not impossible, to see how order could be imposed upon this aggregate. If, then, a transfinite aggregate is to be given as an ordered aggregate, or is to have an order imposed upon it, it would appear to be necessary that the norm which constitutes the defini-

tion of the aggregate should be of such a character that a principle of order is contained therein, or can at all events be adjoined thereto; so that, when any two particular elements are considered, the conditions which they satisfy in virtue of their belonging to the aggregate, when individualized for the particular elements, may be sufficient also to assign relative rank to those elements in accordance with a principle of order. This is, in fact, the case in such aggregates as those of the integral numbers, the rational numbers, or the real numbers. In the case, for example, of the positive rational numbers, defined in accordance with the arithmetic theory as associations of pairs of integers, the relative rank of any two particular elements $(p, q)$, $(p', q')$ is assigned by the system of postulations which defines the aggregate. It may, of course, also be possible in other cases, as in this one, to re-order the aggregate in accordance with some other law extrinsically imposed upon the aggregate; but the nature of the elements must be such that this is possible.

We can now state that:—

*In order that a transfinite aggregate, defined as in* § 2, *may be capable of being ordered, a principle of order must be explicitly or implicitly contained in the norm by which the aggregate is defined.*

In the definition of the order type of an aggregate, given* by Cantor, according to which the order type is obtained by making abstraction of the particular nature of the elements of the aggregate, the conception of their order being retained, it is assumed that the aggregate is given as an ordered one. Again, Cantor has defined a cardinal number as the notion which remains, when, by help of our mental activity, we make abstraction of the nature of the elements of an aggregate, and of the order in which they are given. Here the question arises whether the definition of cardinal numbers should not also be applicable in the case of aggregates which are not given as ordered aggregates. Cantor has himself, in fact, in his theory of exponentials which involve transfinite numbers, contemplated certain aggregates as having cardinal numbers, whilst such aggregates are not given as ordered aggregates, and, *prima facie*, at least, are not capable of being ordered.

The relative order of any two elements of a transfinite ordered aggregate is determined by the individual characteristics of those elements (including, when relevant, their positions in space or time), subject to the defining norm. If then, complete abstraction be made of the nature of the elements, they become indistinguishable from one another, and order in the aggregate is thereby obliterated; all relation to the norm by which the original aggregate was defined having disappeared.

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* *Math. Annalen*, Vol. XLVI., p. 497.

4. Taking the case of an aggregate defined as an ordered aggregate, we now approach the consideration of the fundamental question whether, and under what conditions, if any, such an aggregate can be regarded as having a definite order type and a definite cardinal number. This is equivalent to asking whether, or when, meaning can be given to these terms, of such a character that they can be treated as denoting permanent objects for thought, or as mathematical entities which may themselves be elements in aggregates.

There are two distinct methods of establishing* the existence of a class of mathematical entities :—

(1) Their existence, as definite objects for thought, may be shown to follow as a logical consequence of the existence of other entities already recognized as existing, or of principles already recognized as valid; thus the existence of the new entities in question cannot be denied without coming into contradiction with truths already known. This method may be termed the genetic method.

(2) The existence of the entities may be postulated; their mutual relations and their relations with other entities being defined by means of a complete system of definitions and postulations. Accordingly, the objects in question are a free creation of our mental activity. The validity of the scheme thus set up is established when it is shewn to be free from internal contradiction and from contradiction with other known truths. Its utility is to be judged by its applicability to the general purposes of the science, and the light it may throw upon the fundamental principles of that science, in virtue of the scheme containing a generalization of what was previously known. This method may be termed the method of postulation.

Both these methods have been employed by Cantor in his theory of transfinite numbers and order types. In his earlier treatment of the subject he employed the second of the above methods. The existence of the new number $\omega$, and of the other limiting numbers of the second class, was postulated in accordance with his second principle of generation (*Erzeugungsprinzip*). Freedom from contradiction, and utility in connection with the theory of sets of points, which suggested the postulations, were relied upon as the grounds upon which the scheme of new numbers was to be justified. The first number $\Omega$, of the third class, was introduced by a new postulation.

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* See Hilbert, *Jahresbericht d. deutsch. math. Vereing.*, Vol. VIII., p. 180.

In his later and more abstract treatment of the subject, Cantor applied the genetic method. The existence of the number $\omega$ is not directly postulated, but is taken to follow from the existence of the aggregate $\{n\}$ of integral numbers; $\omega$ is defined to be the order type of this aggregate, and it is assumed that such order type is a definite object which can itself be an element of an aggregate. The existence, as definite entities, of the cardinal numbers being assumed, the successive ordinal numbers of the successive classes are obtained, by assuming the general principle that an ordered aggregate necessarily possesses a definite order type which can be regarded as an object, viz., the ordinal number coming immediately after all those which are the elements of the aggregate of which it is the order type.

5. It has been seen above that the assumptions that an ordered aggregate necessarily possesses a definite order type and a definite cardinal number, which can be treated as objects, lead to the contradiction pointed out by Burali Forti. It appears, therefore, that the classes of entities, the ordinal numbers of all classes, and the similar aggregate of aleph-numbers, do not satisfy the condition of being subject to a scheme of relations which is free from contradiction. In fact, the principle in accordance with which their existence is inferred is in conflict with the definition of the aggregates, as containing every ordinal number, or every aleph-number. It would then appear that the genetic process, which led to the definition of all ordinal numbers and of all aleph-numbers, cannot be a valid one. Thus, the principle that every ordered aggregate has a definite order type which may be regarded as a permanent object for thought cannot be accepted as a universal principle to be used in a genetic mode of establishment of the existence of a class of entities. A denial of the validity of this principle does not, however, preclude the less ambitious procedure of postulating the existence of definite ordinal numbers of a limited number of classes, in accordance with Cantor's earlier method. So long as the postulation of the existence of ordinal numbers does not go beyond some definite point, no contradiction will arise, and the utility of the scheme, for purposes of representation, will suffice to justify the postulations which have been made. An attempt to examine the structure of such a class of ordinal numbers as that of the $\omega$-th class, with cardinal number $\aleph_\omega$, or that of the $\Omega$-th class, with cardinal number $\aleph_\Omega$, will lead to the conviction that such conceptions are unlikely to be capable of useful application in any branch of analysis or geometry for purposes of

representation. Nevertheless, in case inexorable logic compelled us to contemplate the existence of such classes of objects, they would form a very proper field of exploration; we have, however, seen that there are grave doubts as to whether this is the case.

6. The genetic method being rejected on the ground that it leads to the construction of a class of entities which, in its entirety, can have no existence, we have to fall back upon the method of postulation. A consideration of the essential elements in the conceptions which lie at the base of the scheme of finite integral numbers may afford guidance as to how far we may properly proceed in the construction, by postulation, of transfinite ordinal numbers of successive classes. The ordinal numbers of any one particular class are those which belong to rearrangements of the elements of an aggregate of which the order type is the lowest number of that class. We may, therefore, consider primarily the lowest numbers of the classes of which the cardinal numbers are $\aleph_0$, $\aleph_1$, $\aleph_2$, ... respectively. A finite ordinal number is characteristic of a class of aggregates all of which are similar to one another, and a finite cardinal number is characteristic of a class of aggregates all of which are equivalent to one another; thus, in both cases, the notion of correspondence between the elements of different aggregates is an essential factor in the idea of number. The number of each of a class of similar or of equivalent aggregates is considered to be a definite object. The relation of this object to any member of the class of which it is characteristic may be illustrated by the relation of the colour red to a particular red object.

The existence of a particular number does not follow as a mere logical consequence of the existence of the preceding numbers, but from the existence of the class of aggregates of which it is characteristic, the relation between different aggregates of the class being that of (1, 1) correspondence. In the genetic method, as applied to the construction of the whole series of classes of transfinite ordinal numbers, this notion of correspondence plays no part; in fact, the existence of a number is constantly inferred from that of a single unique ordered aggregate. For example, the existence of $\Omega$ and of $\aleph_1$ is inferred from the existence of the single aggregate of numbers of the first and second classes. Generally, in the whole scheme, the existence of a new number is inferred from the existence of that unique aggregate which contains the preceding ordinal numbers. That this procedure leads to contradiction has been already seen. The transfinite numbers must be regarded as obtained or defined in accordance with the same principles as obtain in the case of the finite numbers, if they are to be regarded as numbers, even in an extended sense of the term. It seems, then, highly probable that the neglect of the principle, that correspondence between similar or between equivalent aggregates is essential to our right to consider the numbers belonging to aggregates as definite entities, may be the source of the contradiction which arises from the thorough-going application of the genetic method which leads to Cantor's complete series of ordinal numbers and aleph-numbers. In accordance with this view of the nature of number, finite or transfinite, the postulation of the existence of a definite entity, which entity shall be entitled to be regarded as a number, is only justified when it is shown that other aggregates exist besides the aggregate which consists of the preceding ordinal numbers, of which other aggregates the postulated number is the characteristic number. Thus the postulation of the numbers $\omega$ and $\aleph_0$ requires for its justification the exhibition of other aggregates besides $\{n\}$, that of all finite numbers; in this case the requirement is satisfied by the definition of sets of points or of other geometrical objects, and thus there really exists a class of aggregates which are all similar to the ordered aggregate $1, 2, 3, \ldots, n, \ldots$; and hence the postulated order type $\omega$ and the postulated cardinal number $\aleph_0$ are really entitled to rank as ordinal and cardinal numbers respectively. When we consider the ordinal number $\Omega$, and the cardinal number $\aleph_1$, the state of the case is very different. In order that the existence of $\Omega$ might be on a parity with that of $\omega$, it would require to be shown that it is possible to define a set of objects, say points of the linear continuum, which should be such that to each prescribed ordinal number of the second class there corresponds a definite point of the continuum, *i.e.*, to show that a norm is possible which would define a set of points of order type $\Omega$. This has hitherto* not been accomplished, nor have aggregates having any of the cardinal numbers $\aleph_2, \aleph_3, \ldots$ been defined by means of definite sets of rules. If it be urged that the postulation of the existence of the order type $\Omega$, and of the corresponding cardinal number $\aleph_1$, does not of itself lead to contradiction, it may be replied that such postulation does not entitle $\Omega$ and $\aleph_1$ to rank as numbers in the sense in which $\omega$ and $\aleph_0$ are numbers; for in the latter case the essential elements in the original conceptions of ordinal and cardinal numbers are all present, whereas this has not been shewn to be true of $\Omega$ and $\aleph_1$. Moreover, the postulation of the existence of $\Omega$ and $\aleph_1$, if it does not of itself lead to contradiction, can only be made by means of a principle which, when applied systematically, certainly leads to contradiction. In accordance with the criterion laid down above,

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* An attempt to define a set of points of cardinal number $\aleph_1$ has been made by G. H. Hardy; this will be criticised in § 12.

$\aleph_1$, $\aleph_2$, ... cannot at the present time be regarded as definite entities, and could not be regarded as in any true sense numbers, even if any meaning could be assigned to them.

It may conceivably turn out, in the future, to be possible to justify the postulation of certain of the numbers $\aleph_1$, $\aleph_2$, ... with the classes of ordinal numbers which would belong to them; it will, however, certainly never be possible to do this for the whole class $\{\aleph_\beta\}$, where $\beta$ is any ordinal number of the aggregate of all ordinal numbers, in accordance with Cantor's complete scheme, because such postulation leads to unavoidable contradiction. The setting up of a scale of standards, to which standards no aggregates not consisting of the preceding numbers conform, involving as it does the employment of sphinx-like aggregates, to each of which no other aggregate can be shown to be similar, would, *a priori*, appear to be an illegitimate extension of the notion of number, from which an essential element had dropped out, and, *a posteriori*, it has been shown to lead to contradiction.

It may be urged that no contradiction would ensue if, in single instances, the existence of order types and powers, considered as definite entities, were postulated for aggregates of the unique character referred to above. But, if this were done, such order types and powers would not be entitled to rank as numbers, and such sporadic creations would be of no importance in mathematical theory. Systematic postulation of this character is just what has been shown to lead to a self-contradictory scheme of entities, and is therefore illegitimate.

7. A cardinal number has been defined* by B. Russell to be a class of equivalent aggregates. It may then be urged that such class may contain only one member, and that this is sufficient for the existence of the cardinal number. In fact, Russell infers† the existence of the number $n+1$ from the mere existence of the numbers 0, 1, 2, ..., $n$. In accordance with the view here advocated, this definition, or any other one which allows the existence of a cardinal number to be inferred solely from the existence of a unique aggregate, to which no other aggregates have been shown to be equivalent, must be rejected. Russell objects‡ to the conception of a number as the common characteristic of a family of equivalent aggregates, on the ground that there is no reason to think that such a single entity exists, with which the aggregates have a special relation, but that there may be many such entities. The mind does, however, in point of fact, in the case of finite aggregates at least, recognize the existence of such single

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* *Principles of Mathematics*, Vol. I., p. 111–116.

† *Ibid*, p. 497.

‡ *Ibid*, p. 114.

entity, the number of the aggregates; and this is a valid result of our mental activity, subject to the law of contradiction. Since Russell apparently regards the activities of the mind as irrelevant in questions of existence of entities, his view, and the one here advocated, do not rest upon common premisses. An advantage claimed for the view here supported, over that of Russell, is that it does not lead to such a contradiction as that pointed out by Burali Forti.

The definition of cardinal number, by abstraction, which Cantor employs, can be justified only by the interpretation, that abstraction is made only of those characteristics in which the elements of the aggregate differ from the corresponding elements of equivalent aggregates. The partial abstraction will then consist in taking the cardinal number to be the common characteristic of all such equivalent aggregates. To make a complete abstraction of the nature of the elements of a transfinite aggregate would remove the relation with the cardinal number.

8. The conclusions at which we have arrived in the course of the above discussion may now be summarized as follows:—

(1) The aggregates

$$1,\ 2,\ 3,\ \ldots,\ n,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \Omega,\ \ldots,\ \beta,\ \ldots;$$

$$\aleph_0,\ \aleph_1,\ \ldots,\ \aleph_n,\ \ldots,\ \aleph_\omega,\ \aleph_{\omega+1},\ \ldots,\ \aleph_\Omega,\ \ldots,\ \aleph_\beta,\ \ldots$$

of all ordinal numbers, and of all aleph-numbers, in the sense in which Cantor contemplates them, have no existence. Their existence cannot be established without the assumption of the principle that every normally ordered aggregate necessarily has a definite order type and a definite cardinal number which can themselves be regarded as objects capable of being elements of an aggregate. This principle leads to contradiction, and must therefore be rejected as not being a universally valid truth.

(2) Of the aleph-numbers, the postulation of the existence of $\aleph_0$ has hitherto alone been justified, by showing that it is possible to define aggregates consisting of objects other than the ordinal numbers themselves, of which it is the characteristic cardinal number. The numbers $\omega, \omega+1, \ldots, \omega.2, \ldots, \omega^2, \ldots$ of the second class exist, but it has not yet been shewn that the totality of all these numbers, taken in order, has a definite order type, or a cardinal number. To do this it would be necessary to shew that a finite set of rules can be set up which will suffice to define a definite object corresponding to each ordinal number of the second class.

(3) The existence of individual aleph-numbers, other than $\aleph_0$, with

the classes of ordinal numbers belonging to them, may, in the future, possibly be established, but it is not possible that this should be done beyond some definite point.

It may be observed that an aggregate which consists wholly of distinct physical objects which do not penetrate one another, must be enumerable (*abzählbar*), for each such object occupies some finite volume in space; and it has been shewn by Cantor that any set of non-overlapping portions of space is enumerable. It follows that the objects in an unenumerable aggregate must, with the possible exception of an enumerable component of the aggregate, consist of ideal or abstract objects.

9. The regarding of a collection as a "whole" has been emphasized by Cantor as essential to the notion of an aggregate. It is no doubt true that, in a certain sense, every logical class, or every aggregate as defined in § 2, forms a whole, as being dominated by a norm; but, for the purposes of mathematical science, the important question is under what circumstances such an aggregate may be regarded as having a definite cardinal number, and, if ordered, a definite order type. An attempt has been made above to answer this question in the case of normally ordered aggregates; the answer in the case of other aggregates would be on the same lines.

Ordered aggregrates have been defined which are not normally ordered; and of such aggregates the most important is the arithmetic continuum defined as of the order type $\theta$, considered* in detail by Cantor. That $\theta$ is a definite object, with a corresponding definite cardinal number $c$, must, as has already been pointed out, be regarded as a postulation subject to the law of contradiction. It is known that a class of aggregates exists which are similar to the linear continuum, and thus conform to the type $\theta$, and have $c$ as their common cardinal number; this is in accordance with the regulative principle which we have maintained to be essential to justify our regarding $c$ as a number.

As has been already remarked, aggregates may be defined which are unordered. In such cases no question arises as to the existence of an order type, but there is no reason why such aggregates should not have cardinal numbers, provided that in the case of such an aggregate, equivalent aggregates can be found of which the cardinal number in question is the common characteristic. The aggregates of which the cardinal number is $c^c \equiv f$ are an example of this species of aggregate. Two aggregates which have been independently defined are not necessarily comparable with one another as regards either order type or cardinal number. It cannot be assumed *a priori* that the cardinal number of one of them is necessarily either greater than, equal to, or less than that of the other, in the sense in which these relations are defined in the theory. Further, it cannot be assumed that an ordered aggregate such as, for example, the continuum, is necessarily capable of being normally ordered. There is, as we have seen, no convincing reason for thinking that any unenumerable aggregate is capable of being normally ordered. Two aggregates of abstract objects, which have been independently defined, belong, no doubt, to the same universe of thought; but, nevertheless, any particular category of relations may be too narrow to formulate any nexus between the two systems, so that it is conceivable that, so far as such relations as those of order and cardinal number are concerned, the two aggregates may be completely isolated from one another.

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* *Math. Annalen*, Vol. XLVI., p. 510.

10. In some proofs of theorems which have been given by writers on this subject, which proofs have for their object the establishment of relations of inequality or equality of cardinal numbers, aggregates are employed, the elements of which are regarded as being successively defined by an endless series of separate acts of choice. When we leave the region of the finite, it would, however, appear that we have passed beyond the region in which definitions by arbitrary acts of choice can be regarded as adequate specifications of definite objects; the existence of a norm would appear to be essential to our right to regard an aggregate as really defined, and therefore to justify our making use of the conception of such an aggregate in the proof of a theorem. The point may be illustrated by a discussion given* by P. Du Bois Reymond, in which he contemplates the existence of a number represented by a non-terminating decimal in which the figures are determined by no law. He contemplates that each figure in the decimal may be fixed by a throw of dice, and rejects the conception of such a decimal (*ewig gesetzloses Decimal*) as representing a real number. A non-finite, or endless, process can be conceived of as a completed whole, only when it is subject to some kind of norm; thus a non-terminating decimal represents a number, only under the presupposition that a set of rules can be given which suffice to determine the figure which occupies any assigned place in the decimal. In general, the proof of the possibility of giving

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* *Allgemeine Functionentheorie*, p. 91.

a norm is required before an aggregate of any particular character can be contemplated as existing, or can be legitimately made use of in a demonstration.

Cantor, in his proof that $\aleph_0$ is less than any other transfinite cardinal number, has assumed that it is possible to pick out of any given transfinite aggregate an enumerable component. This proof can only be accepted as valid in case it is possible to define an enumerable component of the aggregate in question. In a large class of cases—possibly in all which are of importance in mathematical theory—this condition can be satisfied: for example, in the case of the continuum. In the aggregate of "all propositions," for example, the enumerable component may be taken to be that aggregate of propositions which asserts the existence of the numbers 1, 2, 3, ....

G. H. Hardy has extended* Cantor's method, for the purpose of shewing that every transfinite cardinal number is either an aleph-number, or is greater than all the aleph-numbers; and, in particular, $2^{\aleph_0} = c \geqslant \aleph_1$. The proof runs as follows:—Having given an aggregate whose cardinal number is $> \aleph_0$, we can choose from it successive individuals

$$u_1,\ u_2,\ \dots,\ u_\omega,\ \dots,\ u_\beta,\ \dots$$

corresponding to all the numbers of the first and second classes; if the process came to an end, the cardinal number would be $\aleph_0$. Its cardinal number is therefore $\geqslant \aleph_1$; and, if $> \aleph_1$, it is $\geqslant \aleph_2$; and so on. And, if $> \aleph_n$, for all finite values of $n$, it must be $\geqslant \aleph_\omega$; for we can choose individuals from the aggregate corresponding to all the numbers of the first, second, ..., $n$-th, ... classes. And, by a repetition of these two arguments, we can shew that, if there is no $\aleph_\beta$ equal to the cardinal number of the aggregate, it must be at least equal to the cardinal number of the aggregate of the $\aleph_\beta$, and so is greater than any $\aleph_\beta$.

Apart altogether from the question as to what constitutes all the aleph-numbers, this argument could only be made valid if it were shewn how the successive individuals $u_1, u_2, \dots, u_\beta, \dots$ are to be defined by means of some norm, and also how the individuals of the aggregate which may correspond to the numbers of the first, second, ..., $n$-th, ... classes can be assigned by a norm. The process can neither come to an end, nor be regarded as, in any sense, a completed one, unless this has been done.

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* *Quarterly Journal of Math.*, Vol. xxxv., 1903, p. 88.

11. Two proofs have been advanced, that every cardinal number is necessarily an aleph-number; this is equivalent to the statement that every aggregate which has a cardinal number can be normally ordered. If these proofs could be accepted as valid, the particular theorem would be established, that the arithmetic continuum is capable of being normally ordered; and the only question which would remain open, as regards this aggregate, would be as to which particular aleph-number is the cardinal number of the continuum.

The first of these proofs, that of Jourdain,* is founded on the assumption that, if a cardinal number is greater than every aleph, there must be a part of the aggregate to which this cardinal number belongs which can be made to have a (1, 1) correspondence with the "inconsistent" aggregate of all the ordinal numbers arranged in normal order. This assumption is regarded as justified by the process of making the successive elements of the aggregate of ordinal numbers correspond to elements of the given aggregate; it is then argued that, if this process comes to an end, the cardinal number of the aggregate is an aleph, and, if it does not come to an end, the given aggregate must contain as a part the "inconsistent" aggregate of all the ordinal numbers; thus in the latter case the aggregate is inconsistent, and has no cardinal number.

The objection to this proof is of the nature which has been already stated, viz., that no norm is forthcoming by which the correspondence in question is defined; and, in default of such norm, there is no meaning in speaking of an essentially endless process as a completed one, or as having come to an end.

In the second proof, due† to E. Zermelo, no account is taken of the possibility that an aggregate may have no cardinal number, or of the existence of "inconsistent" aggregates. The proof, which is fundamentally of a similar character to that of Jourdain, is represented as demonstrating that every aggregate can be normally ordered, and thus has an aleph as its cardinal number.

It is assumed that in each part $M'$ of a given aggregate $M$, one element $m'$, called the special (*ausgezeichnetes*) element of $M'$, can be chosen. A part $M'$ must contain one element of $M$ at least, and may contain all the elements; and the aggregate $\{M'\}$ of all parts of $M$ is considered. Each element $M'$ of $\{M'\}$, corresponds to a special element which belongs to $M'$, and this particular mode of covering the elements of $\{M'\}$ by elements of $M$, is called a "covering" $\gamma$; the employment of a particular "covering" $\gamma$ is essential to the proof. A $\gamma$-aggregate

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* *Phil. Mag.*, January, 1904, pp. 67, 70.

† *Math. Annalen*, Vol. LIX, 1904, "Beweis, dass jede Menge wohlgeordnet werden kann."

is then defined as follows :—Let $M_\gamma$ be a normally ordered aggregate consisting of different elements of $M$, such that, if $a$ is any arbitrarily chosen element of $M_\gamma$, and $A$ is the segment (*Abschnitt*) of $M_\gamma$ defined by $a$, which segment consists of all the elements of $M_\gamma$ which precede $A$, then $a$ is always the special element of $M-A$. Every such aggregate $M_\gamma$ is a $\gamma$-aggregate. If every element of $M$ which occurs in a $\gamma$-aggregate be called a $\gamma$-element of $M$, it is shewn that the aggregate $L_\gamma$ of all $\gamma$-elements can be so ordered that it is itself a $\gamma$-aggregate, and contains all the elements of the original aggregate $M$. It follows then that $M$ can be normally ordered.

Zermelo himself expressly recognizes the assumption made as to the existence of a definite "covering" $\gamma$. The objection to this assumption is of the same character as before, viz., that for its validity a norm must be shewn to be possible; this norm must assign to each part of the given aggregate, a definite "special" element belonging to that part. In the case of such an aggregate as the arithmetic continuum, it is not clear how such a norm could be devised; indeed, it seems probable that a proof of the possibility of establishing such a norm involves difficulties comparable with those which occur in any attempt to prove the original theorem. The non-recognition of the existence of "inconsistent" aggregates, which existence, on the assumption of Cantor's theory, cannot be denied, introduces an additional element of doubt as regards this proof. The aggregate $L_\gamma$ here employed is parallel with the normally ordered aggregate which occurs in Jourdain's earlier proof.

12. [*Added February 25th,* 1905.]—A method has been given* by G. H. Hardy for the construction of a set of points of cardinal number $\aleph_1$. He proposes to determine for each ordinal number $\beta$ of the first or of the second class, a sequence $b_1, b_2, b_3, \ldots, b_n, \ldots$, consisting of increasing integers, so that to each number $\beta$ a unique sequence corresponds which is not identical with the sequence which corresponds with any other ordinal number of the first two classes. The sequence is then correlated with a point of the linear continuum (0, 1), by taking that binary radix fraction in which the $b_1$-th, $b_2$-th, $b_3$-th, ... figures are all 1, whilst the remaining figures are all 0; the point of the linear continuum so defined being taken to correspond to $\beta$. The sequence corresponding to the number 1 being taken to be 1, 2, 3, 4, 5, ..., the other sequences are determined by the following modes of formation :—

(1) If $b_1, b_2, b_3, \ldots, b_n, \ldots$ corresponds to $\beta$, then $b_2, b_3, b_4, \ldots, b_{n+1}, \ldots$ is the sequence which corresponds to $\beta+1$.

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* *Quarterly Journal of Mathematics*, Vol. xxxv., 1903, p. 88.

(2) The sequence which corresponds to a limiting number $\gamma$ of the second class is obtained from the sequences

$$\begin{array}{l} a_{11},\ a_{12},\ a_{13},\ \ldots,\ a_{1n},\ \ldots, \\ a_{21},\ a_{22},\ a_{23},\ \ldots,\ a_{2n},\ \ldots, \\ a_{31},\ a_{32},\ a_{33},\ \ldots,\ a_{3n},\ \ldots, \end{array}$$

which correspond to a sequence $\beta_1, \beta_2, \beta_3, \ldots$ of ordinal numbers of which $\gamma$ is the limiting number, by traversing the sequences diagonally, and thus $a_{11}, a_{22}, a_{33}, \ldots$ is the sequence which corresponds to $\gamma$.

Every sequence $b_1, b_2, b_3, \ldots$ is to be such that $b_1 < b_2 < b_3, \ldots$. It is pointed out by Hardy that an infinite freedom of choice arises in the case of each limiting number $\gamma$, since there are an indefinite number of sequences of the preceding ordinal numbers, of each of which sequences $\gamma$ is the limiting number. Thus, for example, $\omega^2$ is not only the limit of $\omega, \omega.2, \omega.3, \ldots$, but also of $\omega+1, \omega.2+2, \omega.3+3, \ldots$. In the case of the smaller limiting numbers of the second class, Hardy shews how to exercise this freedom of choice so as to obtain distinct sequences; thus $\omega$ is taken as the limit of $1, 2, 3, \ldots$; $\omega^2$ is taken as the limit of $\omega, \omega.2, \omega.3, \ldots$; in particular, $1, 5, 17, 49, 129, \ldots$ is obtained as the sequence corresponding to $\omega^2$.

In order that the method should really suffice to define a sequence which shall correspond uniquely to each prescribed number $\beta$ of the first or of the second class, it would be necessary to replace the freedom of choice by a definite norm, or finite set of rules, which would decide in the case of any particular limiting number $\gamma$, of which particular sequence of the preceding ordinal numbers it must be regarded as the limit, for the purpose of forming the sequence of integers which is to correspond to $\gamma$, in accordance with the mode of formation prescribed in (2).

Hardy has given no norm of this character, but confines himself to the selection of the sequences which are to correspond to some of the lower limiting numbers of the second class; thus he finds

| | |
|---|---|
| $(\omega^2.2)$ | $1, 17, \ldots,$ |
| $(\omega^3)$ | $1, 17, \ldots,$ |
| $(\omega^\omega)$ | $1, 5, \ldots,$ |
| $(\epsilon_1)$ | $1, 5, \ldots.$ |

When we reach the region of the $\epsilon$-numbers of the second class, it is difficult, if not impossible, to imagine the nature of the norm which would suffice to make the decision referred to above, and no such norm is in fact forthcoming. On this ground, the method cannot be regarded as really defining a set of points corresponding to all the numbers of the first and second classes.

Moreover, it can be shown that, if sequences such as those contemplated by Hardy are employed, the second figure $b_2$ would have indefinitely great values for numbers $\beta$ of the second class, and thus that for sufficiently great ordinal numbers of the second class the corresponding sequences can have no existence.*

Corresponding to $\omega^{\omega}$, Hardy finds the sequence 1, 5, $p$, $q$, ... ($p > 5$). With the mode of defining the sequences, as in (2), corresponding to higher limiting numbers, for all numbers $> \omega^{\omega}$, the second figure in the sequences will in no case be $< 5$, and, in fact, Hardy finds for the number $\epsilon_1$ defined as the limit of $\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots$, a sequence 1, 5, $p'$, $q'$, ... ($p' > 5$). For any number $> \epsilon_1$ the second figure will be not less than $p'$; thus $\epsilon_1 + 1$ has 5, $p'$, $q'$, ... for its sequence, and for $\epsilon_1 + \omega$ the second figure of the sequence is $p'$. It follows that, with the mode of formation contemplated, the second figure of the sequence corresponding to $\epsilon_2$, defined as the limit of $\epsilon_1, \epsilon_1^{\epsilon_1}, \epsilon_1^{\epsilon_1^{\epsilon_1}}, \ldots$, is not less than $p'$. Proceeding in this way, we see that for the numbers $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_n, \ldots$ the second figure of the sequence must continually increase as $n$ is increased. Now consider any sequence $k_1, k_2, \ldots, k_n, \ldots$ of increasing integers; there is a number $\beta_1$, from and after which the $a_2$ is always $> k_1$; again, there is a number $\beta_2$, from and after which the $a_2$ is always $> k_2$, and so on. The number $\beta_\omega$, which is also a number of the second class, being the limiting number of the sequence $\beta_1, \beta_2, \beta_3, \ldots$, is such that in the sequence of integers which corresponds with it, the $a_2$ is greater than $k_n$ for every value of $n$, and is therefore indefinitely great. There can therefore exist no sequence which corresponds to $\beta_\omega$; hence it is impossible to construct in the manner contemplated an unenumerable set of points. The point here discussed was not considered by Hardy, probably simply because he did not attempt to set up a definite norm for the sequences.

W. H. Young has employed† Bernstein's diagrammatic method, to correlate ordinal numbers of the second class with points of the continuum. The method does not, however, amount to the construction of a set of points of cardinal number $\aleph_1$, in accordance with the view maintained in the present communication as to the necessity for a definite norm which shall determine the correspondence.

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* Prof. W. Burnside communicated to me that he had convinced himself of the truth of this statement. I then devised the proof of its correctness given in the text.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 243.

18. In case the criticism which has been here given of the general theory of order types and of aleph-numbers be accepted as wholly, or in part, valid, the debt which mathematical science owes to the genius of G. Cantor will be in no respect diminished. The fundamental distinction between enumerable and unenumerable aggregates, the ordinal theory of the continuum, the theory of irrational numbers, with the interpretation of the arithmetical theory of limits, and the conception of the transfinite ordinal numbers of the second class, with their applications to the theory of sets of points, remain as permanent acquisitions which rest upon a firm logical basis. This order of ideas has already become indispensable for purposes of formulation in analysis and in geometry; it is constantly receiving new applications, owing to its admirable power in providing the language requisite for expressing results in the theory of functions with the highest degree of rigour and generality. Cantor's creations have rendered inestimable service in formulating the limitations to which many results in analysis, formerly supposed to be universally valid, are subject.

The outlying parts of the theory to which exception has been taken in the present communication, do not appear to be comparable in importance, for the general purposes of analysis, with those parts to which the criticisms made are inapplicable. The latter involve only a natural extension of the notion of numbers, in which account is taken of all the elements which are essential in the conception of number in its original form; whereas an endeavour has here been made to shew that the more speculative general theory of aleph-numbers and order types depends upon an extension of the notion of numbers which leaves out of account an essential element in that conception, viz., the notion of correspondence; and that this is the origin of the contradiction which ensues when an endeavour is made to contemplate the totality of these new entities.

The criticisms here given are advanced with some diffidence, on account of the great logical difficulties of the subject, and especially of the philosophical difficulties involved in questions relating to existential propositions. It is hoped, however, that the remarks made in this communication may, in any case, be of some utility as a contribution towards the discussion of questions of great interest which, at the present time, cannot be regarded as having been decisively settled.

# ON THE EVALUATION OF CERTAIN DEFINITE INTEGRALS BY MEANS OF GAMMA FUNCTIONS

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[Received May 2nd, 1904.—Read May 12th, 1904.—Received, in revised form, January 10th, 1905.]

THE following communication consists of two distinct parts. In Part I., I investigate a formula for the multiple integral

$$\int_{(n-1)} \frac{\xi_1^{\beta_1-1}\,\xi_2^{\beta_2-1}\ldots\xi_n^{\beta_n-1}}{\prod\limits_{t=1}^{t=m}\left(\sum\limits_{s=1}^{s=n} A_{st}\,\xi_s\right)^{\gamma_t}}\,d\xi_1\,d\xi_2\ldots d\xi_{n-1},$$

where $\xi_n = 1-\xi_1-\xi_2-\ldots-\xi_{n-1}$, and in which $\beta_s$, $\gamma_t$, $A_{st}$ are quantities which have their real parts positive, and in which, moreover, $\Sigma\beta-\Sigma\gamma$ is either zero or has its real part positive. This formula is modelled on Lejeune Dirichlet's formula

$$\int_{(n-1)} \xi_1^{\beta_1-1}\,\xi_2^{\beta_2-1}\ldots\xi_n^{\beta_n-1}\,d\xi_1\ldots d\xi_{n-1} = \frac{\prod\limits_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma(\Sigma\beta_s)},$$

both integrals being taken for all real positive values of $\xi_1, \xi_2, \ldots, \xi_{n-1}$ which make $\xi_n$ positive.

In Part II. I wish to find integrals which will take the place of the two given above in the cases when $\beta_s$ and $\gamma_t$ are not restricted to have their real parts positive. This second part is founded on Pochhammer's investigation[*] of a double circuit integral constructed to replace the Eulerian integral of the second kind $\int_0^1 x^{l-1}(1-x)^{m-1}\,dx$, in cases when the real parts of $l$ and $m$ are not positive.

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[*] *Math. Ann.*, Vol. xxxv., p. 495, "Zur Theorie der Euler'schen Integrale."

## I.

1. The multiple integral

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \prod_{t=1}^{t=m} \left(\sum_{s=1}^{s=n} A_{st}\xi_s\right)^{-\gamma_t} d\xi_1 d\xi_2 \dots d\xi_{n-1},$$

in which $$\xi_n = 1-\xi_1-\xi_2-\dots-\xi_{n-1},$$

and $$\sum_{s=1}^{s=n} \beta_s = \sum_{t=1}^{t=m} \gamma_t,$$

and in which $\beta_s$, $\gamma_t$, $A_{st}$ denote constants which have their real parts positive for all values of $s$ and $t$, taken over a field of integration for which $\xi_1, \xi_2, \dots, \xi_n$ are all real and positive, is equal to

$$\frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\prod_{t=1}^{t=m}\Gamma(\gamma_t)} \int_{(m-1)} \prod_{t=1}^{t=m} \theta_t^{\gamma_t-1} \prod_{s=1}^{s=n} \left(\sum_{t=1}^{t=m} A_{st}\theta_t\right)^{-\beta_s} d\theta_1 d\theta_2 \dots d\theta_{m-1},$$

in which $$\theta_m = 1-\theta_1-\theta_2-\dots-\theta_{m-1},$$

taken over a field of integration for which $\theta_1, \theta_2, \dots, \theta_m$ are all real and positive. In particular when $m = 1$,

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \left(\sum_{s=1}^{s=m} A_s\xi_s\right)^{-\Sigma\beta} d\xi_1 d\xi_2 \dots d\xi_{n-1} = \frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma\left(\sum_{s=1}^{s=n}\beta_s\right)} \prod_{s=1}^{s=n} A_s^{-\beta_s}.$$

2. Consider first the case when $m = 1$, and put $\gamma = \Sigma\beta_s$. In the integral

$$\int_{(n-1)} \prod_1^n \xi_s^{\beta_s-1} \left(\sum_1^n A_s\xi_s\right)^{-\gamma} d\xi_1 d\xi_2 \dots d\xi_{n-1},$$

make in the first place the transformation

$$\xi_s = \xi_n\eta_s \quad (s = 1, 2, \dots, n-1),$$

so that $$\xi_n = 1-\sum_1^{n-1}\xi_s = \left(1+\sum_1^{n-1}\eta_s\right)^{-1},$$

and $$\xi_s = \eta_s/(1+\Sigma\eta_s).$$

Then the limits of $\eta_s$ are 0 and $\infty$, and the transformed integral is

$$\int_0^\infty\int_0^\infty \cdots \int_0^\infty \prod_{s=1}^{s=n-1} \eta_s^{\beta_s-1} \left(A_n + \sum_{s=1}^{s=n} A_s\eta_s\right)^{-\gamma} d\eta_1 d\eta_2 \dots d\eta_{n-1},$$

the factor $(1+\Sigma\eta_s)^{\gamma-\Sigma\beta}$ disappearing, since $\gamma = \Sigma\beta$.

Now substitute for $\left(A_n+\sum\limits_1^{n-1} A_s\eta_s\right)^{-\gamma}$ by the formula

$$\left(A_n+\sum_1^{n-1} A_s\eta_s\right)^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty \zeta^{\gamma-1}\, e^{-\zeta(A_n+\Sigma A_s\eta_s)}\, d\zeta.$$

Then we shall get

$$\frac{1}{\Gamma(\gamma)}\int_0^\infty \cdots \int_0^\infty \prod_1^{n-1} \eta_s^{\beta_s-1} \zeta^{\gamma-1} e^{-\zeta(A_n+\Sigma A_s\eta_s)}\, d\eta_1 \ldots d\eta_{n-1}\, d\zeta.$$

In this change the order of integration,* and integrate with respect to the $\eta$ by the formula

$$\int_0^\infty \eta_s^{\beta_s-1} e^{-\eta_s A_s\zeta}\, d\eta_s = \Gamma(\beta_s)\, A_s^{-\beta_s} \zeta^{-\beta_s},$$

and we shall get
$$\frac{\prod\limits_{s=1}^{s=n-1} \Gamma(\beta_s) A_s^{-\beta_s}}{\Gamma(\gamma)} \int_0^\infty \zeta^{\beta_n-1} e^{-\zeta A_n}\, d\zeta,$$

since
$$\gamma - \sum_{s=1}^{s=n-1} \beta_s = \beta_n.$$

That is, we have, finally,

$$\frac{\prod\limits_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma\left(\sum\limits_{s=1}^{s=n} \beta_s\right)} \prod_{s=1}^{s=n} A_s^{-\beta_s}$$

as the value of the integral

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \left(\sum_{s=1}^{s=n} A_s\xi_s\right)^{-\Sigma\beta} d\xi_1\, d\xi_2 \ldots d\xi_{n-1}.$$

3. This result being established, let us apply it to the multiple integral

$$\int_{(n+m-2)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \prod_{t=1}^{t=m} \theta_t^{\gamma_t-1} \left(\sum_{s=1}^{s=n}\sum_{t=1}^{t=m} A_{st}\xi_s\theta_t\right)^{-\sigma} d\xi_1 \ldots d\xi_{n-1}\, d\theta_1 \ldots d\theta_{m-1}$$

where
$$\sigma = \sum_{s=1}^{s=n} \beta_s = \sum_{t=1}^{t=m} \gamma_t$$

and
$$\sum_{s=1}^{s=n} \xi_s = 1, \quad \sum_{t=1}^{t=m} \theta_t = 1,$$

the field of integration being for all real positive values of

$$\xi_1,\ \xi_2,\ \ldots,\ \xi_{n-1},\ \xi_n,\ \theta_1,\ \theta_2,\ \ldots,\ \theta_{m-1},\ \theta_m.$$

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* For a justification of this see later, § 5.

Since $A_{st}$ has its real part positive for all values of $s$ and $t$, $\sum\limits_{s=1}^{s=n} A_{st}\xi_s$ and $\sum\limits_{t=1}^{t=m} A_{st}\theta_t$ also have their real parts positive, and so we get, integrating first with regard to the $\theta$, and secondly with regard to the $\xi$, that

$$\prod_{t=1}^{t=m}\Gamma(\gamma_t)\int_{(n-1)}\prod_{s=1}^{s=n}\xi_s^{\beta_s-1}\prod_{t=1}^{t=m}\left(\sum_{s=1}^{s=n}A_{st}\xi_s\right)^{-\gamma_t}d\xi_1\,d\xi_2\dots d\xi_{n-1}$$

is equal to

$$\prod_{s=1}^{s=n}\Gamma(\beta_s)\int_{(m-1)}\prod_{t=1}^{t=m}\theta_t^{\gamma_t-1}\prod_{s=1}^{s=n}\left(\sum_{t=1}^{t=m}A_{st}\theta_t\right)^{-\beta_s}d\theta_1\,d\theta_2\dots d\theta_{m-1}.$$

4. Another formula of the same type is obtained by integrating

$$e^{-(\Sigma A_r x_r+\Sigma B_s y_s+\Sigma\Sigma C_{rs}x_r y_s)}\prod_{r=1}^{r=n}x_r^{\beta_r-1}\prod_{s=1}^{s=m}y_s^{\gamma_s-1}$$

first with regard to the $x$, and then with regard to the $y$, and equating the two results obtained. In this way we get the equality

$$\prod_{r=1}^{r=n}\Gamma(\beta_r)\int_0^\infty\int_0^\infty\dots\int_0^\infty\frac{\prod\limits_{s=1}^{s=m}e^{-B_s y_s}y_s^{\gamma_s-1}}{\prod\limits_{r=1}^{r=n}(A_r+\Sigma_s C_{rs}y_s)^{\beta_r}}\,dy_1\,dy_2\dots dy_m$$

$$=\prod_{s=1}^{s=m}\Gamma(\gamma_s)\int_0^\infty\int_0^\infty\dots\int_0^\infty\frac{\prod\limits_{r=1}^{r=n}e^{-A_r x_r}x_r^{\beta_r-1}}{\prod\limits_{s=1}^{s=m}(B_s+\Sigma_r C_{rs}x_r)^{\gamma_s}}\,dx_1\,dx_2\dots dx_n,$$

in which $\beta_r$, $\gamma_s$, $A_r$, $B_s$, $C_{rs}$ are constants which have their real parts positive for all values of $s$ and $t$, and the field of integration in both integrals is for all positive real values of the variables.

5. It is necessary to justify the inversion of the order of integration in the multiple integral with infinite limits.

Taking the integral just considered in § 4, I suppose the limits of $x_r$ to be 0 to $h_r$ and of $y_s$ to be 0 to $k_s$, where $h_r$ and $k_s$ are real positive quantities; then it is sufficient to show that, as $h_r$ and $k_s$ are indefinitely increased, in any order and in any manner, the integral

$$\int_{h_1}^\infty\int_{h_2}^\infty\dots\int_{h_n}^\infty\int_{k_1}^\infty\int_{k_2}^\infty\dots\int_{k_m}^\infty e^{-V}\prod_r x_r^{\beta_r-1}\prod_s y_s^{\gamma_s-1}\,dx_1\dots dx_n\,dy_1\dots dy_m,$$

where $$V=\sum_r A_r x_r+\sum_s B_s y_s+\sum_r\sum_s C_{rs}x_r y_s,$$

can be made less than any assignable quantity.

In order to include the integral in § 2, zero values of $A_r$ and $B_s$ will ot be excluded from consideration. This we may prove as follows:— hroughout the range of integration we have

$$x_r > h_r \quad \text{and} \quad y_s > k_s,$$

nd therefore
$$x_r y_s > \tfrac{1}{2}(h_r y_s + k_s x_r),$$

nd so
$$V > \sum_r x_r(A_r + \tfrac{1}{2}\sum_s C_{rs} k_s) + \sum_s y_s(B_s + \tfrac{1}{2}\sum_r C_{rs} h_r);$$

that the above integral is less than

$$\prod_{r=1}^{r=n} \int_{h_r}^{\infty} e^{-x_r(A_r+\frac{1}{2}\sum_s C_{rs}k_s)} x_r^{\beta_r-1}\, dx_r \prod_{s=1}^{s=m} \int_{k_s}^{\infty} e^{-y_s(B_s+\frac{1}{2}\sum_r C_{rs}h_r)} y_s^{\gamma_s-1}\, dy_s.$$

This is a product of factors which tend to zero separately and inde- endently as $h_r$ and $k_s$ are indefinitely increased, and so the inversion of e order of integration is justified.

6. If in the formula of § 1 we take $A_{sm} = 1$ for all values of $s$, we hall have $\sum_s A_{sm}\xi_s = \sum_s \xi_s = 1$, and so, changing $m-1$ into $m$, we shall et the following result:—

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \prod_{t=1}^{t=m} \left(\sum_{s=1}^{s=n} A_{st}\xi_s\right)^{-\gamma_t} d\xi_1 d\xi_2 \dots d\xi_{n-1},$$

which
$$\xi_n = 1-\xi_1-\xi_2-\dots-\xi_{n-1},$$

nd $\beta_s$, $\gamma_t$, $A_{st}$ denote constants which have their real parts positive for all alues of $s$ and $t$, with the condition that the real part of $\Sigma\beta-\Sigma\gamma$ is ositive, taken over a field of integration for which $\xi_1, \xi_2, \dots, \xi_n$ are all eal and positive, is equal to

$$\frac{\prod_{s=1}^{s=n} \Gamma(\beta_s)}{(\Sigma\beta-\Sigma\gamma) \prod_{t=1}^{t=m} \Gamma(\gamma_t)} \int_{(m)} \prod_{t=1}^{t=m+1} \theta_t^{\gamma_t-1} \prod_{s=1}^{s=n} \left(\theta_{m+1} + \sum_{t=1}^{t=m} A_{st}\theta_t\right)^{-\beta_s} d\theta_1 d\theta_2 \dots d\theta_m,$$

here
$$\theta_{m+1} = 1 - \sum_{t=1}^{t=m} \theta_t \quad \text{and} \quad \gamma_{m+1} = \sum_{s=1}^{s=n} \beta_s - \sum_{t=1}^{t=m} \gamma_t,$$

he field of integration being for all positive real values of

$$\theta_1,\ \theta_2,\ \dots,\ \theta_{m+1}.$$

By making the substitution $\theta_t = \theta'\theta_{m+1}$, this last expression may be written in the equivalent form

$$\frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma(\Sigma\beta-\Sigma\gamma)\prod_{t=1}^{t=m}\Gamma(\gamma_t)}\int_0^\infty\int_0^\infty\dots\int_0^\infty\prod_{t=1}^{t=m}\theta_t'^{\gamma_t-1}\prod_{s=1}^{s=n}\left(1+\sum_{t=1}^{t=m}A_{st}\theta'\right)^{-\beta_s}$$

$$\times\, d\theta_1'\, d\theta_2' \dots d\theta_m'.$$

In particular, when $m = 1$ we have that

$$\int_{(n-1)}\prod_{s=1}^{s=n}\xi_s^{\beta_s-1}\{\Sigma A_s\xi_s\}^{-\gamma}d\xi_1\,d\xi_2\dots d\xi_{n-1}$$

is equal to
$$\frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma(\Sigma\beta-\gamma)\,\Gamma(\gamma)}\int_0^1\frac{\theta^{\gamma-1}(1-\theta)^{\Sigma\beta-\gamma-1}d\theta}{\prod_{s=1}^{s=n}(1-\theta+A_s\theta)^{\beta_s}},$$

which may also be written

$$\frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma(\Sigma\beta-\gamma)\,\Gamma(\gamma)}\int_0^\infty\frac{\theta^{\gamma-1}d\theta}{\prod_{s=1}^{s=n}(1+A_s\theta)^{\beta_s}}.$$

7. Generalisations of the known formulæ for definite integrals of Lejeune Dirichlet's type, in which $\Phi(\xi_1+\xi_2+\dots\xi_{n-1})$ takes the place of $\xi_n^{\beta_n-1}$, are easily obtained from the fundamental formulæ of § 3. Thus, for example,

$$\prod_{t=1}^{t=m}\Gamma(\gamma_t)\int_{(n)}\frac{\Phi\left(\sum_{s=1}^{s=n}\xi_s\right)\prod_{s=1}^{s=n}\xi_s^{\beta_s-1}}{\prod_{t=1}^{t=m}\left(A+\sum_{s=1}^{s=n}A_{st}\xi_s\right)^{\gamma_t}}\,d\xi_1\,d\xi_2\dots d\xi_n$$

is equal to
$$\prod_{s=1}^{s=n}\Gamma(\beta_s)\int_{(m)}\frac{\Phi\left(\sum_{t=1}^{t=m}\theta_t\right)\prod_{t=1}^{t=m}\theta_t^{\gamma_t-1}}{\prod_{s=1}^{s=n}\left(A+\sum_{t=1}^{t=m}A_{st}\theta_t\right)^{\beta_s}}\,d\theta_1\,d\theta_2\dots d\theta_m,$$

where
$$\Sigma\beta_s = \Sigma\gamma_t\,;$$

the field of integration for both integrals being over all real positive values of the variables, such that their sum is less than $h$ (*i.e.*, we have $\Sigma\xi_s \not> h$ and $\Sigma\theta_t \not> h$) and $\beta_s$, $\gamma_t$, $A+A_{st}h$ being constants which have their real parts positive for all values of $s$ and $t$.

To prove this it is only necessary to make the substitutions $\Sigma\xi_s = t$, $\xi_s = tx_s$, and $\Sigma\theta_t = u$, $\theta_t = uy_t$, so that $\Sigma x_s = 1$ and $\Sigma y_t = 1$.

If in this formula we take $m = 2$, write $\gamma$ for $\gamma_1$, and put $A_{s2} = 0$ for all values of $s$, we shall get that

$$\int_{(n)} \frac{\Phi\left(\sum\limits_{s=1}^{s=n} \xi_s\right) \prod\limits_{s=1}^{s=n} \xi_s^{\beta_s-1}}{\left(A + \sum\limits_{s=1}^{s=n} A_s \xi_s\right)^{\gamma}} \, d\xi_1 \, d\xi_2 \ldots d\xi_n$$

is equal to

$$\frac{A^{\Sigma\beta-\gamma} \prod\limits_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma(\gamma)\,\Gamma(\Sigma\beta-\gamma)} \int_0^h d\theta_1 \, \frac{\theta_1^{\gamma-1}}{\prod\limits_{s=1}^{s=n} (A+A_s\theta_1)^{\beta_s}} \int_0^{h-\theta_1} d\theta_2 \, \Phi(\theta_1+\theta_2)\, \theta_2^{\Sigma\beta-\gamma-1}$$

when the real part of $\Sigma\beta-\gamma$ is positive.

## II.

8. In a paper in the *Mathematische Annalen*, Vol. xxxv., p. 495, Pochhammer has investigated a double circuit integral, constructed to take the place of the Eulerian integral of the first kind,

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx,$$

when the real parts of $a$ and $b$ are not restricted to be positive.

The double circuit integral in question is

$$\int^{(1+,\,0+,\,1-,\,0-)} x^{a-1}(1-x)^{b-1}\,dx,$$

by which is denoted an integral taken along a path which starts from some point $c$, lying on the real axis between 0 and 1, makes a circuit round 1 in the positive direction, and returns to $c$, then a circuit round 0 in the positive direction in the same way, and then circuits round 1 and 0, respectively, in the negative direction so that the integrand returns to $c$ with its initial value.

The value of this double circuit integral is independent of the position of $c$, and we may suppose the path of the variable to be the real axis except at points in the immediate neighbourhood of the points 0 and 1, where we may suppose it to describe small circles about 0 and 1.

Now when the real parts of $a$ and $b$ are positive the values of the integral round these small circles become infinitely small at the same time as the radii of the circles become infinitely small, and the circles may be reduced to their centres so that the integral is reduced to one taken along the real axis between the points 0 and 1. Moreover the integrand is multiplied by $e^{2\pi i a}$ when a circuit is made round 0, and by $e^{2\pi i b}$ when a circuit is made round 1.

Denoting this integral by $e^{\pi i (a+b)}\,\mathfrak{E}(a, b)$, Pochhammer shows that

$$\mathfrak{E}(a+1, b) = -\frac{a}{a+b}\,\mathfrak{E}(a, b),$$

$$\mathfrak{E}(a, b+1) = -\frac{b}{a+b}\,\mathfrak{E}(a, b);$$

and further that when the real parts of $a$ and $b$ are positive

$$e^{\pi i (a+b)}\,\mathfrak{E}(a, b) = (e^{2\pi i a}-1)(e^{2\pi i b}-1)\int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

Leaving out of consideration the cases in which $a$ or $b$ is an integer, for which some modification is necessary, it is easy to see that these equations establish the formula

$$\int^{(1+,\,0+,\,1-,\,0-)} x^{a-1}(1-x)^{b-1}\,dx = (e^{2\pi i a}-1)(e^{2\pi i b}-1)\,\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)},$$

for all values of $a$ and $b$.

This particular double circuit integral is, of course, not the only one which can be so used; any circuit which encloses each of the points 0 and 1, the same number of times positively and negatively, and for which, when the real parts of $a$ and $b$ are positive, the integral reduces to

$$f(e^{2\pi i a}, e^{2\pi i b})\int_0^1 x^{a-1}(1-x)^{b-1}\,dx,$$

where $f$ is a rational integral function, will do as well. Thus, for instance, if the circuit $(1+,\ 0+,\ 1-,\ 0-)$ be denoted by $C$, the circuit $1+,\ C,\ 1-,\ C$ will make $f(e^{2\pi i a}, e^{2\pi i b})$ equal to $(e^{2\pi i b}-1)^2(e^{2\pi i a}-1)$; the circuit $1+,\ C,\ 1-$ will make $f$ equal to $e^{2\pi i b}(e^{2\pi i b}-1)(e^{2\pi i a}-1)$; and the circuit $1+,\ 0+,\ C,\ 0-,\ 1-,\ C-$ will make $f$ equal to

$$(e^{2\pi i (a+b)}-1)(e^{2\pi i b}-1)(e^{2\pi i a}-1).$$

9. Consider the integral

$$\int_{(n-1)} x_1^{\beta_1-1} x_2^{\beta_2-1} \ldots x_{n-1}^{\beta_{n-1}-1} x_n^{\beta_n-1}\,dx_1\,dx_2 \ldots dx_{n-1},$$

where $$x_n \equiv 1-x_1-x_2-\ldots-x_{n-1},$$

and the field of integration is such that each of the boundaries $x_1=0$, $x_2=0$, ..., $x_n=0$ is enclosed by the same number of positive and negative circuits, these circuits being drawn so as not to meet any of the boundaries, so that after the field has been covered each factor of the integrand returns to its original value, and, moreover, is such that, when $\beta_1, \beta_2, \ldots$ have their real parts positive, the integral can be reduced to one taken throughout the content bounded by $x_1=0$, $x_2=0$, ..., $x_n=0$, where $x_1, x_2, \ldots,$ are real and positive. Since the integrand is multiplied by a factor $e^{2\pi i\beta_r}$ when a circuit is made round $x_r=0$, the value of the integral, when $\beta_1, \beta_2, \ldots$ have their real parts positive, will be

$$f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \ldots)\int_{(n-1)} x_1^{\beta_1-1}\ldots dx_1\,dx_2\ldots dx_{n-1},$$

taken throughout the content bounded by $x_1=0$, $x_2=0$, ..., that is, it will be

$$f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \ldots)\frac{\prod\limits_{r=1}^{r=n}\Gamma(\beta_r)}{\Gamma\left(\sum\limits_{r=1}^{r=n}\beta_r\right)},$$

$f$ being the appropriate rational integral function which belongs to the particular field chosen, and I wish to show that the value of the integral will be given by the same formula for all values of $\beta_1, \beta_2, \ldots$.

The form of the field of integration will in general be independent of the values of $\beta_1, \beta_2, \ldots,$ but modifications are necessary if any of them are positive integers, in order to avoid the occurrence of zero factors in $f$. If $\beta_r$ is a positive integer, the field of integration will start from and end in $x_r=0$, just as in the case of Pochhammer's integrals $\int x^{a-1}(1-x)^{b-1}\,dx$, in which, if $a$ is a positive integer, the double circuit is replaced by a single contour which starts from 0, makes a circuit round 1, and returns to 0 again. Negative integral values of $\beta_r$ are excluded from consideration.

10. One such field of integration is obtained, and, moreover, the formula is proved for the particular case by applying to the integral

$$\int_{(n-1)}\prod_{s=1}^{s=n} x_s^{\beta_s-1}\,dx_1\,dx_2\ldots dx_{n-1}$$

the transformation which is commonly used to obtain its value when $\beta_s$ is restricted to be positive for all values of $s$.

If we put

$$\begin{aligned} x_1 &= 1-v_1, \\ x_2 &= v_1(1-v_2), \\ x_3 &= v_1 v_2(1-v_3), \\ &\dots \quad \dots \quad \dots \quad \dots \\ x_{n-1} &= v_1 v_2 \dots v_{n-2}(1-v_{n-1}), \\ x_n &= v_1 v_2 \dots v_{n-2} v_{n-1}, \end{aligned}$$

$$\int_{(n-1)} \prod_{s=1}^{s=n} x_s^{\beta_s-1}\, dx_1\, dx_2 \dots dx_{n-1}$$

is transformed into

$$\int (1-v_1)^{\beta_1-1} v_1^{\beta_2+\beta_3+\dots+\beta_n-1}\, dv_1 \int (1-v_2)^{\beta_2-1} v_2^{\beta_3+\beta_4+\dots+\beta_n-1}\, dv_2 \dots$$
$$\times \int (1-v_{n-1})^{\beta_{n-1}-1} v_{n-1}^{\beta_n-1}\, dv_{n-1},$$

and now, if we suppose all the variables $v$ to make double circuits about the points 0 and 1, so that each factor of the above expression is a Pochhammer integral, we get as the value of the expression

$$(e^{2\pi i\beta_1}-1)(e^{2\pi i(\beta_2+\beta_3+\dots+\beta_n)}-1)(e^{2\pi i\beta_2}-1)(e^{2\pi i(\beta_3+\dots+\beta_n)}-1)\dots \frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma\left(\sum_{s=1}^{s=n}\beta_s\right)}.$$

The original variables $x$ will in this case describe a closed continuum of $(n-1)$ dimensions in a space of $2(n-1)$ dimensions, such that each of the boundaries $x_r = 0$ is enclosed by the same number of positive and negative circuits, because this is true of the variables $v$, and, moreover, this continuum will not meet any of the boundaries. Also any factor $x_s^{\beta_s-1}$ of the integrand will return to its original value after the continuum has been covered, since this is true of any of its factors $v_1^{\beta_s-1} v_2^{\beta_s-1} \dots (1-v_s)^{\beta_s-1}$. Further, when the real part of $\beta_s$ is positive for all values of $s$, each of the double circuits can be reduced to the part of the real axis between the points 0 and 1, described four times with different values of the integrand, as explained by Pochhammer; so that the continuum described by the variables $x$ can be reduced to that bounded by $x_1 = 0$, $x_2 = 0$, ..., where all the variables may be taken to be real and positive. This field of integration, therefore, is such a one as has been described, and the value of the function $f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots)$ for this particular field is

$$(e^{2\pi i\beta_1}-1)(e^{2\pi i(\beta_2+\beta_3+\dots+\beta_n)}-1)(e^{2\pi i\beta_2}-1)(e^{2\pi i(\beta_3+\dots+\beta_n)}-1)\dots$$
$$\times (e^{2\pi i\beta_{n-1}}-1)(e^{2\pi i\beta_n}-1).$$

If any of the $\beta$ are positive integers, it is necessary to modify the field of integration. As the order of the letters is immaterial, we may suppose all the positive integral indices to come at the end; thus, if $\beta_n$ is a positive integer, the path of the variable $v_{n-1}$ is a contour which starts from 0 and makes a circuit round 1 and returns to 0*; if $\beta_{n-1}$ and $\beta_n$ are both positive integers, $v_{n-2}$ makes the circuit just mentioned, while the path of $v_{n-1}$ is the real axis from 0 to 1.

11. It is also possible to construct such surfaces geometrically. Consider the double integral

$$\iint x_1^{\beta_1-1} x_2^{\beta_2-1} x_3^{\beta_3-1}\, dx_1\, dx_2,$$

where

$$x_3 \equiv 1 - x_1 - x_2.$$

If we put $x_1 = y_1 + z_1 \iota$, $x_2 = y_2 + z_2 \iota$, then $y_1$, $z_1$, $y_2$, $z_2$ are the coordinates of a point in four-dimensional space. The boundaries are the planes $y_1 = 0$, $z_1 = 0$; $y_2 = 0$, $z_2 = 0$; $y_1 + y_2 = 1$, $z_1 + z_2 = 0$. When the real parts of $\beta_1$, $\beta_2$, and $\beta_3$ are positive, the integration is over the area of the triangle in the plane $z_1 = 0$, $z_2 = 0$ which is bounded by $y_1 = 0$, $y_2 = 0$, $y_1 + y_2 = 1$.

Let this triangle be $OAB$ in the figure. Take a point $P_1$ in the

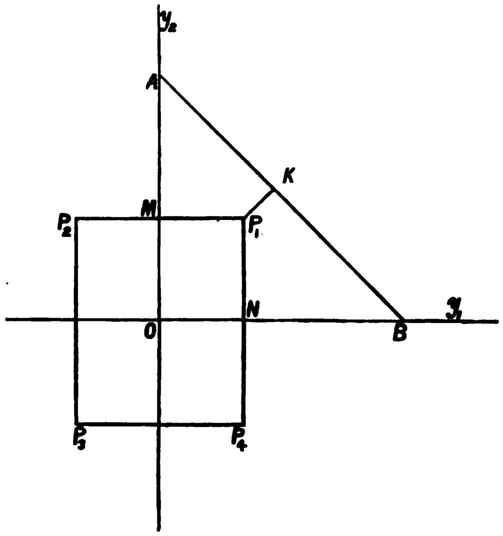

FIG. 1.

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[*] *Cf.* Pochhammer, *loc. cit.*, p. 510.

triangle whose co-ordinates are $(a, b, 0, 0)$; and its images $P_2$, $P_3$, $P_4$ in the other three quadrants.

Consider the surface whose equations are

$$y_1^2+z_1^2 = a^2, \qquad y_2^2+z_2^2 = b^2.$$

This can be represented by two circles, a point on the surface being represented by two points taken together, one on each circle. Thus the points $P_1$, $P_2$, $P_3$, $P_4$ in Fig. 1 are represented respectively by the pairs $(A_1 A_2)$, $(A_1' A_2)$, $(A_1' A_2')$, $(A_1 A_2')$.

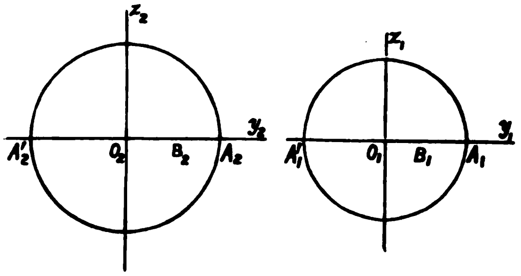

Fig. 2.

This surface can be described in four distinct ways, since the circles $O_1$ and $O_2$ may each be described either in a positive or negative direction. If, starting from $(A_1, A_2)$, we describe each circle in the positive direction, $x_1$ and $x_2$ will each have taken the factor $e^{2\pi i}$, and $x_1^{\beta_1-1}x_2^{\beta_2-1}$, therefore, the factor $e^{2\pi i(\beta_1+\beta_2)}$, and the whole surface will have been described in a positive sense. If, now, we describe each circle in the negative sense, the whole surface will have been described positively again, but $x_1$ and $x_2$ will each have taken the factor $e^{-2\pi i}$, and $x_1^{\beta_1-1}x_2^{\beta_2-1}$ will have returned to its original value.*

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* This may be illustrated by putting $\beta_3 = 1$, $y_1 = a\cos\theta$, $z_1 = a\sin\theta$, $y_2 = b\cos\phi$, $z_2 = b\sin\phi$. The integral $\iint x_1^{\beta_1-1}x_2^{\beta_2-1}\,dx_1\,dx_2$ becomes in this case $-a^{\beta_1}b^{\beta_2}\iint e^{\beta_1 i\theta}e^{\beta_2 i\phi}\,d\theta\,d\phi$, and its four values will be $-a^{\beta_1}b^{\beta_2}(e^{\pm 2\pi i\beta_1}-1)(e^{\pm 2\pi i\beta_2}-1)$ according to the directions in which the circles round $O_1$ and $O_2$ are described, and the integrand will return to its starting point $(A_1, A_2)$ with the corresponding values $a^{\beta_1-1}b^{\beta_2-1}e^{2\pi i(\pm\beta_1\pm\beta_2)}$. If, starting with these values as the initial value of the integrand, the surface were described in the negative sense, that is if the integral were taken over the opposite side of the surface—or over a *variété opposée*—the value of the integral would be changed in sign.

In the case of an anchor ring in ordinary space of three dimensions, we have similarly a surface which has an inside and an outside, and in which two sets of circles drawn on it, viz., those about the axis of revolution and those about the circular axis, can be described independently in either direction.

Let such a double description of the surface be denoted by $C_{12}$, the surface itself being called $S_{12}$.

We may deform this surface $S_{12}$ into the surface represented by two contours, one of which starting from $A_1$ describes the line $A_1O_1$ to some point near to $O_1$, then makes a small circle about $O_1$, and returns to $A_1$ on the other side of the axis $O_1A_1$; the other being a similar contour starting from $A_2$, and making a small circle about $O_2$. Then corresponding to any point $Q$ in the area $OMP_1N$ we shall have four points on this deformed surface, viz., if the point $Q$ is represented by the two points $B_1$ and $B_2$, and if the points infinitesimally near to $B_1$ and $B_2$ on opposite sides of $O_1A_1$ and $O_2A_2$ respectively are denoted by $B_1+$, $B_1-$ and $B_2+$, $B_2-$, we have corresponding to $Q$ the four points represented by $(B_1\pm,\ B_2\pm)$.

If $x_1^{\beta_1-1}x_2^{\beta_2-1}$ start from $P$, with its natural value, viz.,

$$\exp\left[(\beta_1-1)\log a+(\beta_2-1)\log b\right],$$

where $\log a$ and $\log b$ are real, and if its value at $(B_1+,\ B_2+)$ be $x_1^{\beta_1-1}x_2^{\beta_2-1}$, then its value at $(B_1-,\ B_2+)$ will be $e^{2\pi\iota\beta_1}x_1^{\beta_1-1}x_2^{\beta_2-1}$, at $(B_1+,\ B_2-)$ will be $e^{2\pi\iota\beta_2}x_1^{\beta_1-1}x_2^{\beta_2-1}$, and at $(B_1-,\ B_2-)$ will be $e^{2\pi\iota(\beta_1+\beta_2)}x_1^{\beta_1-1}x_2^{\beta_2-1}$, and, if we consider the area $OMP_1N$ to be described positively as the describing point is passing through $(B_1+,\ B_2+)$, it will be described negatively at $(B_1+,\ B_2-)$ and $(B_1-,\ B_2+)$, but positively at $(B_1-,\ B_2-)$.

So that, when the real parts of $\beta_1$ and $\beta_2$ are positive and the circles round $O_1$ and $O_2$ may be reduced to their centres, we have that the integral over the surface $S_{12}$ is equivalent to $(e^{2\pi\iota\beta_1}-1)(e^{2\pi\iota\beta_2}-1)$ times the integral over the area $OMP_1N$. Thus the integral over $C_{12}$ is equivalent to $2(e^{2\pi\iota\beta_1}-1)(e^{2\pi\iota\beta_2}-1)$ times the integral over $OMP_1N$.

An exactly similar field of integration enclosing the corner $A$, viz., that over the surface $S_{23}$ whose equations are

$$y_2^2+z_2^2=b^2, \qquad y_3^2+z_3^2=c^2,$$

where

$$y_3+\iota z_3\equiv 1-y_1-y_2-\iota(z_1+z_2),$$

will be denoted by $C_{23}$, and over a similar surface enclosing the corner $B$ by $C_{31}$.

In order that when $z_1$ and $z_2$ are taken infinitesimally small, the whole area of the triangle $OAB$, in Fig. 1, may be covered, these surfaces must be so deformed that each meets the others in a single line. That this can be done may be seen by moving $B$ out to a very great distance along $Oy_1$; then the two surfaces $S_{12}$ and $S_{23}$ become

$$y_1^2+z_1^2 = a^2, \qquad y_2^2+z_2^2 = b^2,$$

and
$$(1-y_1)^2+z_1^2 = (1-a)^2, \qquad y_2^2+z_2^2 = b^2,$$

which meet one another along the circle

$$y_1 = a, \qquad z_1 = 0, \qquad y_2^2+z_2^2 = b^2,$$

and by continuous deformation we may bring $B$ back again, and still have the two surfaces meeting one another along a single line. We may suppose the portions of the triangle $OAB$ which correspond to the three fields of integration $C_{12}$, $C_{23}$, $C_{31}$ to be $OMP_1N$, $MAKP_1$, $KBNP_1$ respectively, and now, if we assume the element of area which describes the field to start from $P_1$, then move along the line $P_1K$, make a circuit round $AB$ at $K$, and return to $P_1$—so that the integrand has taken the factor $e^{2\pi i\beta_3}$—then to describe $C_{12}$ in a positive sense, then the circuit round $K$ in a negative sense, and finally $C_{12}$ in a negative sense, then we shall have as the equivalent of the integral over this field, when the real parts of $\beta_1$ and $\beta_2$ are positive,

$$2(e^{2\pi i\beta_3}-1)(e^{2\pi i\beta_2}-1)(e^{2\pi i\beta_1}-1)$$

times the integral over the area $OMP_1N$ in the plane $z_1 = 0$, $z_2 = 0$. The integrals along the positive and negative circuits around $K$ destroy one another, since the integrand returns to its original value after $C_{12}$ has been described.

If we construct similar fields corresponding to the other two regions $AMP_1K$ and $BNP_1K$, these three taken together will give a field of integration such as is required, and the factor $f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, e^{2\pi i\beta_3})$ corresponding to it will be

$$2(e^{2\pi i\beta_1}-1)(e^{2\pi i\beta_2}-1)(e^{2\pi i\beta_3}-1).$$

If $\beta_3$ is a positive integer, it is necessary to avoid the factor $(e^{2\pi i\beta_3}-1)$. The field of integration may be taken to be over the surface $S_{12}$, so deformed that $P_1M$ and $P_1N$ lie along $KA$ and $KB$ respectively, or rather over the part of such a surface cut off by a boundary curve, which lies in the plane $y_1+y_2 = 1$, $z_1+z_2 = 0$, and encloses the line $AB$ of Fig. 1.

12. I now propose to show that for any field of integration of this sort the value of the integral

$$\int_{(n-1)} x_1^{\beta_1-1} x_2^{\beta_2-1} \dots x_n^{\beta_n-1}\, dx_1\, dx_2 \dots dx_{n-1},$$

where $$x_n = 1-x_1-x_2-\dots-x_{n-1},$$

is $$f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots)\frac{\prod\limits_{r=1}^{r=n}\Gamma(\beta_r)}{\Gamma(\Sigma\beta_r)}$$

for all values of $\beta_1, \beta_2, \dots$ (negative integral values excluded), $f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots)$ being the appropriate function for the field chosen.

This I shall do by the method of induction, so that the formula, being true for any values of $\beta_1, \beta_2, \dots$, which have their real parts all positive, is also true for any values of $\beta_1, \beta_2, \dots$ whatever.

For shortness I write $F$ for the integral

$$\int_{(n-1)} x_1^{\beta_1-1}x_2^{\beta_2-1}\dots x_n^{\beta_n-1}\,dx_1\,dx_2\dots dx_{n-1},$$

taken over a field of integration such that each factor of the integrand $x_r^{\beta_r-1}$ returns to its initial value after the field has been covered.

Further, I write $F_r$ for the integral obtained by replacing $\beta_r$ in $F$ by $\beta_r+1$. Then, since

$$\begin{aligned}\frac{d}{dx_r}(x_r^{\beta_r}x_n^{\beta_n}) &= \beta_r x_r^{\beta_r-1}x_n^{\beta_n}-\beta_n x_r^{\beta_r}x_n^{\beta_n-1}\\ &= x_r^{\beta_r-1}x_n^{\beta_n-1}\{\beta_r(1-x_1-x_2-\dots-x_{n-1})-\beta_n x_r\},\end{aligned}$$

I get, on multiplying by $x_1^{\beta_1-1}x_2^{\beta_2-1}\dots x_{r-1}^{\beta_{r-1}-1}x_{r+1}^{\beta_{r+1}-1}\dots x_{n-1}^{\beta_{n-1}-1}$ and integrating throughout the field, the two relations

$$0 = \beta_r F_n-\beta_n F_r, \tag{1}$$

$$0 = \beta_r F-\beta_r F_1-\beta_r F_2-\dots-(\beta_r+\beta_n)F_r-\dots-\beta_r F_{n-1}, \tag{2}$$

the left-hand side vanishing since $x_r^{\beta_r}x_n^{\beta_n}$ returns to its initial value.*

In these equations $r$ may have any one of the values $1, 2, \dots, (n-1)$, and, if we write down the series of $n-1$ equations so obtained from (2), and solve them, we get at once

$$F_r = (\beta_r/\Sigma\beta)F \quad (r = 1, 2, 3, \dots, n).$$

13. By the help of this relation the evaluation of the integral

$$\int_{(n-1)}\prod_{s=1}^{s=n} x_s^{\beta_s-1}\,dx_1\dots dx_{n-1}$$

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* Or, if $\beta_r$ is a positive integer, has zero for its initial and final values.

is reduced to that of another integral of the same form in which $\beta_r$ can be replaced by $\beta_r+n$, where $n$ is a positive integer, that is, finally, to the evaluation of the integral when all the $\beta$ have their real parts positive, and so the result is established that, if the value of

$$\int_{(n-1)} \prod_{s=1}^{s=n} x_s^{\beta_s-1}\, dx_1 \ldots dx_{n-1}$$

taken over a suitable field is

$$f(e^{2\pi\iota\beta_1}, e^{2\pi\iota\beta_2}, \ldots)\, \frac{\prod\limits_{r=1}^{r=n} \Gamma(\beta_r)}{\Gamma(\Sigma\beta_r)},$$

when the real parts of $\beta_1, \beta_2, \ldots$ are positive, it is given by the same formula for all values of $\beta_1, \beta_2, \ldots$.

14. The same reasoning may be applied to extend to all values of $\beta_s$ and $\gamma_t$—with the exception of negative integral values—the formula proved in Part I., § 3, of this paper.

For, if we have

$$\prod_{t=1}^{t=m} \Gamma(\gamma_t)\phi(e^{2\pi\iota\gamma_1}, e^{2\pi\iota\gamma_2}, \ldots) \int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \prod_{t=1}^{t=m} \left(\sum_{s=1}^{s=n} A_{st}\xi_s\right)^{-\gamma_t} d\xi_1 d\xi_2 \ldots d\xi_{n-1}$$

$$= \prod_{s=1}^{s=n} \Gamma(\beta_s) f(e^{2\pi\iota\beta_1}, e^{2\pi\iota\beta_2}, \ldots) \int_{(m-1)} \prod_{t=1}^{t=m} \theta_t^{\gamma_t-1} \prod_{s=1}^{s=n} \left(\sum_{t=1}^{t=m} A_{st}\theta_t\right)^{-\beta_s}$$

$$\times d\theta_1 d\theta_2 \ldots d\theta_{m-1},$$

the integrals being taken over suitable fields, and $f$ and $\phi$ being the rational integral functions corresponding to the fields of integration which have been chosen, then we shall get on integrating each side with respect to some one coefficient, say $A_{s't'}$, the same formula with $\beta_{s'}$ and $\gamma_{t'}$ changed into $\beta_{s'}-1$ and $\gamma_{t'}-1$ respectively. That the constant of integration introduced in this way is zero may be seen as follows. It is not a function of $A_{s't'}$, and therefore it cannot be a function of $A_{st}$ for any values of $s$ or $t$, since the integrals are unaltered, if, throughout them both, any two suffixes $s$, or any two suffixes $t$, are interchanged.

As it is therefore independent of the coefficients $A_{st}$, we may put $A_{st}$ equal to unity for all values of $s$ and $t$, and we get integrals of the form considered in the last paragraph, and the relation is at once verified.

The formula, then, having been proved for values of $\beta_s$ and $\gamma_t$ which have their real parts positive, is also true when $\beta_s$ and $\gamma_t$ are replaced by

$\beta_s - n$ and $\gamma_t - n$ where $n$ is any integer, and so is true, in general, for all values of $\beta_s$ and $\gamma_t$.

15. A similar extension may be made for the formula of § 4, the field of integration beginning and ending at infinity ($x_r = +\infty$, for all values of $r$), and making circuits about $x_r = 0$. In fact the formula is at once proved for one such field, by making use of the expression for $\Gamma(p)$ as a contour integral, viz., $\dfrac{1}{e^{2\pi ip}-1}\displaystyle\int x^{p-1}e^{-x}\,dx$, taken round a contour which starts from infinity on the positive side of the axis of real quantity, makes a circuit about the origin, and returns to infinity on the negative side of the real axis.

# GENERALISATIONS OF LEGENDRE'S FORMULA

## $KE'-(K-E)K' = \frac{1}{2}\pi$

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[Received May 2nd, 1904.—Read May 12th, 1904.—Received in revised form January 10th, 1905.]

IN a paper in the *Messenger* (Vol. XXXIII., Part 2, p. 31) entitled " A Formula including Legendre's $EK'+KE'-KK' = \frac{1}{2}\pi$," Prof. E. B. Elliott has proved that, if we write $L(a, b, c;\, \kappa)$ for

$$\int_0^K \operatorname{sn}^a u \operatorname{cn}^b u \operatorname{dn}^c u\, du \quad (\text{mod } \kappa),$$

the following relation will hold good, viz.,

$$L(a, b, c+2;\, \kappa)\, L(c, b, a;\, \kappa') + L(a, b, c;\, \kappa)\, L(c, b, a+2;\, \kappa')$$
$$-L(a, b, c;\, \kappa)\, L(c, b, a;\, \kappa') = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{4\Gamma\left(\frac{a+b+c+3}{2}\right)}.$$

In the present communication I propose to apply the method used by Prof. Elliott to the proof of similar relations for more general integrals.

Legendre's formula may be written*

$$\begin{vmatrix} \int_{a_2}^{a_1} \prod_{s=1}^{s=3} (x-a_s)^{-\frac{1}{2}}x\, dx, & \int_{a_2}^{a_1} \prod_{s=1}^{s=3} (x-a_s)^{-\frac{1}{2}}\, dx \\ \int_{a_3}^{a_2} \prod_{s=1}^{s=3} (x-a_s)^{-\frac{1}{2}}x\, dx, & \int_{a_3}^{a_2} \prod_{s=1}^{s=3} (x-a_s)^{-\frac{1}{2}}\, dx \end{vmatrix} = 2\pi\iota\, ;$$

and in the first place I investigate a similar formula by which a determinant of the $(n-1)$-th order, whose elements are of the form

$$\int_{a_{r+1}}^{a_r} \prod_{s=1}^{s=n} (x-a_s)^{\beta_s-1} x^p\, dx,$$

is equal to a product of gamma functions; secondly, I prove that the ratio of a determinant of the $(n-r-1)$-th order to a similar one of the $r$-th order, the elements of both being of the same form as that just given, can be expressed as a product of gamma functions.

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* Weierstrass, " Beitrag zur Theorie der Abel'schen Integrale " (*Werke*, Vol. I., p. 118).

From these two results together I deduce some other formulæ, which correspond to Weierstrass's relations for hyperelliptic integrals.*

Finally, I show that similar relations exist for integrals of the form†

$$\int e^{-x} \prod_{s=1}^{s=n} (x-a_s)^{\beta_s-1} dx.$$

1. Writing $\qquad f(\theta) \equiv (\theta-a_1)(\theta-a_2)\dots(\theta-a_n),$

where $a_1, a_2, \dots, a_n$ are taken to be all real, and arranged in descending order of magnitude, so that

$$a_1 > a_2 > a_3 \dots > a_n,$$

consider the substitution

$$\left.\begin{array}{l}
(-1)^{n-1} f'(a_1)\xi_1 = (x_1-a_1)(x_2-a_1)\dots(x_{n-1}-a_1), \\
(-1)^{n-1} f'(a_2)\xi_2 = (x_1-a_2)(x_2-a_2)\dots(x_{n-1}-a_2), \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
(-1)^{n-1} f'(a_r)\xi_r = (x_1-a_r)(x_2-a_r)\dots(x_{n-1}-a_r), \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
(-1)^{n-1} f'(a_n)\xi_n = (x_1-a_n)(x_2-a_n)\dots(x_{n-1}-a_n)
\end{array}\right\}, \tag{1}$$

where the number of the $x$ is one less than the number of the roots of the equation $f(\theta) = 0$.

The identical relation between the $\xi$ is

$$\xi_1+\xi_2+\dots+\xi_n = 1, \tag{2}$$

which is immediately deduced from the relations

$$\Sigma \frac{a_r^p}{f'(a_r)} = 0 \quad (p < n-1); \qquad \Sigma \frac{a_r^{n-1}}{f'(a_r)} = 1.$$

Transform the multiple integral

$$\iint\dots\int \xi_1^{\beta_1-1}\xi_2^{\beta_2-1}\dots\xi_{r-1}^{\beta_{r-1}-1}\xi_r^{\beta_r-1}\xi_{r+1}^{\beta_{r+1}-1}\dots\xi_n^{\beta_n-1} d\xi_1 d\xi_2\dots d\xi_{r-1} d\xi_{r+1}\dots d\xi_n, \tag{3}$$

in which $\qquad \xi_r = 1-\xi_1-\xi_2\dots-\xi_{r-1}-\xi_{r+1}\dots-\xi_n,$

and $\beta_s$ has its real part positive for all values of $s$, by the substitution (1).

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* Weierstrass, *loc. cit.*, p. 127.

† Cf. Abel, "Petite Contribution à la Théorie de quelques Fonctions Transcendentales" (*Œuvres*, Vol. I., pp. 40–60).

It becomes

$$\prod_{s=1}^{s=n} \{(-1)^{(n-1)} f'(a_s)\}^{1-\beta_s} \iint \cdots \int \prod_{s=1}^{s=n-1} R(x_s) J \, dx_1 \, dx_2 \ldots dx_{n-1}, \tag{4}$$

where
$$R(x) = \prod_{t=1}^{t=n} (x-a_t)^{\beta_t - 1}$$

and
$$J \equiv \frac{d(\xi_1, \xi_2, \ldots, \xi_{r-1}, \xi_{r+1}, \ldots, \xi_n)}{d(x_1, x_2, \ldots, x_{n-1})}.$$

The limits are chosen so that $x_1$ varies from $a_2$ to $a_1$, $x_2$ from $a_3$ to $a_2$, ..., $x_s$ from $a_{s+1}$ to $a_s$. All these limits correspond to the limit zero for one of the $\xi$, and, moreover, the value of any $\xi$ is always positive for this range of values of the $x$.

Since $\xi_r$ is always positive, we have

$$1 > \xi_1 + \xi_2 + \ldots + \xi_{r-1} + \xi_{r+1} + \ldots + \xi_n \tag{5}$$

throughout the integration, and so the field of integration for $\xi_1, \ldots, \xi_{r-1}, \xi_{r+1}, \ldots, \xi_n$ is for all positive values subject to this condition (5).

The value of the $\xi$ integral is therefore by Dirichlet's theorem

$$\frac{\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_n)}{\Gamma(\beta_1+\beta_2+\ldots+\beta_n)},$$

and I have the result

$$\int_{a_2}^{a_1}\int_{a_3}^{a_2} \cdots \int_{a_n}^{a_{n-1}} \prod_{s=1}^{s=n-1} R(x_s) J \, dx_1 \, dx_2 \ldots dx_{n-1} = \prod_{s=1}^{s=n} \{(-1)^{n-1} f'(a_s)\}^{\beta_s - 1} \frac{\Pi\,\Gamma(\beta_s)}{\Gamma(\Sigma\beta_s)}. \tag{6}$$

2. Now

$$J \equiv \frac{d(\xi_1, \xi_2, \ldots, \xi_{r-1}, \xi_{r+1}, \ldots, \xi_n)}{d(x_1, x_2, \ldots, x_{n-1})}$$

$$= \prod_{s=1}^{s=n}{}' \xi_s \begin{vmatrix} \frac{1}{x_1-a_1}, & \frac{1}{x_2-a_1}, & \cdots & \frac{1}{x_{n-1}-a_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_1-a_{r-1}}, & \frac{1}{x_2-a_{r-1}}, & \ldots, & \frac{1}{x_{n-1}-a_{r-1}} \\ \frac{1}{x_1-a_{r+1}}, & \frac{1}{x_2-a_{r+1}}, & \ldots, & \frac{1}{x_{n-1}-a_{r+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_1-a_n}, & \frac{1}{x_2-a_n}, & \ldots, & \frac{1}{x_{n-1}-a_n} \end{vmatrix}, \tag{7}$$

where in $\overset{s=n}{\underset{s=1}{\Pi'}}$ the dash denotes that the term corresponding to $s=r$ is omitted from the product.

Further, $J\Pi'f'(a_s)$ vanishes when $x_s=x_t$ and when $a_s=a_t$, and it is therefore equal to a numerical multiple of $\Pi(x_s-x_t)\,\Pi'(a_{s'}-a_{t'})$; and so I get

$$J=\frac{\Pi(x_s-x_t)}{\Pi(a_{s'}-a_{t'})}$$

$$(t>s;\ t'>s';\ s,t=1,2,3,\ldots,(n-1);\ s',t'=1,2,3,\ldots,n),$$

which may be written in the form

$$J=\frac{1}{\Pi(a_{s'}-a_{t'})}\begin{vmatrix} x_1^{n-2}, & x_1^{n-3}, & x_1^{n-4}, & \ldots, & x_1, & 1 \\ x_2^{n-2}, & x_2^{n-3}, & x_2^{n-4}, & \ldots, & x_2, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ x_{n-1}^{n-2}, & x_{n-1}^{n-3}, & x_{n-1}^{n-4}, & \ldots, & x_{n-1}, & 1 \end{vmatrix}. \tag{8}$$

3. Now, writing $H(a_s, a_{s+1}, m)$ for

$$\int_{a_{s+1}}^{a_s}\prod_{t=1}^{t=n}(x-a_t)^{\beta_t-1}x^m\,dx, \tag{9}$$

I shall get as the value of the multiple integral, when expressed in $x_1, x_2, \ldots, x_n$—taking for $J$ the form (8)—the determinant

$$\begin{vmatrix} H(a_1, a_2, n-2), & H(a_1, a_2, n-3), & \ldots, & H(a_1, a_2, 1), & H(a_1, a_2, 0) \\ H(a_2, a_3, n-2), & H(a_2, a_3, n-3), & \ldots, & H(a_2, a_3, 1), & H(a_2, a_3, 0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H(a_{n-1}, a_n, n-2), & H(a_{n-1}, a_n, n-3), & \ldots, & H(a_{n-1}, a_n, 1), & H(a_{n-1}, a_n, 0) \end{vmatrix} \tag{10}$$

divided by $\quad \Pi\{(-1)^{n-1}f'(a_s)\}^{\beta_s-1}\Pi(a_s-a_t) \quad (s>t);$

so that I have as the final result: the determinant (10) is equal to

$$\Pi\{(-1)^{n-1}f'(a_s)\}^{\beta_s-1}\Pi(a_s-a_t)\frac{\Pi\Gamma(\beta_s)}{\Gamma(\Sigma\beta_s)},$$

where $s$ and $t$ take all the values $1, 2, 3, \ldots, n$, subject to the condition $s>t$.

It should be noticed that the question of the sign of $J(\xi/x)$ has been disregarded, so that it is necessary to determine what sign should be given to the result in any particular case.

As an example of this formula, put

$$n=5, \qquad \beta_s=\tfrac{1}{2} \quad (s=1, 2, 3, 4, 5);$$

then, writing $R(x)$ for $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$, we get for hyperelliptic functions of genus two the relation

$$\begin{vmatrix} \displaystyle\int_{a_2}^{a_1}\frac{x^3\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_2}^{a_1}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_2}^{a_1}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_2}^{a_1}\frac{dx}{\sqrt{\{R(x)\}}} \\ \displaystyle\int_{a_3}^{a_2}\frac{x^3\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_3}^{a_2}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_3}^{a_2}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_3}^{a_2}\frac{dx}{\sqrt{\{R(x)\}}} \\ \displaystyle\int_{a_4}^{a_3}\frac{x^3\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_4}^{a_3}\frac{x^3\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_4}^{a_3}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_4}^{a_3}\frac{dx}{\sqrt{\{R(x)\}}} \\ \displaystyle\int_{a_5}^{a_4}\frac{x^3\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_5}^{a_4}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_5}^{a_3}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_5}^{a_4}\frac{dx}{\sqrt{\{R(x)\}}} \end{vmatrix} = -\tfrac{4}{3}\pi^2.$$

4. Taking the form (7) for $J$, and writing $I_p(a_s, a_{s+1})$ for

$$\int_{a_{s+1}}^{a_s}\prod_{t=1}^{t=n}(x-a_t)^{\beta_t}\frac{dx}{(x-a_p)(x-a_r)},$$

I get

$$\begin{vmatrix} I_1(a_1, a_2), & I_1(a_2, a_3), & \dots, & I_1(a_{n-1}, a_n) \\ I_2(a_1, a_2), & I_2(a_2, a_3), & \dots, & I_2(a_{n-1}, a_n) \\ \dots & \dots & \dots & \dots \\ I_{r-1}(a_1, a_2), & I_{r-1}(a_2, a_3), & \dots, & I_{r-1}(a_{n-1}, a_n) \\ I_{r+1}(a_1, a_2), & I_{r+1}(a_2, a_3), & \dots, & I_{r+1}(a_{n-1}, a_n) \\ \dots & \dots & \dots & \dots \\ I_n(a_1, a_2), & I_n(a_2, a_3), & \dots, & I_n(a_{n-1}, a_n) \end{vmatrix} = \frac{\Pi\{(-1)^{n-1}f'(a_s)\}^{\beta_s}\,\Pi\,\Gamma(\beta_s)}{(-1)^{n-1}f'(a_r)\,\Gamma(\Sigma\beta_s)}$$

$$(s = 1, 2, 3, \dots, n). \quad (11)$$

5. I now go on to investigate the result of applying this method of transformation to evaluating a determinant of the $(n-1)$-th order, whose elements are integrals of the form

$$\int_{a_{r+1}}^{a_r}\prod_{s=1}^{s=n}(x-a_s)^{\beta_s-1}\prod_{t=1}^{t=m}(x-b_t)^{-\gamma_t}x^p\,dx,$$

in which $b_1, b_2, \dots, b_m, a_1, a_2, \dots, a_n$ are taken to be all real and arranged in descending order of magnitude, so that

$$a_1 > a_2 > a_3 \dots > a_n$$

and

$$\infty > b_1 > b_2 > b_3 \dots > b_m > a_1,$$

and in which $\beta_s$ and $\gamma_t$ and $\Sigma\beta - \Sigma\gamma$ denote quantities whose real part is positive for all values of $s$ and $t$.

Under these conditions $b_t-a_s$ is always positive, and $x-b_t$ does not change sign throughout the integration.

Consider the transformation of the integral

$$\int_{a_2}^{a_1}\int_{a_3}^{a_2}\dots\int_{a_n}^{a_{n-1}}\prod_{s=1}^{s=n-1}\prod_{t=1}^{t=n}(x_s-a_t)^{\beta_t-1}\prod_{s=1}^{s=n-1}\prod_{t=1}^{t=m}(x_s-b_t)^{-\gamma_t}J\,dx_1\,dx_2\dots dx_n \tag{12}$$

by the substitution (1).

$J$ is $\dfrac{d(\xi_1, \xi_2, \dots, \xi_{r-1}, \xi_{r+1}, \dots, \xi_n)}{d(x_1, x_2, \dots, x_{n-1})}$, and is considered to be expressed in terms of $x_1, x_2, \dots, x_{n-1}$.

Expanding $\prod_{s=1}^{s=n-1}(x_s-b_t)\div(-1)^{n-1}f(b_t)$ by partial fractions, I have

$$\frac{\prod_{s=1}^{s=n-1}(x_s-b_t)}{(-1)^{n-1}f(b_t)}=\sum_{p=1}^{p=n}\frac{\prod_{s=1}^{s=n-1}(x_s-a_p)}{(-1)^{n-1}f'(a_p)(b_t-a_p)}=\sum_{s=1}^{s=n}\frac{\xi_s}{b_t-a_s}; \tag{13}$$

so that by this transformation the given integral becomes

$$\prod_{s=1}^{s=n}\left\{(-1)^{n-1}f'(a_s)\right\}^{\beta_s-1}\prod_{t=1}^{t=m}\left\{(-1)^{n-1}f(b_t)\right\}^{-\gamma_t}$$

$$\times\iint\dots\int\prod_{s=1}^{s=n}\xi^{\beta_s-1}\prod_{t=1}^{t=m}\left(\sum_{s=1}^{s=n}\frac{\xi_s}{b_t-a_s}\right)^{-\gamma_t}d\xi_1\,d\xi_2\dots d\xi_{r-1}\,d\xi_{r+1}\dots d\xi_n, \tag{14}$$

the range of integration being for all positive values of the $\xi$, with the condition $\Sigma\,\xi_s=1$.

6. In the preceding paper (Part I., § 6) it has been shown that this multiple integral can be replaced by

$$\frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma(\Sigma\beta-\Sigma\gamma)\prod_{t=1}^{t=m}\Gamma(\gamma_t)}\int_0^\infty\int_0^\infty\dots\int_0^\infty\prod_{t=1}^{t=m}\theta_t^{\gamma_t-1}\prod_{s=1}^{s=n}\left(1+\sum_{t=1}^{t=m}\frac{\theta_t}{b_t-a_s}\right)^{-\beta_s}d\theta_1\dots d\theta_m. \tag{15}$$

Transform this integral by a substitution exactly similar to (1), viz.,

$$(-1)^{m-1}\phi'(b_t)\theta_t=(x_1-b_t)(x_2-b_t)\dots(x_m-b_t),$$

where

$$\phi(x)\equiv(x-b_1)(x-b_2)\dots(x-b_m)$$

and the number of the variables $\theta$ is equal to the number of the $x$.

The limits of $x_1$ are from $b_1$ to $\infty$, of $x_2$ from $b_2$ to $b_1$, …, of $x_t$ from $b_t$ to $b_{t-1}$. Then, as in § 1,

$$J'=J\left(\frac{\theta}{x}\right)=\frac{\Pi(x_s-x_t)}{\Pi(b_s-b_t)}\quad(t>s;\ s, t=1, 2, \dots, m),$$

and can be expressed in various forms as a determinant, just as $J(\xi/x)$ in § 2. In particular, the form (8) of § 2 is the same for the two cases, and, corresponding to the form (7), I have

$$J' = \prod_{t=1}^{t=m} \theta_t \begin{vmatrix} \frac{1}{x_1-b_1}, & \frac{1}{x_2-b_1}, & \dots, & \frac{1}{x_m-b_1} \\ \dots & \dots & \dots \quad \dots & \dots \\ \frac{1}{x_1-b_m}, & \frac{1}{x_2-b_m}, & \dots, & \frac{1}{x_m-b_m} \end{vmatrix}. \tag{16}$$

Also, I get at once

$$\frac{\prod\limits_{t=1}^{t=m} (x_t-a_s)}{(-1)^m\phi(a_s)} = 1-\Sigma\frac{\theta_t}{a_s-b_t} = 1+\Sigma\frac{\theta_t}{b_t-a_s}, \tag{17}$$

so that the integrand takes the form

$$J' \prod_{t=1}^{t=m}\left(\frac{\prod\limits_{r=1}^{r=m}(x_r-b_t)}{(-1)^{m-1}\phi'(b_t)}\right)^{\gamma_t-1} \prod_{s=1}^{s=n}\left(\frac{\prod\limits_{r=1}^{r=m}(x_r-a_s)}{(-1)^m\phi(a_s)}\right)^{-\beta_s}. \tag{18}$$

Finally, then, I get that the integral

$$\int_{a_2}^{a_1}\int_{a_3}^{a_2}\dots\int_{a_n}^{a_{n-1}} \prod_{s=1}^{s=n-1}\prod_{t=1}^{t=n}(x_s-a_t)^{\beta_t-1}\prod_{s=1}^{s=n-1}\prod_{t=1}^{t=m}(x_s-b_t)^{-\gamma_t} J\left(\frac{\xi}{x}\right) dx_1\,dx_2\dots dx_{n-1}$$

is equal to

$$\int_{b_1}^{\infty}\int_{b_2}^{b_1}\dots\int_{b_m}^{b_{m-1}} \prod_{r=1}^{r=m}\prod_{t=1}^{t=m}(x_r-b_t)^{\gamma_t-1}\prod_{r=1}^{r=m}\prod_{s=1}^{s=n}(x_r-a_s)^{-\beta_s} J\left(\frac{\theta}{x}\right) dx_1\,dx_2\dots dx_m \tag{19}$$

multiplied by

$$\frac{\prod\limits_{s=1}^{s=n}\{(-1)^m\phi(a_s)\}^{\beta_s}\{(-1)^{n-1}f'(a_s)\}^{\beta_s-1}}{\prod\limits_{t=1}^{t=m}\{(-1)^{n-1}f(b_t)\}^{\gamma_t}\{(-1)^{m-1}\phi'(b_t)\}^{\gamma_t-1}} \frac{\prod\limits_{s=1}^{s=n}\Gamma(b_s)}{\prod\limits_{t=1}^{t=m}\Gamma(\gamma_t)} \frac{1}{\Gamma(\Sigma\beta-\Sigma\gamma)}.$$

7. Giving to $J(\xi/x)$ and $J(\theta/x)$ the form (8), and writing $H(a_s, a_{s+1}, r)$ for

$$\int_{a_{s+1}}^{a_s} \prod_{t=1}^{t=n}(x-a_t)^{\beta_t-1}\prod_{t=1}^{t=m}(x-b_t)^{-\gamma_t}x^r\,dx \tag{20}$$

and $H'(b_s, b_{s+1}, r)$ for

$$\int_{b_{s+1}}^{b_s} \prod_{t=1}^{t=m}(x-b_t)^{\gamma_t-1}\prod_{t=1}^{t=n}(x-a_t)^{-\beta_t}x^r\,dx, \tag{21}$$

I get—writing down in each case only the first and last rows of the determinants—that the determinant

$$\begin{vmatrix} H(a_1, a_2, n-2), & H(a_1, a_2, n-3), & \dots, & H(a_1, a_2, 1), & H(a_1, a_2, 0) \\ \dots & \dots & \dots & \dots & \dots \\ H(a_{n-1}, a_n, n-2), & H(a_{n-1}, a_n, n-3), & \dots, & H(a_{n-1}, a_n, 1), & H(a_{n-1}, a_n, 0) \end{vmatrix}$$

is equal to the determinant

$$\begin{vmatrix} H'(\infty, b_1, m-1), & H'(\infty, b_1, m-2), & \dots, & H'(\infty, b_1, 1), & H'(\infty, b_1, 0) \\ \dots & \dots & \dots & \dots & \dots \\ H'(b_{m-1}, b_m, m-1), & H'(b_{m-1}, b_m, m-2), & \dots, & H'(b_{m-1}, b_m, 1), & H'(b_{m-1}, b_m, 0) \end{vmatrix}$$

multiplied by

$$\frac{1}{\Gamma(\Sigma\beta-\Sigma\gamma)} \frac{\Pi(a_s-a_{s'})}{\Pi(b_t-b_{t'})} \frac{\Pi\{(-1)^m\phi(a_s)\}^{\beta_s}\{(-1)^{n-1}f'(a_s)\}^{\beta_s-1}\Gamma(\beta_s)}{\Pi\{(-1)^{n-1}f(b_t)\}^{\gamma_t}\{(-1)^{m-1}\phi'(b_t)\}^{\gamma_t-1}\Gamma(\gamma_t)}; \tag{22}$$

where $s$ and $s'$ take the values 1, 2, 3, ..., $n$, and $s' > s$; $t$ and $t'$ the values 1, 2, 3, ..., $m$, and $t' > t$.

As examples of this formula, putting $n=3$, $m=2$, $\beta_s=\gamma_t=\frac{1}{2}$, and writing $a_4$ and $a_5$ for $b_2$ and $b_1$, and $R(x)$ for $\prod_{r=1}^{r=5}(x-a_r)$, we have

$$\begin{vmatrix} \int_{a_2}^{a_1}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_2}^{a_1}\frac{dx}{\sqrt{\{R(x)\}}} \\ \int_{a_3}^{a_2}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_3}^{a_2}\frac{dx}{\sqrt{\{R(x)\}}} \end{vmatrix} = \begin{vmatrix} \int_{a_5}^{\infty}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_5}^{\infty}\frac{x\,dx}{\sqrt{\{R(x)\}}} \\ \int_{a_4}^{a_5}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_4}^{a_5}\frac{dx}{\sqrt{\{R(x)\}}} \end{vmatrix};$$

similarly, putting $R(x)$ for $\prod_{r=1}^{r=7}(x-a_r)$, we have

$$\begin{vmatrix} \int_{a_2}^{a_1}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_2}^{a_1}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_2}^{a_1}\frac{dx}{\sqrt{\{R(x)\}}} \\ \int_{a_3}^{a_2}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_3}^{a_2}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_3}^{a_2}\frac{dx}{\sqrt{\{R(x)\}}} \\ \int_{a_4}^{a_3}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_4}^{a_3}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_4}^{a_3}\frac{dx}{\sqrt{\{R(x)\}}} \end{vmatrix}$$

$$= \begin{vmatrix} \int_{a_7}^{\infty}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_7}^{\infty}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_7}^{\infty}\frac{dx}{\sqrt{\{R(x)\}}} \\ \int_{a_6}^{a_7}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_6}^{a_7}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_6}^{a_7}\frac{dx}{\sqrt{\{R(x)\}}} \\ \int_{a_5}^{a_6}\frac{x^2\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_5}^{a_6}\frac{x\,dx}{\sqrt{\{R(x)\}}}, & \int_{a_5}^{a_6}\frac{dx}{\sqrt{\{R(x)\}}} \end{vmatrix},$$

and there are similar formulæ for hyperelliptic functions of any order. If we put $m = 1$, $\gamma = 1$, $n = 4$, $\beta_1 = \beta_2 = \beta_3 = \frac{1}{2}$, $\beta_4 = 1$, and write $a$ for $a_4$, $R(x)$ for $(x-a_1)(x-a_2)(x-a_3)$, we shall get, after some slight reduction,

$$\begin{vmatrix} \displaystyle\int_{a_2}^{a_1} \frac{x\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_2}^{a_1} \frac{dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_2}^{a_1} \frac{dx}{(x-b)\sqrt{\{R(x)\}}} \\ \displaystyle\int_{a_3}^{a_2} \frac{x\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_3}^{a_2} \frac{dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a_3}^{a_2} \frac{dx}{(x-b)\sqrt{\{R(x)\}}} \\ \displaystyle\int_{a}^{a_3} \frac{x\,dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a}^{a_3} \frac{dx}{\sqrt{\{R(x)\}}}, & \displaystyle\int_{a}^{a_3} \frac{dx}{(x-b)\sqrt{\{R(x)\}}} \end{vmatrix}$$

$$= \pi\iota \frac{\sqrt{\{R(a)\}}}{\sqrt{\{R(b)\}}} \int_b^\infty \frac{dx}{(x-a)\sqrt{\{R(x)\}}},$$

which is equivalent to $\Pi(u, v) - \Pi(v, u) = uE(v) - vE(u)$, in the notation of elliptic functions.

8. Giving to $J(\xi/x)$ and $J(\theta/x)$ the forms (7) and (16) respectively, and writing $I_p(a_q, a_{q+1})$ for

$$\int_{a_{q+1}}^{a_q} \prod_{s=1}^{s=n} (x-a_s)^{\beta_s} \prod_{t=1}^{t=m} (x-b_t)^{-\gamma_t} \frac{dx}{(x-a_p)(x-a_r)}, \tag{23}$$

where $r$ is any chosen number out of 1, 2, 3, ..., $n$; and $I'_p(b_q, b_{q+1})$ for

$$\int_{b_{q+1}}^{b_q} \prod_{t=1}^{t=m} (x-b_t)^{\gamma_t} \prod_{s=1}^{s=n} (x-a_s)^{-\beta_s} \frac{dx}{x-b_p}, \tag{24}$$

I get—again writing down only the first and last rows of the determinants—that

$$\begin{vmatrix} I_1(a_1, a_2), & I_1(a_2, a_3), & \ldots, & I_1(a_{n-1}, a_n) \\ \ldots & \ldots & \ldots & \ldots \\ I_n(a_1, a_2), & I_n(a_2, a_3), & \ldots, & I_n(a_{n-1}, a_n) \end{vmatrix}{}^{*}$$

is equal to

$$\begin{vmatrix} I'_1(\infty, b_1), & I'_1(b_1, b_2), & \ldots, & I'_1(b_{m-1}, b_m) \\ \ldots & \ldots & \ldots & \ldots \\ I'_m(\infty, b_1), & I'_m(b_1, b_2), & \ldots, & I'_m(b_{m-1}, b_m) \end{vmatrix} \tag{25}$$

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* In this determinant there is, of course, no row $I_r$.

multiplied by

$$\frac{1}{\Gamma(\Sigma\beta-\Sigma\gamma)(-1)^{n-1}f'(a_r)}\,\frac{\prod\limits_{s=1}^{s=n}\{(-1)^m\phi(a_s)\}^{\beta_s}\{(-1)^{n-1}f'(a_s)\}^{\beta_s}\Gamma(\beta_s)}{\prod\limits_{t=1}^{t=m}\{(-1)^{n-1}f(b_t)\}^{\gamma_t}\{(-1)^{m-1}\phi'(b_t)\}^{\gamma_t}\Gamma(\gamma_t)}.$$

9. It is convenient, for the sake of extensions to be made immediately, to restate these formulæ in the form they take when the integrals which form the constituents of the determinants are transformed by a general homographic transformation, or, what is the same thing, when $\Sigma\beta-\Sigma\gamma=0$. The results are most easily obtained by using transformations similar to (1), § 1, on the formulæ of § 1 of the preceding paper.

If $b_1, b_2, \ldots, b_m, a_1, a_2, \ldots, a_n$ are real quantities arranged in order of magnitude, and $\beta_s$ and $\gamma_t$ denote quantities whose real part is positive for all values of $s$ and $t$, such that $\sum\limits_{s=1}^{s=n}\beta_s=\sum\limits_{t=1}^{t=m}\gamma_t$, and if

$$H(a_p, a_{p+1}, r)\equiv\int_{a_{p+1}}^{a_p}\prod_{s=1}^{s=n}(x-a_s)^{\beta_s-1}\prod_{t=1}^{t=m}(x-b_t)^{-\gamma_t}x^r\,dx,$$

and $$H'(b_p, b_{p+1}, r)\equiv\int_{b_{p+1}}^{b_p}\prod_{t=1}^{t=m}(x-b_t)^{\gamma_t-1}\prod_{s=1}^{s=n}(x-a_s)^{-\beta_s}x^r\,dx,$$

then the determinant

$$\begin{vmatrix} H(a_1, a_2, n-2), & H(a_1, a_2, n-3), & \ldots, & H(a_1, a_2, 1), & H(a_1, a_2, 0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H(a_{n-1}, a_n, n-2), & H(a_{n-1}, a_n, n-3), & \ldots, & H(a_{n-1}, a_n, 1), & H(a_{n-1}, a_n, 0) \end{vmatrix}$$

is equal to the determinant $(m>1)$ (A)

$$\begin{vmatrix} H'(b_1, b_2, m-2), & H'(b_1, b_2, m-3), & \ldots, & H'(b_1, b_2, 1), & H'(b_1, b_2, 0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H'(b_{m-1}, b_m, m-2), & H'(b_{m-1}, b_m, m-3), & \ldots, & H'(b_{m-1}, b_m, 1), & H'(b_{m-1}, b_m, 0) \end{vmatrix}$$

multiplied by

$$\frac{\Pi(a_s-a_{s'})}{\Pi(b_t-b_{t'})}\,\frac{\Pi\{(-1)^m\phi(a_s)\}^{\beta_s}\{(-1)^{n-1}f'(a_s)\}^{\beta_s-1}\Gamma(\beta_s)}{\Pi\{(-1)^{n-1}f(b_t)\}^{\gamma_t}\{(-1)^{m-1}\phi'(b_t)\}^{\gamma_t-1}\Gamma(\gamma_t)},$$

where $$f(\theta)\equiv\prod_{s=1}^{s=n}(\theta-a_s),\qquad \phi(\theta)\equiv\prod_{t=1}^{t=m}(\theta-b_t),$$

$s$ and $s'$ take the values 1, 2, 3, ..., $n$, and $s>s'$, $t$ and $t'$ the values 1, 2, 3, ..., $m$, and $t>t'$.

When $m = 1$ and $\gamma = \Sigma\beta$ the above first written determinant is equal to

$$\Pi(a_s - a_{s'}) \frac{\Pi(b-a_s)^{\beta_s}\{(-1)^{n-1}f'(a_s)\}^{\beta_s-1}\Gamma(\beta_s)}{\{(-1)^{n-1}f(b)\}^{\gamma}\Gamma(\gamma)}. \tag{B}$$

10. Moreover, if

$$I_k(a_p, a_{p+1}) \equiv \int_{a_{p+1}}^{a_p} \frac{\prod\limits_{s=1}^{s=n}(x-a_s)^{\beta_s}}{\prod\limits_{t=1}^{t=m}(x-b_t)^{\gamma_t}} \frac{dx}{(x-a_k)(x-a_r)},$$

where $r$ is some selected number from 1 to $n$, and

$$I'_k(b_p, b_{p+1}) \equiv \int_{b_{p+1}}^{b_p} \frac{\prod\limits_{t=1}^{t=m}(x-b_t)^{\gamma_t}}{\prod\limits_{s=1}^{s=n}(x-a_s)^{\beta_s}} \frac{dx}{(x-b_k)(x-b_q)},$$

where $q$ is some selected number from 1 to $m$, then the determinant

$$\begin{vmatrix} I_1(a_1, a_2), & I_1(a_2, a_3), & \dots, & I_1(a_{n-1}, a_n) \\ \dots & \dots & \dots & \dots \\ I_n(a_1, a_2), & I_n(a_2, a_3), & \dots, & I_n(a_{n-1}, a_n) \end{vmatrix}$$

(in which there is no row $I_r$) is equal to the determinant $(m > 1)$

$$\begin{vmatrix} I'_1(b_1, b_2), & I'_1(b_2, b_3), & \dots, & I'_1(b_{m-1}, b_m) \\ \dots & \dots & \dots & \dots \\ I'_m(b_1, b_2), & I'_m(b_2, b_3), & \dots, & I'_m(b_{m-1}, b_m) \end{vmatrix} \tag{C}$$

(in which there is no row $I'_q$) multiplied by

$$\frac{(-1)^{m-1}\phi'(b_q)}{(-1)^{n-1}f'(a_r)} \frac{\prod\limits_{s=1}^{s=n}\{(-1)^m\phi(a_s)(-1)^{n-1}f'(a_s)\}^{\beta_s}\Gamma(\beta_s)}{\prod\limits_{t=1}^{t=m}\{(-1)^{n-1}f(b_t)(-1)^{m-1}\phi'(b_t)\}^{\gamma_t}\Gamma(\gamma_t)}.$$

When $m = 1$ and $\gamma = \Sigma\beta$, the above first determinant is equal to

$$\frac{1}{(-1)^{n-1}f'(a_r)} \frac{\prod\limits_{s=1}^{s=n}\{(b-a_s)(-1)^{n-1}f'(a_s)\}^{\beta_s}\Gamma(\beta_s)}{\{(-1)^{n-1}f(b)\}^{\gamma}\Gamma(\gamma)}. \tag{D}$$

11. The results (A), (B), (C), (D) of the last paragraphs, obtained on the supposition that $\beta_s$ and $\gamma_t$ are quantities whose real parts are positive, and that $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$ are arranged in order of

magnitude, may now be extended to include all values of these quantities, the only condition which is retained being that $b_t - a_s$ must be positive as to its real part, for all values of $t$ and $s$.

This is done by replacing the definite integrals between fixed limits by double-circuit integrals.

Thus, for example, if $H(a_s, a_{s+1}, r)$ in formula (A) or (B), be defined as

$$\frac{1}{(e^{2\pi\iota\beta_s}-1)(e^{2\pi\iota\beta_{s+1}}-1)} \int^{(a_s+,\, a_{s+1}+,\, a_s-,\, a_{s+1}-)} \prod_{t=1}^{t=n} (x-a_t)^{\beta_t-1} \prod_{t=1}^{t=m} (x-b_t)^{-\gamma_t} x^r \, dx,$$

in which, of course, the double circuit is so drawn as not to pass through any of the points $a$ or $b$, and, if similar modifications be made in the definitions of $I_p(a_q, a_{q+1})$, ..., the relations (A), (B), (C), and (D) will still be true.

For it is obvious that, corresponding to these double-circuit integrals for $x_1, x_2, \ldots$, the field of integration for $\xi_1, \xi_2, \ldots$ as given by the transformation (1) will be such a one as has been considered in Part II. of the preceding paper, that is, it is one such that each factor of the integrand will return to its initial value after the field has been covered, and also such that, when $\beta_1, \beta_2, \ldots$ are positive as to their real parts, the integral can be reduced to one taken over the content bounded by

$$\xi_1 = 0, \; \xi_2 = 0, \; \ldots, \; \xi_n = 0,$$

where $\xi_1, \xi_2, \xi_3, \ldots$ are all real and positive.

This will be true for all values of $\beta_s$ and $\gamma_t$, $a_s$ and $b_t$, and so in accordance with what has been shown in Part II. of the preceding paper, the formulæ (A), (B), (C), and (D) will hold good for all values of $\beta_s$ and $\gamma_t$, $a_s$ and $b_t$; for, when $\beta_s$ and $\gamma_t$ are positive as to their real parts, the two expressions for $H(a_s, a_{s+1}, r)$, viz.,

$$\frac{1}{(e^{2\pi\iota\beta_s}-1)(e^{2\pi\iota\beta_{s+1}}-1)} \int^{(a_s+,\, a_{s+1}+,\, a_s-,\, a_{s+1}-)} \prod_{t=1}^{t=n} (x-a_t)^{\beta_t-1} \prod_{t=1}^{t=m} (x-b_t)^{-\gamma_t} x^r \, dx$$

and
$$\int_{a_{s+1}}^{a_s} \prod_{t=1}^{t=n} (x-a_t)^{\beta_t-1} \prod_{t=1}^{t=m} (x-b_t)^{-\gamma_t} x^r \, dx,$$

become identical,* and the same thing is true with regard to

$$I_k(a_p, a_{p+1}), \quad \ldots.$$

The condition that the two sets of quantities, $a$ and $b$, are separated from one another by a straight line drawn perpendicular to the axis of real quantities, may be changed by a homographic transformation into the condition that a circle can be drawn to separate the two sets from one another, and I get, finally, that, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$

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* Cf. Pochhammer, *Math. Ann.*, Vol. xxxv., p. 470, "Ueber ein Integral mit doppeltem Umlauf."

are two sets of quantities such that a circle can be drawn separating one set from the other, and if $\beta_s$ and $\gamma_t$ denote quantities such that

$$\sum_{s=1}^{s=n} \beta_s = \sum_{t=1}^{t=m} \gamma_t,$$

and if

$$H(a_p, a_{p+1}, r) \equiv \frac{1}{(e^{2\pi i\beta_p}-1)(e^{2\pi i\beta_{p+1}}-1)} \int^{(a_p+,\, a_{p+1}+,\, a_p-,\, a_{p+1}-)} \frac{\prod\limits_{s=1}^{s=n} (x-a_s)^{\beta_s-1}}{\prod\limits_{t=1}^{t=m} (x-b_t)^{\gamma_t}}\, x^r dx$$

and $I_k(a_p, a_{p+1})$

$$\equiv \frac{1}{(e^{2\pi i\beta_p}-1)(e^{2\pi i\beta_{p+1}}-1)} \int^{(a_p+,\, a_{p+1}+,\, a_p-,\, a_{p+1}-)} \frac{\prod\limits_{s=1}^{s=n} (x-a_s)^{\beta_s}}{\prod\limits_{t=1}^{t=m} (x-b_t)^{\gamma_t}}\; \frac{dx}{(x-a_k)(x-a_r)},$$

with similar definitions for $H'(b_p, b_{p+1}, r)$ and $I'_k(b_p, b_{p+1})$, then the equations (A), (B), (C), and (D) of §§ 9, 10 will still be true.

12. These two relations (C) and (D) taken together will furnish identities in other forms.

In the formula (D) suppose $a_1$ and $b$ to lie outside a circle which can be drawn to enclose all the other points $a_2, a_3, \ldots, a_n$, and take $r = 1$; then we have

$$\frac{\prod\limits_{s=1}^{s=n} \{(b-a_s)(-1)^{n-1}f'(a_s)\}^{\beta_s}\,\Gamma(\beta_s)}{(-1)^{n-1}f'(a_1)\{(-1)^{n-1}f(b)\}^{\gamma}\,\Gamma(\gamma)}$$

$$= \begin{vmatrix} I_2(a_1, a_2), & I_2(a_2, a_3), & \ldots, & I_2(a_{n-1}, a_n) \\ \cdots & \cdots & \cdots \quad \cdots & \cdots \\ I_n(a_1, a_2), & I_n(a_2, a_3), & \ldots, & I_n(a_{n-1}, a_n) \end{vmatrix}$$

$$= \sum_{s=2}^{s=n} (-1)^s I_s(a_1, a_2)\,\Delta_s,$$

where $\Delta_s$ is the minor of $I_s(a_1, a_2)$ in the determinant. Now

$$\Delta_s = \begin{vmatrix} I_2(a_2, a_3), & \ldots, & I_2(a_{n-1}, a_n) \\ \cdots \quad \cdots & & \cdots \quad \cdots \\ I_{s-1}(a_2, a_3), & \ldots, & I_{s-1}(a_{n-1}, a_n) \\ I_{s+1}(a_2, a_3), & \ldots, & I_{s+1}(a_{n-1}, a_n) \\ \cdots \quad \cdots & & \cdots \quad \cdots \\ I_n(a_2, a_3), & \ldots, & I_n(a_{n-1}, a_n) \end{vmatrix},$$

and can be evaluated by the formula (C), $n-1$ being written for $n$, $\beta_s$ being changed to $\beta_s+1$, $a_1$ taking the place of $b_2$, and $b$ of $b_1$, $m$ being 2, $\gamma_2$ being $1-\beta_1$, and $\gamma_1$ being $\gamma$, and $a_{t+1}$ taking the place of $a_t$.

In this way we get

$$\begin{aligned}\pm\Delta_s = (a_s-b)(a_s-a_1)\beta_s \\ \times\frac{\prod\limits_{t=2}^{t=n}\{(b-a_t)(-1)^{n-1}f'(a_t)\}^{\beta_t}\Gamma(\beta_t)}{\{(-1)^{n-1}f(b)\}^{\gamma}\{(-1)^{n}(b-a_1)f'(a_1)\}^{1-\beta_1}\Gamma(\gamma)\Gamma(1-\beta_1)} \\ \times\int_{a_1}^{b}\frac{(x-a_1)^{1-\beta_1}(x-b)^{\gamma}}{\prod\limits_{t=2}^{t=n}(x-a_t)^{\beta_t}(x-a_s)}\,\frac{dx}{(x-a_1)(x-b)}.^{*}\end{aligned}$$

Substituting, then, for $\Delta_s$, and noticing that the sign of $\Delta_s$, as given by this formula, will be different according as $s$ is odd or even, we get

$$\begin{aligned}\sum_{s=2}^{s=n}(a_s-b)(a_s-a_1)\beta_s\int_{a_1}^{b}\frac{(x-b)^{\gamma}}{\prod\limits_{t=1}^{t=n}(x-a_t)^{\beta_t}}\,\frac{dx}{(x-a_s)(x-b)}\int_{a_s}^{a_1}\frac{\prod\limits_{t=1}^{t=n}(x-a_t)^{\beta_t}}{(x-b)^{\gamma}}\,\frac{dx}{(x-a_s)(x-a_1)} \\ = (-1)^{\beta_1}\Gamma(\beta_1)\Gamma(1-\beta_1) = (-1)^{\beta_1}\frac{\pi}{\sin\beta_1\pi}.\end{aligned}$$

Now, changing the notation a little by writing $a_0$ for $b$, and $-\beta_0$ for $\gamma$, so that $\sum\limits_{s=0}^{s=n}\beta_s=0$, I get

$$\sum_{s=2}^{s=n}(a_s-a_0)(a_s-a_1)\int_{a_1}^{a_0}\frac{\prod\limits_{t=0}^{t=n}(x-a_t)^{-\beta_t}\,dx}{(x-a_s)(x-a_0)}\int_{a_s}^{a_1}\frac{\prod\limits_{t=0}^{t=n}(x-a_t)^{\beta_t}\,dx}{(x-a_s)(x-a_1)} = (-1)^{\beta_1}\frac{\pi}{\sin\beta_1\pi}. \tag{27}$$

18. This result may be given a more symmetrical form, for, writing

$$\frac{a_s-a_1}{(x-a_s)(x-a_1)} = \frac{1}{x-a_s}-\frac{1}{x-a_1},$$

$$\frac{a_s-a_0}{(x-a_s)(x-a_0)} = \frac{1}{x-a_s}-\frac{1}{x-a_0},$$

and noticing that $\quad\sum\limits_{s=0}^{s=n}\beta_s\int_{a_1}^{a_0}\frac{dx}{(x-a_s)\prod\limits_{t=0}^{t=n}(x-a_t)^{\beta_t}} = 0,$

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* Here and afterwards I write $\int_a^b (x-a)^c(x-b)^d f(x)\,dx$ for

$$\frac{1}{(e^{2\pi ic}-1)(e^{2\pi id}-1)}\int^{(b+,\,a+,\,b-,\,a-)}(x-a)^c(x-b)^d f(x)\,dx.$$

and $$\sum_{s=0}^{s=n} \beta_s \int_{a_2}^{a_1} \prod_{t=0}^{t=n} (x-a_t)^{\beta_t} \frac{dx}{x-a_s} = 0,$$

I get at once

$$\sum_{s=0}^{s=n} \beta_s \int_{a_1}^{a_0} \frac{dx}{(x-a_s) \prod\limits_{t=0}^{t=n} (x-a_t)^{\beta_t}} \int_{a_2}^{a_1} \frac{\prod\limits_{t=0}^{t=n} (x-a_t)^{\beta_t} dx}{x-a_s} = (-1)^{\beta_1} \frac{\pi}{\sin \beta_1 \pi}. \tag{28}$$

Again, keeping the same meaning as before for $\Delta_s$, I have

$$\sum_{s=2}^{s=n} (-1)^s I_s(a_{r-1}, a_r) \Delta_s = 0 \quad (r \neq 2),$$

from which $$\sum_{s=0}^{s=n} \beta_s \int_{a_1}^{a_0} \frac{dx}{(x-a_s) \Pi (x-a_t)^{\beta_t}} \int_{a_r}^{a_{r-1}} \frac{\Pi (x-a_t)^{\beta_t} dx}{(x-a_s)} = 0. \tag{29}$$

In these formulæ the condition $\Sigma\beta = 0$ may be removed, by moving one of the points $a$ to $\infty$ by an appropriate homographic transformation; so that I have in general, putting

$$R(x) = \prod_{t=1}^{t=n} (x-a_t)^{\beta_t},$$

$$\sum_{s=1}^{s=n} \beta_s \int_{a_p}^{a_{p-1}} \frac{dx}{R(x)(x-a_s)} \int_{a_{q+1}}^{a_q} \frac{R(x)\,dx}{x-a_s} = (-1)^{\beta_p} \frac{\pi}{\sin \beta_p \pi}, \tag{30}$$

when $p = q$; and $= 0$ when $p \neq q$, $a_p$ and $a_{p-1}$ being so chosen that it is possible to draw a circle to separate them from all the other points $a$.

14. As a verification put, in (27), $a_0 = \infty$, $n = 3$, $\beta_s = \frac{1}{2}$, and write

$$x = \wp(u), \qquad \wp'(u)^2 = 4[\wp(u)-e_1][\wp(u)-e_2][\wp(u)-e_3];$$

then it becomes

$$2(e_2-e_1) \int_0^{\omega} \frac{du}{\wp(u)-e_2} \int_{\omega}^{\omega+\omega'} [\wp(u)-e_3]\,du$$
$$+ 2(e_3-e_1) \int_0^{\omega} \frac{du}{\wp(u)-e_3} \int_{\omega}^{\omega+\omega} [\wp(u)-e_2]\,du = \pi\iota,$$

i.e., $$\frac{1}{e_2-e_3} \int_0^{\omega} [\wp(u+\omega+\omega')-e_2]\,du \int_{\omega}^{\omega+\omega'} [\wp(u)-e_3]\,du$$
$$+ \frac{1}{e_3-e_2} \int_0^{\omega} [\wp(u+\omega')-e_3]\,du \int_{\omega}^{\omega+\omega'} [\wp(u)-e_2]\,du = \frac{\pi}{2}\iota,$$

or $$(\eta-e_2\omega)(\eta'-e_3\omega')-(\eta-e_3\omega)(\eta'-e_2\omega') = \tfrac{1}{2}\pi\iota(e_2-e_3)$$

or $$\eta\omega'-\eta'\omega = \tfrac{1}{2}\pi\iota.$$

15. These formulæ (30) may be established directly by the method used by Weierstrass (*loc. cit.*, *Werke*, Vol. I., p. 118). For writing the left-hand side as a double integral in the form

$$\iint \frac{R(x)}{R(y)} \Sigma \frac{\beta_s}{(x-a_s)(y-a_s)} \, dx \, dy,$$

where $R(x) = \Pi (x-a_t)^{\beta_t}$, the integrand is equal to

$$\frac{R(x)}{R(y)} \frac{1}{(x-y)} \left\{ \frac{R'(y)}{R(y)} - \frac{R'(x)}{R(x)} \right\},$$

that is, to
$$-\frac{\partial}{\partial y}\left(\frac{R(x)}{(x-y)\, R(y)}\right) - \frac{\partial}{\partial x}\left(\frac{R(x)}{(x-y)\, R(y)}\right),$$

so that the integral will vanish except when $x = y$ within the field of integration, and its value in the exceptional case may be determined.*

By this method also, by taking the limits of $x$ to be $c$ and $c'$, and of $y$, $d$ and $d'$, we most easily obtain the formula

$$\begin{aligned} \frac{1}{R(d)} \int_c^{c'} \frac{R(x)\, dx}{x-d} - \frac{1}{R(d')} \int_c^{c'} \frac{R(x)}{x-d'} \, dx - R(c) \int_d^{d'} \frac{dy}{(y-c)\, R(y)} & \\ + R(c') \int_d^{d'} \frac{dy}{(y-c')\, R(y)} & \\ = \sum_{s=1}^{s=n} \beta_s \int_c^{c'} \frac{R(x)\, dx}{x-a_s} \int_d^{d'} \frac{dy}{(y-a_s)\, R(y)}, & \qquad (31) \end{aligned}$$

where
$$R(x) = \prod_{t=1}^{t=n} (x-a_t)^{\beta_t}.$$

16. The method of this paper may also be used to obtain similar results with respect to integrals of the form

$$\int e^{-x} \Pi (x-b_t)^{\gamma_t - 1} \, dx.$$

For with the substitution used in § 6, viz.,

$$(-1)^{m-1} \phi'(b_t)\, \theta_t = (x_1 - b_t)(x_2 - b_t) \ldots (x_m - b_t),$$

where
$$\phi(x) = (x-b_1)(x-b_2) \ldots (x-b_m),$$

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* *Cf.* also Picard et Simart, *Théorie des fonctions de deux variables indépendantes*, Tome II., pp. 198–201.

we have, as before

$$\frac{\prod\limits_{t=1}^{t=m}(x_t-a_s)}{(-1)^m\phi(a_s)}=1+\Sigma\frac{\theta_t}{b_t-a_s},$$

and also $$\sum_{t=1}^{t=m}x_t-\sum_{t=1}^{t=m}b_t=\Sigma\theta_t.$$

Therefore the integral

$$\int_{b_1}^{\infty}\int_{b_2}^{b_1}\cdots\int_{b_m}^{b_{m-1}}\prod_{r=1}^{r=m}\prod_{t=1}^{t=m}(x_r-b_t)^{\gamma_t-1}e^{-\Sigma(x_t-b_t)}J\left(\frac{\theta}{x}\right)dx_1\,dx_2\ldots dx_m$$

is transformed into

$$\Pi\left\{(-1)^{m-1}\phi'(b_t)\right\}^{\gamma_t-1}\int_0^{\infty}\int\cdots\int\Pi\theta_t^{\gamma_t-1}e^{-\Sigma\theta_t}\,d\theta_1\,d\theta_2\ldots d\theta_m,$$

which is equal to $$\Pi\left\{(-1)^{m-1}\phi'(b_t)\right\}^{\gamma_t-1}\Gamma(\gamma_t),$$

and so, writing $I_p(b_q,\,b_{q+1})$ for

$$\int_{b_{q+1}}^{b_q}e^{-(x-b_{q+1})}\prod_{t=1}^{t=m}(x-b_t)^{\gamma_t}\frac{dx}{x-b_p},\tag{32}$$

I get that the determinant

$$\begin{vmatrix} I_1(\infty,\,b_1), & I_1(b_1,\,b_2), & \ldots, & I_1(b_{m-1},\,b_m)\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ I_m(\infty,\,b_1), & I_m(b_1,\,b_2), & \ldots, & I_m(b_{m-1},\,b_m)\end{vmatrix}$$

$$=\Pi\left\{(-1)^{m-1}\phi'(b_t)\right\}^{\gamma_t}\Gamma(\gamma_t).\tag{33}$$

Similarly the integral

$$\int_{b_1}^{\infty}\int_{b_2}^{b_1}\cdots\int_{b_m}^{b_{m-1}}\prod_{r=1}^{r=m}\prod_{t=1}^{t=m}(x_r-b_t)^{\gamma_t-1}\prod_{r=1}^{r=m}\prod_{s=1}^{s=n}(x_r-a_s)^{-\beta_s}e^{-\Sigma(x_t-b_t)}J\left(\frac{\theta}{x}\right)$$
$$\times\,dx_1\,dx_2\ldots dx_m$$

is transformed into

$$\frac{\Pi\left\{(-1)^{m-1}\phi'(b_t)\right\}^{\gamma_t-1}}{\Pi\left\{(-1)^m\phi(a_s)\right\}^{\beta_s}}\int_0^{\infty}\int\cdots\int\frac{\prod\limits_{t=1}^{t=m}\theta_t^{\gamma_t-1}e^{-\Sigma\theta_t}}{\prod\limits_{s=1}^{s=n}\left(1+\Sigma_t\dfrac{\theta_t}{b_t-a_s}\right)^{\beta_s}}\,d\theta_1\,d\theta_2\ldots d\theta_m,$$

which by the help of the formula in § 4 of the preceding paper is equal to

$$\frac{\Pi\left\{(-1)^{m-1}\phi'(b_t)\right\}^{\gamma_t-1}\Gamma(\gamma_t)}{\Pi\left\{(-1)^{m}\phi(a_s)\right\}^{\beta_s}\Gamma(\beta_s)}\int_0^\infty\int\dots\int\frac{\prod\limits_{s=1}^{s=n}\xi_s^{\beta_s-1}\,e^{-\Sigma\xi}}{\prod\limits_{t=1}^{t=m}\left(1+\Sigma_s\dfrac{\xi_s}{b_t-a_s}\right)^{\gamma_t}}\,d\xi_1\,d\xi_2\dots d\xi_n.$$

If in this integral we put

$$(-1)^n f'(a_s)\,\xi_s=(x_1-a_s)(x_2-a_s)\dots(x_n-a_s), \tag{34}$$

where $$f(x)=(x-a_1)(x-a_2)\dots(x-a_n),$$

we obtain

$$\int_{a_2}^{a_1}\int_{a_3}^{a_2}\dots\int_{-\infty}^{a_n}\prod_{r=1}^{r=n}\prod_{s=1}^{s=n}(x_r-a_s)^{\beta_s-1}\prod_{r=1}^{r=n}\prod_{t=1}^{t=m}(x_r-b_t)^{-\gamma_t}\,e^{\Sigma(x_s-a_s)}\,J\left(\frac{\xi}{x}\right)$$
$$\times\,dx_1\,dx_2\dots dx_n,$$

multiplied by $\dfrac{\Pi\left\{(-1)^{m-1}\phi'(b_t)\right\}^{\gamma_t-1}\left\{(-1)^n f(b_t)\right\}^{\gamma_t}\Gamma(\gamma_t)}{\Pi\left\{(-1)^n f'(a_s)\right\}^{\beta_s-1}\left\{(-1)^m\phi(a_s)\right\}^{\beta_s}\Gamma(\beta_s)}$,

so that, writing $I'_p(b_q,\,b_{q+1})$ for

$$\int_{b_{q+1}}^{b_q}e^{-(x-b_{q+1})}\frac{\prod\limits_{t=1}^{t=m}(x-b_t)^{\gamma}}{\prod\limits_{s=1}^{s=n}(x-a_s)^{\beta_s}}\,\frac{dx}{x-b_p}, \tag{35}$$

and $I_p(a_q,\,a_{q+1})$ for $$\int_{a_{q+1}}^{a_q}e^{(x-a_q)}\frac{\prod\limits_{s=1}^{s=n}(x-a_s)^{\beta_s}}{\prod\limits_{t=1}^{t=m}(x-b_t)^{\gamma_t}}\,\frac{dx}{x-a_p}, \tag{36}$$

I get that the determinant

$$\begin{vmatrix} I'_1(\infty,\,b_1), & I'_1(b_1,\,b_2), & \dots, & I'_1(b_{m-1},\,b_m) \\ \dots & \dots & \dots & \dots \\ I_m(\infty,\,b_1), & I'_m(b_1,\,b_2), & \dots, & I'_m(b_{m-1},\,b_m) \end{vmatrix}$$

is equal to the determinant

$$\begin{vmatrix} I_1(a_1,\,a_2), & I_1(a_2,\,a_3), & \dots, & I_1(a_n,\,-\infty) \\ \dots & \dots & \dots & \dots \\ I_n(a_1,\,a_2), & I_n(a_2,\,a_3), & \dots, & I_n(a_n,\,-\infty) \end{vmatrix} \tag{37}$$

multiplied by $\dfrac{\prod\limits_{t=1}^{t=m}\left\{(-1)^{m-1}\phi'(b_t)\right\}^{\gamma_t}\left\{(-1)^n f(b_t)\right\}^{\gamma_t}\Gamma(\gamma_t)}{\prod\limits_{s=1}^{s=n}\left\{(-1)^n f'(a_s)\right\}^{\beta_s}\left\{(-1)^m\phi(a_s)\right\}^{\beta_s}\Gamma(\beta_s)}$.

From these two, by a method exactly similar to that used in § 12, I deduce

$$\sum_{t=1}^{t=m} \gamma_t \int_{b_m}^{b_{m-1}} e^{-(x-b_m)} \frac{\prod_{s=1}^{s=m} (x-b_s)^{\gamma_s}}{x-b_t}\, dx \int_{-\infty}^{b_m} \frac{e^{(x-b_m)}\, dx}{\prod_{s=1}^{s=m} (x-b_s)^{\gamma_s} (x-b_t)}$$

$$= (-1)_{-\gamma^m} \frac{\pi}{\sin \gamma_m \pi},$$

or, changing the notation, so as to agree with that used before, and writing

$$R(x) = \prod_{s=1}^{s=n} (x-a_s)^{\beta_s},$$

I have $$\sum_{s=1}^{s=n} \beta_s \int_{a_p}^{\infty} \frac{e^{-x}\, dx}{R(x)(x-a_s)} \int_{-\infty}^{a_q} \frac{e^x R(x)\, dx}{x-a_s} = (-1)^{\beta_p} \frac{\pi}{\sin \beta_p \pi}, \qquad (38)$$

when $p=q$; and $=0$ when $p \neq q$.

# THE WEDDLE QUARTIC SURFACE

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[Received February 10th, 1905.—Read March 9th, 1905.]

## 1.

In a note at the end of a paper in the *Camb. and Dubl. Math. Journal*, Vol. v., 1850, p. 69, Weddle remarks that the locus of the vertex of a quadric cone passing through six given points is a quartic surface and not a twisted cubic as Chasles had once stated. This note was called attention to by Cayley (*Comptes Rendus*, 1861 ; *Collected Works*, Vol. v., p. 4), and a symmetrical equation for the surface was given by him. The subject was again taken up by Chasles (*Comptes Rendus*, t. LII., 1861, pp. 1157–62), where he shows that the locus of the vertex of a cone which divides six given segments harmonically is also a quartic surface. The properties of this more general surface have been very fully discussed by Cayley ("A Memoir on Quartic Surfaces," *Collected Works*, Vol. VII.), who identifies it with the Jacobian of four quadrics, the Weddle surface corresponding to the case in which the four quadrics have six common points.

From the definition of the surface it is easy to see that the surface contains the fifteen lines joining two of the points and also the lines of intersection of the ten pairs of planes which can be drawn through the six points, making twenty-five lines in all.*

The object of this paper is to obtain some of the tangential properties of the surface ; a construction for the tangent plane at any point is obtained geometrically, but, in the latter half of the paper, considerable use is made of a parametric representation of the surface which was shown to the author by Mr. Richmond.

The surface is obtained as the envelope of a certain system of planes, and is also shown to belong to a general class of surfaces which may be defined as the projections of surfaces formed by the translation of a curve in four dimensions.

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* The existence of the twenty-five lines is noticed by both Cayley and Chasles, but, as Cayley afterwards refers to Chasles' paper, it is probable that they were first discovered by Chasles.

It appears in the course of the work that the twisted cubic is an asymptotic line on the surface. I have obtained a differential equation for the asymptotic lines, but have not succeeded in integrating it.

### 2. *The Jacobian of a System of Quadrics.*

The Jacobian of a linear system of quadrics of four terms may be described geometrically[*] as (1) the locus of the vertices of cones belonging to the system, (2) the locus of pairs of points which are conjugate with regard to all quadrics of the system, (3) the locus of points at which two of the quadrics touch, (4) the locus of the corners of tetrahedra self-conjugate with regard to two of the quadrics. If we denote the four quadrics by $S$, $T$, $U$, $V$, then the equation

$$\alpha S+\beta T+\gamma U+\delta V = 0$$

will represent a cone if

$$\frac{\partial}{\partial x}(\alpha S+-) = 0, \quad \frac{\partial}{\partial y}(\ ) = 0, \quad \frac{\partial}{\partial z}(\ ) = 0, \quad \frac{\partial}{\partial t}(\ ) = 0.$$

The result of eliminating $\alpha$, $\beta$, $\gamma$, $\delta$ gives us a quartic equation, viz., the Jacobian, while the result of eliminating $x$, $y$, $z$, $t$ is a quartic equation in $\alpha$, $\beta$, $\gamma$, $\delta$ which may be considered as that of another surface which Cayley has called the "symmetroid."

In the ordinary case the Jacobian surface contains ten straight lines and the symmetroid possesses ten nodes; and the above equations may be considered as establishing a correspondence between points on the two surfaces. The case in which the four quadrics have six common points is particularly interesting, for then the Jacobian is the Weddle surface, and the symmetroid has sixteen nodes and is no other than Kummer's famous quartic surface.

The second mode of generation tells us that the polar planes of any point $P$ on the Jacobian meet in another point $P'$; the relation between the points $P$ and $P'$ is evidently a reciprocal one, and so we have pairs of corresponding points on the surface.

When the quadrics have six common points, there will be three linearly independent quadrics containing the twisted cubic through the six points; the polar planes of a point $P$ with regard to these quadrics meet on the chord of the cubic through $P$. It follows, therefore, that the line joining two corresponding points on the surface is a chord of the twisted cubic and is cut harmonically by it.[†]

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[*] Cf. Darboux, *Bulletin des Sciences Mathématiques*, Ser. 1, t. I., p. 348, 1870.

[†] This fact is mentioned by Darboux, *loc. cit.*, p. 356; he also remarks that in the general case, the lines joining corresponding points form a congruence (7, 3).

Since any chord of the cubic is cut harmonically by the surface, it follows that any tangent to the cubic meets the surface in three consecutive points; the cubic is therefore an asymptotic line on the surface.

The system of corresponding points is obtained in another way by considering the system of quadrics apolar* to the original system of quadrics; it is built up by taking linear combinations of the tangential equations of any six linearly independent quadrics apolar to the given set.

Let $U_1 = 0$, $U_2 = 0$, ..., $U_6 = 0$ be the tangential equations of the six base points: then the six quadrics $U_1^2 = 0$, ..., $U_6^2 = 0$ are linearly independent and are apolar to any quadrics through the six points; accordingly, the general equation of a quadric apolar to the first system is

$$k_1 U_1^2 + k_2 U_2^2 + \ldots + k_6 U_6^2 = 0.$$

Whenever this equation represents two points they will be conjugate with regard to any quadric through the six points, and will therefore be corresponding points on the Weddle surface; if, then, $L$ and $L'$ be the tangential equations of two such points, we have a relation of the form†

$$(1) \qquad LL' \equiv k_1 U_1^2 + k_2 U_2^2 + k_3 U_3^2 + k_4 U_4^2 + k_5 U_5^2 + k_6 U_6^2.$$

By giving special values to the $k$'s we may obtain a number of points on the surface: for instance, by taking $k_3 = k_4 = k_5 = k_6 = 0$, we recognize that the straight line joining 1 and 2 lies entirely on the surface; by taking $k_2$ also zero, the point 1 appears as a self-corresponding point. Since we have five lines through each of these points, they must all be nodes on the surface.

Now let $M$ and $N$ be the equations of two points dividing $LL'$ harmonically; then clearly

$$LL' = M^2 - N^2,$$

so that

$$M^2 - N^2 \equiv k_1 U_1^2 + \ldots + k_6 U_6^2.$$

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* Two quadrics whose point and plane equations are respectively

$$S = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2 + 2px_1x_4 + 2qx_2x_4 + 2rx_3x_4 = 0,$$

$$\Sigma = Au_1^2 + Bu_2^2 + Cu_3^2 + Du_4^2 + 2Fu_2u_3 + 2Gu_3u_1 + 2Hu_1u_2 + 2Pu_1u_4 + 2Qu_2u_4 + 2Ru_3u_4 = 0,$$

are said to be apolar when the invariant

$$\Theta' = aA + bB + cC + dD + 2Ff + 2Gg + 2Hh + 2Pp + 2Qq + 2Rr$$

vanishes; it is easily seen that, when $\Sigma = 0$ represents two points, this is the condition that they should be conjugate with regard to $S = 0$.

† Cf. Serret, *Nouvelles Annales*, Sér. 2, t. IV., 1865, p. 438.

It has been remarked by Serret* that a relation of this kind is the condition that the eight points $M$, $N$, $U_1, \ldots, U_6$ are such that any quadric through seven of the points should pass through the eighth; accordingly, every quadric through $U_1, \ldots, U_6$, $M$ passes through $N$, and every pair of points $M$, $N$ on the line $LL'$ which possess this property divide $LL'$ harmonically. A pair of such points can only coincide at one of the points $L$ and $L'$. Therefore, if we consider a transformation in which a point $N$ is derived from $M$ by drawing quadrics through $U_1, U_2, \ldots, U_6$, and $M$, and finding their eighth common point of intersection, the Weddle surface will appear as the locus of a point $M$ which coincides with its corresponding point $N$. This method of obtaining the surface is due to Geiser (*Crelle*, Vol. LXVII., 1867, p. 78).

If we consider only planes which pass through the point $L$, the relation (1) gives

$$k_1U_1^2+k_2U_2^2+\ldots+k_6U_6^2 = 0.$$

Now $U_1 = 0$ may be regarded as the tangential equation of the point in which any plane is met by the line $LU_1$, and a relation of this form between the tangential equations of six points in a plane is the necessary and sufficient condition that they should lie on a conic.† This shows that $LU_1, LU_2, \ldots, LU_6$ are generators of a quadric cone, so that $L$ actually is the vertex of a cone through the six points $U_1, U_2, \ldots, U_6$.

### 3. *Parametric Representation of the Surface.*

Let $A$ and $B$ be any two points upon the twisted cubic through the six nodes; then any quadric through $U_1, U_2, \ldots, U_6$ and $A$ contains the cubic, and therefore passes through $B$; accordingly, there must be an identical relation of the form

$$A^2-B^2 = k_1U_1^2+k_2U_2^2+\ldots k_6U_6^2,$$

where $A = 0$ and $B = 0$ are the tangential equations of $A$ and $B$. If, then, we break $A^2-B^2$ into factors, we shall obtain the tangential equations of a pair of corresponding points on the surface.

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* Serret, "Géometrie de Direction," *Nouvelles Annales*, t. IV., 1865, p. 440. An admirable account of theorems of this type, with full extensions to curves and surfaces of any order, is given by Clifford, *Collected Works*, pp. 119–129.

† This theorem is commonly ascribed to Serret, but, as Darboux points out (*Bulletin des Sciences Mathématiques*, Ser. 1, t. I., 1870, p. 11), it was first given by Hesse (Extrait du *Zeitschrift für Mathematik und Physik*, 1866; Leipzig, Teubner).

The co-ordinates of points on the twisted cubic may be expressed in terms of a parameter $\theta$ by the relations

$$x:y:z:t = \theta^3:\theta^2:\theta:1.$$

Denoting the parameters of $U_1, \dots, U_6, A, B$ by $\theta_1, \theta_2, \dots, \theta_6, \theta, \phi$, the identical relation between the squares of the tangential equations of the eight points may easily be verified to be

$$\frac{(l\theta^3+m\theta^2+n\theta+p)^2}{(\theta-\phi)(\theta-\theta_1)\dots(\theta-\theta_6)}+\frac{(l\phi^3+m\phi^2+n\phi+p)^2}{(\phi-\theta)(\phi-\theta_1)\dots(\phi-\theta_6)}$$
$$+\sum_{r=1}^{6}\frac{(l\theta_r^3+m\theta_r^2+n\theta_r+p)^2}{(\theta_r-\theta)(\theta_r-\phi)(\theta_r-\theta_1)\dots(\theta_r-\theta_6)}=0.$$

Comparing this with the previous relation, we have at once

$$LL' = A^2-B^2 = \frac{(l\theta^3+m\theta^2+n\theta+p)^2}{(\theta-\phi)(\theta-\theta_1)\dots(\theta-\theta_6)}+\frac{(l\phi^3+m\phi^2+n\phi+p)^2}{(\phi-\theta)(\phi-\theta_1)\dots(\phi-\theta_6)}.$$

Rejecting the common factor $\theta-\phi$, we see that the tangential equation of a pair of corresponding points $L$ and $L'$ may be written in the form

$$\frac{(l\theta^3+m\theta^2+n\theta+p)^2}{f(\theta)}=\frac{(l\phi^3+m\phi^2+n\phi+p)^2}{f(\phi)},$$

where $$f(a)=(a-\theta_1)(a-\theta_2)\dots(a-\theta_6).$$

Taking the square root of each side, we obtain the parametric representation

$$x:y:z:t=\frac{\theta^3}{\sqrt{f(\theta)}}\pm\frac{\phi^3}{\sqrt{f(\phi)}}:\frac{\theta^2}{\sqrt{f(\theta)}}\pm\frac{\phi^2}{\sqrt{f(\phi)}}:\frac{\theta}{\sqrt{f(\theta)}}\pm\frac{\phi}{\sqrt{f(\phi)}}$$
$$:\frac{1}{\sqrt{f(\theta)}}\pm\frac{1}{\sqrt{f(\phi)}}.$$

This representation was obtained by Mr. Richmond by writing down the co-ordinates of $L$ and $L'$ as

$$x:y:z:t=\frac{\theta^3}{\Theta}\pm\frac{\phi^3}{\Phi}:\frac{\theta^2}{\Theta}\pm\frac{\phi^2}{\Phi}:\frac{\theta}{\Theta}\pm\frac{\phi}{\Phi}:\frac{1}{\Theta}\pm\frac{1}{\Phi}$$

(the co-ordinates necessarily being of this form, since $AB$ is cut harmonically by $L$ and $L'$), and expressing that $L$ and $L'$ are conjugate with regard to quadrics through the six points. The hyperelliptic parametric representation is considered by Humbert (*Liouville*, Vol. IX., 1893, pp. 466–473). Methods of obtaining a representation of this kind directly are suggested in a recent paper by Dr. Baker (*Proc. London Math. Soc.*, 1903, Vol. I., p. 261) and in Hudson's treatise on *Kummer's Surface*, p. 199.

### 4. *Systems of Points on the Surface.*

If we denote the quantity $(l\theta^3+m\theta^2+n\theta+p)^2/f(\theta)$ by $F(\theta)$, then $F(\theta)-F(\phi)$ is the tangential equation of a pair of corresponding points on the surface.

Let $\theta$, $\phi$, $\psi$ be the parameters of any three points on the twisted cubic; they give rise to three pairs of points

$$\alpha\alpha' \equiv F(\phi)-F(\psi), \qquad \beta\beta' \equiv F(\psi)-F(\theta), \qquad \gamma\gamma' \equiv F(\theta)-F(\phi),$$

connected by the relation

$$\alpha\alpha'+\beta\beta'+\gamma\gamma' = 0.$$

This signifies that the six points lie by threes on four lines, *i.e.*, are at the corners of a plane quadrilateral. Further, if $\alpha$, $\alpha'$ and $\beta$, $\beta'$ are two pairs of corresponding points in a plane, they are conjugate with regard to all quadrics of the system, and so by a well known theorem due to Hesse the other two points $\gamma$, $\gamma'$ in which the four lines $\alpha\beta$, $\alpha\beta'$, $\alpha'\beta$, $\alpha'\beta'$ intersect are also conjugate and are therefore corresponding points on the surface. If, however, the plane meets the cubic curve in $A$, $B$, $C$, it is easily seen that $(\alpha, \alpha')$, $(\beta, \beta')$, $(\gamma, \gamma')$ must lie on $BC$, $CA$, $AB$ respectively; so that there are only three pairs of corresponding points in each plane.

Again, if $\theta$, $\phi$, $\psi$, $\chi$ are the parameters of any four points on the cubic, we obtain six pairs of points,

$$\begin{aligned} \alpha\alpha' &= F(\phi)-F(\psi), & xx' &= F(\theta)-F(\chi), \\ \beta\beta' &= F(\psi)-F(\theta), & yy' &= F(\phi)-F(\chi), \\ \gamma\gamma' &= F(\theta)-F(\phi), & zz' &= F(\psi)-F(\chi), \end{aligned}$$

and we have the relation

$$\alpha\alpha'xx'+\beta\beta'yy'+\gamma\gamma'zz' = 0,$$

which signifies that the three tetrahedra whose corners are $(\alpha, \alpha', x, x')$, $(\beta, \beta', y, y')$, $(\gamma, \gamma', z, z')$ respectively form a desmic system.*

If we take any six points in space and draw the twisted cubic through them, we may choose the coefficients in $f(\theta)$ so that the quantities $F(\theta)$ are any given multiples of the tangential equations of the given points; accordingly, if the line joining each pair of points is divided internally and externally in the ratio of the numbers corresponding to each point, the thirty points thus found will lie on a Weddle surface. With each point as centre let a sphere, whose radius is proportional to the corresponding number, be described; then the thirty points are the centres of similitude of these six spheres taken in pairs; hence the centres of similitude of any six spheres lie on a Weddle surface.

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* For an account of desmic tetrahedra I may refer to Hudson's treatise on *Kummer's Surface*, pp. 1, 3.

If the six nodes of a Weddle surface lie on a sphere, the surface may be described as the locus of the centres of similitude of pairs of spheres whose centres lie on a twisted cubic and which cut a given sphere orthogonally; for the tangential equation of such a sphere is of the form

$$(l\theta^3+m\theta^2+n\theta+p)^2 = kf(\theta),$$

where $f(\theta) = 0$ is the result of substituting $x = \theta^3$, $y = \theta^2$, $z = \theta$, $t = 1$, in the equation of the given sphere, and the centres of similitude of a pair of such spheres are given by

$$\frac{(l\theta^3+m\theta^2+n\theta+p)^2}{f(\theta)} = \frac{(l\phi^3+m\phi^2+n\phi+p)^2}{f(\phi)};$$

so that they are corresponding points on the Weddle surface whose nodes are the points of intersection of the given sphere with the twisted cubic.

The surface which is the locus of the centres of similitude of spheres belonging respectively to two singly infinite sets possesses a number of interesting properties. It is easily seen that the co-ordinates of any point on such a surface can be expressed in the form

$$x : y : z : t = f_1(u)+\phi_1(v) : f_2(u)+\phi_2(v) : f_3(u)+\phi_3(v) : f_4(u)+\phi_4(v).$$

If now we regard $(x, y, z, t)$ as the actual Cartesian co-ordinates of a point $P$ in space of four dimensions, the corresponding point $Q$ in our space will be obtained by projecting $P$ from some outside point. If, moreover, the co-ordinates of $P$ are given by the above equations, $P$ will lie on a certain surface obtained by the translation of a given curve.

The lines $u =$ constant, $v =$ constant form a system of conjugate lines on such a surface and will project into a conjugate system on the surface in our space. In the case of the Weddle surface the space curve by means of which the surface of translation is generated is easily seen to be of the sixth degree;* it is symmetrical with regard to the vertex of projection in one position for which it projects into the twisted cubic through the nodes of the Weddle surface; in any other position it passes through the origin and so projects into a curve of the fifth degree. We thus obtain a conjugate system of quintic curves on the Weddle surface.

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* The surface may be considered to be generated by the curve $\phi =$ constant moving so that each point on it describes a similar curve; the symmetrical position is given by $\phi = 0$, while in any other position the origin is one of the points given by $\theta = \phi$.

The class of surfaces obtained by reciprocation are considered by Darboux (*Théorie Générale des Surfaces*, Vol. I., p. 123), where it is shown that they possess two families of plane curves forming a conjugate system: a surface of this type may be described geometrically as the envelope of the radical plane of two spheres belonging respectively to two singly infinite systems.

### 5. *Tangential Properties of the Surface.*

Let $S$ and $S'$ be two consecutive points on the twisted cubic through the six nodes, $R$ any other point on this cubic; then the sides of the triangle $RSS'$ will meet the surface again in three pairs of points $PQ$, $P'Q'$, $TT'$, lying by threes on four lines (§ 4). Hence $PP'$ and $QQ'$, which are ultimately tangents at $P$ and $Q$, intersect in a point $T$ on the surface, and, since the corresponding point $T'$ ultimately coincides with $S$, the polar planes of $S$ with regard to quadrics through the six nodes meet in the point $T$, which lies upon the tangent at $S$.

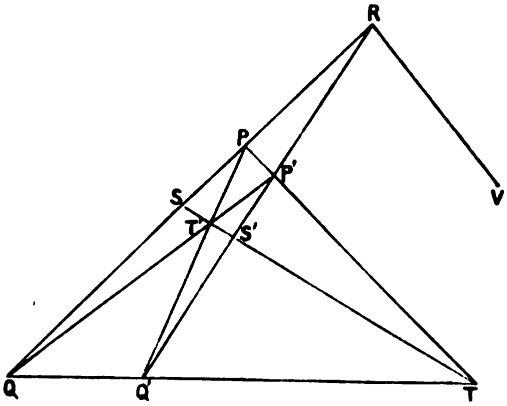

FIG. 1.

As $R$ moves along the cubic, the point $T$ will remain fixed; the points $P$ and $P'$ will therefore describe the curve of contact of the tangent cone from $T$ to the surface.

Again, if $U$ be the point derived from $R$ in the same way that $T$ was derived from $S$, and we keep $R$ fixed and vary $S$, the points $P$ and $Q$ will describe the curve of contact of the tangent cone from $U$; also it is easily seen from the above figure that $TP$ and $TQ$ are the tangents at $P$ and $Q$ to this curve of contact.

Now $UP$ and $UQ$ are generators of this cone; hence it follows that $PU$ and $PT$ are conjugate tangents to the surface at $P$, and so the curves obtained by keeping one point on the cubic fixed form a conjugate system. They are in fact the curves of the fifth degree obtained previously.

The tangent planes to the surface at $P$ and $Q$ are the planes $PUT$ and $QUT$ respectively. Now $P$ and $Q$, being points on the surface, will be vertices of cones passing through the six double points; the polar planes of $R$ and $S$ with regard to these cones will therefore pass through $U$ and $T$ respectively; but any point on $RS$ has the same polar plane with regard to a cone vertex $P$; hence the polar plane of $Q$ with regard to this cone is the plane $PUT$, that is, the tangent plane at $P$.

The tangent plane to the surface at any point $P$ is therefore the polar plane of the corresponding point $Q$ with regard to the cone whose vertex is $P$, and which passes through the six double points.*

## 6. *Analytical Treatment.*

The plane $lx+my+nz+pt = 0$ will pass through the point $(\theta, \phi)$ on the Weddle surface, if

$$\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}} = \frac{l\phi^3+m\phi^2+n\phi+p}{\sqrt{f(\phi)}}.$$

It will pass through the consecutive point $(\theta+\delta\theta, \phi+\delta\phi)$ for all values of $\delta\theta$ and $\delta\phi$. If

$$\frac{\partial}{\partial\theta}\left\{\frac{l\theta^3+m\theta^3+n\theta+p}{\sqrt{f(\theta)}}\right\} = 0$$

and

$$\frac{\partial}{\partial\phi}\left\{\frac{l\phi^3+m\phi^2+n\phi+p}{\sqrt{f(\phi)}}\right\} = 0,$$

it will then be the tangent plane at the point $\theta$, $\phi$. These equations signify that the tangent plane passes through the point whose tangential equation is

$$\frac{\partial}{\partial\theta}\left\{\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}}\right\} = 0,$$

for all values of $\phi$.

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* This construction is analogous to that for the tangent plane at any point on the Hessian of a cubic surface. Darboux remarks that many of the properties of the Hessian can be extended to the Jacobian of any four quadrics; it is probable therefore that this construction was known to him.

If in Fig. 1 $S$ is the point $\theta$, and $R$ the point $\phi$, then the equation just written down represents the point $T$. The co-ordinates of $T$ may therefore be represented by

$$x : y : z : t = \frac{\partial}{\partial\theta}\left\{\frac{\theta^3}{\sqrt{f(\theta)}}\right\} : \frac{\partial}{\partial\theta}\left\{\frac{\theta^2}{\sqrt{f(\theta)}}\right\} : \frac{\partial}{\partial\theta}\left\{\frac{\theta}{\sqrt{f(\theta)}}\right\} : \frac{\partial}{\partial\theta}\left\{\frac{1}{\sqrt{f(\theta)}}\right\};$$

so that the locus of $T$ is a rational curve of degree 7 lying on the surface.

If $P$ is any point on this curve, the chord $RS$ is a tangent to the twisted cubic and so the points $P$, $U$ and $T$ coincide; but $PU$ and $PT$ are conjugate directions on the surface; therefore $PU$ is a tangent to one of the asymptotic lines at $P$; hence this rational curve of degree 7 is an asymptotic line on the surface.

We saw that the tangent plane at any point must satisfy the relations

$$\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}} = \frac{l\phi^3+m\phi^2+n\phi+p}{\sqrt{f(\phi)}},$$

$$\frac{\partial}{\partial\theta}\left\{\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}}\right\} = 0, \qquad \frac{\partial}{\partial\phi}\left\{\frac{l\phi^3+m\phi^2+n\phi+p}{\sqrt{f(\phi)}}\right\} = 0.$$

Consider the sextic $(l\theta^3+m\theta^2+n\theta+p)^2-kf(\theta) = 0.$

If we eliminate $\theta$ between this equation and the second equation, we shall obtain the values of $k$ for which this sextic has equal roots; the first and third equation then signify that this occurs twice for each value of $k$, since $\theta$ and $\phi$ are different; accordingly, with such a value of $k$ the sextic has two pairs of equal roots, and so the tangential equation of Weddle's surface may be obtained by expressing the condition that the equation

$$(l\theta^3+m\theta^2+n\theta+p)^2-kf(\theta) = 0$$

may have two pairs of equal roots for some value of $k$.[*]

The tangential equation will be of the twenty-fourth degree, so I shall not stop to work it out; we may, however, obtain a geometrical interpretation of the last result.

Let $U = 0$ be any quadric through the six nodes; then the equation

$$V = (lx+my+nz+pt)^2-kU = 0$$

represents a quadric touching $U = 0$ round a conic. The points where

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[*] This method may be adopted whenever a surface has a parametric representation

$$x : y : z : t = f_1(\theta)+f_1(\phi) : f_2(\theta)+f_2(\phi) : f_3(\theta)+f_3(\phi) : f_4(\theta)+f_4(\phi).$$

$V = 0$ meets the cubic are given by

$$(l\theta^3+m\theta^2+n\theta+p)^2-kf(\theta) = 0\,;$$

so that, if $k$ is properly chosen, $V$ will touch the twisted cubic twice. If therefore we describe a quadric $V$ to touch a given quadric $U$ all round a conic and to have double contact with a twisted cubic, then the envelope of the plane of the conic is a Weddle surface.

The tangent plane at $(\theta, \phi)$ will pass through a consecutive point $(\theta+\delta\theta, \phi+\delta\phi)$ if

$$\frac{\partial^2}{\partial\theta^2}\left\{\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}}\right\}\partial\theta^2 = \frac{\partial^2}{\partial\phi^2}\left\{\frac{l\phi^3+m\phi^2+n\phi+p}{\sqrt{f(\phi)}}\right\}\delta\phi^2.$$

Now, since the equation

$$(lx^3+mx^2+nx+p)^2-kf(x) = 0$$

has two pairs of equal roots, we may write

$$(lx^3+mx^2+nx+p)^2-kf(x) = \lambda\,(x-\theta)^2\,(x-\phi)^2\,(x-\alpha)(x-\beta)\,;$$

therefore

$$\left[\frac{lx^3+mx^2+nx+p}{\sqrt{f(x)}}\right]^2-k = \lambda\,\frac{(x-\theta)^2\,(x-\phi)^2\,(x-\alpha)(x-\beta)}{f(x)}\,.$$

Differentiate twice with regard to $x$ and put $x = 0$; then, since

$$\frac{\partial}{\partial\theta}\left[\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}}\right] = 0,$$

we obtain the relation

$$2\,\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}}\,\frac{\partial^2}{\partial\theta^2}\left[\frac{l\theta^3+m\theta^2+n\theta+p}{\sqrt{f(\theta)}}\right] = 2\,\frac{(\theta-\phi)^2\,(\theta-\alpha)(\theta-\beta)}{f(\theta)},$$

with a similar one in $\phi$; accordingly the differential equation for the asymptotic lines becomes

$$\frac{(\theta-\alpha)(\theta-\beta)}{f(\theta)}\,d\theta^2 = \frac{(\phi-\alpha)(\phi-\beta)}{f(\phi)}\,d\phi^2,$$

where $\alpha$ and $\beta$ may be regarded as defined in terms of $\theta$ and $\phi$ by the fact that

$$kf(x)+\lambda\,(x-\theta)^2\,(x-\phi)^2\,(x-\alpha)(x-\beta)$$

is a perfect square.

The asymptotic lines at a point $(\theta, \phi)$ will coincide only when one of the coefficients of $d\theta^2$ and $d\phi^2$ is zero; this will occur either when $f(\phi) = 0$, that is, at the double points, or when, say, $\phi = \alpha$; this, therefore, may be regarded as the equation of the parabolic curve.

The first form of the equation shows that the tangent plane at a point on the parabolic curve satisfies the two equations

$$\frac{\partial}{\partial\phi}\left\{\frac{l\phi^3+m\phi^2+n\phi+p}{\sqrt{f(\phi)}}\right\}=0,\qquad \frac{\partial^2}{\partial\phi^2}\left\{\frac{l\phi^3+m\phi^2+n\phi+p}{\sqrt{f(\phi)}}\right\}=0;$$

the developable generated by them therefore contains the rational curve of degree 7.

A particular integral of the differential equation for the asymptotic lines is evidently given by $\theta=\phi$: this consists of the twisted cubic and the rational curve of degree 7 already found.

## 7. *Curves on the Surface.*

The principal types of curves on the surface known to us at present may be described as follows:—

(1) *Sextic Curves.*—These include the Jacobian of three quadrics, a curve which may be described geometrically as the locus of the vertex of a cone through the six nodes and one other point; the cones necessarily pass through an eighth point, and the curve is symmetrically situated with regard to these points. In general the curve does not pass through any of the points, but, if the seventh point is on the Weddle surface, the eighth point coincides with it and the sextic has an actual double point there.*

These curves are of great importance in one of the birational transformations between the Weddle surface and the Kummer surface (cf. Hudson, pp. 169–172). A beautiful method of obtaining this birational transformation was given by Dr. Baker (*loc. cit.*, p. 249): it has the advantage of exhibiting the relation between the two surfaces in a clear form.

A point on the Weddle surface is made to correspond by means of the transformation

$$x=\frac{\eta'(\zeta'-\zeta)}{\xi(\eta-\eta')},\quad y=\frac{\zeta'(\xi'-\eta)}{\eta(\zeta-\zeta')},\quad z=\frac{\xi'(\eta'-\zeta)}{\zeta(\xi-\xi')},\quad \xi+\eta+\zeta=\xi'+\eta'+\zeta'$$

to a point on the curve of contact of the tangent cone to the variety $\xi\eta\zeta=\xi'\eta'\zeta'$ from an arbitrary point upon it: the section of the cone by an arbitrary plane is a Kummer surface.

Any quadric through the six nodes of $W$ is shown to correspond to a planar three-fold through the vertex of projection and therefore to a plane section of $K$, a cone through the nodes of $W$ corresponding to a tangent plane to $K$.

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* See Cayley, *loc. cit.*; also Chasles' paper in *Comptes Rendus*.

A system of quadrics through the six nodes and one other point corresponds to planar three-folds passing through a line, and therefore to plane sections of $K$ passing through a fixed point. The sextic curve which is the locus of the vertices of cones of the family will therefore correspond to the curve of contact of the tangent cone from this point to $K$; the corners of the tetrahedron which is self-conjugate with regard to two of the quadrics correspond to the points of contact of the tangent planes through a line to $K$.

Further properties of these sextic curves are given by Humbert (*loc. cit.*) where their hyperelliptic equations are obtained.

Another type of sextic curve is cut out by a cubic surface with nodes at four of the nodes of $W$, but they are not of much interest.

(2) *Octavic Curves cut out by Quadrics through the six Nodes.*—These correspond to plane sections of $K$. When the quadric contains the twisted cubic through the six nodes the octavic breaks up into the cubic and a quintic; if, moreover, it is a cone, we obtain the type of quintic curve already considered; since this quintic lies on the polar cubic of the point $T$ (Fig. 1), it appears that the cone whose vertex is $P$ and the polar cubic have one line in common.

(3) *Curves of the tenth degree cut out by Weddle Surfaces whose Nodes lie on the same Twisted Cubic.*—These curves are mentioned in Darboux's paper in connection with a linear system of five independent quadrics. If $f(x) = 0$ and $\phi(x) = 0$ are the equations giving the parameters on the twisted cubic of the two sets of nodes, the curve will cut the twisted cubic in ten points given by the equation

$$f(x)\,\phi'(x) - f'(x)\,\phi(x) = 0,$$

and will cut each surface again in a configuration of thirty points similar to that considered in § 4.

If $W$ and $W'$ are the two surfaces, we may obtain the curve as the intersection of any two surfaces of the system

$$W + kW' = 0;$$

accordingly we shall obtain a series of groups of thirty points upon the curve.

The straight lines on $W + kW'$ which do not meet the cubic will be quadrisecants of the curve, and these can be shown to lie upon a ruled surface of degree 10 (Darboux).

If the nodes of the second Weddle surface coincide in pairs, it degenerates into a plane and a cubic surface having three nodes; the $C_{10}$ then breaks up into a plane section and a sextic curve; the thirty points in any one configuration divide into two sets of fifteen, one set lying on the sextic and the other upon the plane quartic. We thus obtain configurations of fifteen points upon the plane quartic lying by threes on twenty lines.

Another degenerate case occurs when $W$ and $W'$ have four common nodes: they then have six lines in common and touch all along the twisted cubic; the residual intersection is therefore a quartic curve.

There are also a number of special curves of low degree: for instance, Darboux remarks the existence of (1) ten twisted cubics having nine of the lines on the surface as chords; (2) twenty-five twisted cubics having three of the lines as chords; (3) forty-five quartics meeting eight lines, forty-five quartics meeting two lines.

We may obtain other types of curves upon the surface by using the birational transformation in which a point $P$ corresponds to a point $P'$ on the same chord of the cubic, and such that $PP'$ divides this chord harmonically; it will, however, be found that some of the curves mentioned transform into themselves.

[*May* 19*th*, 1905.—The reciprocal of Weddle's surface being the envelope of the plane of a conic which touches six planes, it is natural to expect that the properties of the surface will take a simpler form when the six planes touch the circle at infinity. The surface may then be obtained either as the envelope of the principal planes of a quadric inscribed in a quartic developable, or as the locus of the centre of a sphere having double contact with this developable.

If the equation of a rational curve of degree seven with six cusps be written in the form

$$x : y : z : 1 = \frac{\partial}{\partial\theta}\left\{\frac{\theta^3}{\sqrt{f(\theta)}}\right\} : \frac{\partial}{\partial\theta}\left\{\frac{\theta^2}{\sqrt{f(\theta)}}\right\} : \frac{\partial}{\partial\theta}\left\{\frac{\theta}{\sqrt{f(\theta)}}\right\} : \frac{\partial}{\partial\theta}\left\{\frac{1}{\sqrt{f(\theta)}}\right\},$$

and we regard it as having a density $\rho$ given by the equation

$$\rho\, ds = d\left\{\frac{1}{\sqrt{f(\theta)}}\right\},$$

the Weddle surface will be the locus of the centre of gravity of any portion of this curve.]

# ON THE COMPLETE REDUCTION OF ANY TRANSITIVE PERMUTATION - GROUP; AND ON THE ARITHMETICAL NATURE OF THE COEFFICIENTS IN ITS IRREDUCIBLE COMPONENTS

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[Received and Read March 9th, 1905.]

THE present paper consists of two quite distinct parts, though the second is closely connected with the first. If a group of finite order $N$ is represented as a regular permutation-group in $N$ symbols, and if this permutation-group is completely reduced, each irreducible component occurs a number of times equal to the number of symbols on which it operates. This result is due to Herr Frobenius. I have recently given a direct proof* of it, depending only on two fundamental properties of finite groups. The first part of the present paper determines the number of times that any given irreducible component occurs, when any representation of a group of finite order as a transitive permutation-group is completely reduced. The result, of course, involves the above mentioned theorem as a particular case. In the second part of the paper the actual reduction of the permutation-group is dealt with: first, on the supposition that the domain of rationality is defined by the characteristics; secondly, that it is defined by the roots of unity of which the characteristics are functions. Conditions are obtained under which it is possible to exhibit the completely reduced groups so that the coefficients in their transformations shall be (i) rational functions of the characteristics, (ii) rational functions of roots of unity.

## I.

*Notation.*—$G$ is an abstract group of finite order $N$, with $r$ sets of conjugate operations.

$$\Gamma_1, \Gamma_2, \ldots, \Gamma_r$$

are the $r$ distinct representations of $G$ as an irreducible group of linear substitutions; $\Gamma_1$ being the so-called identical representation in which every operation of $G$ corresponds to the identical substitution.

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* *Acta Mathematica*, Vol. XXVIII., pp. 369–387.

The number of operations in the $i$-th conjugate set is $h_i$, and the characteristic of any operation of the $i$-th conjugate set in $\Gamma_k$ is $\chi_i^k$.

If $g$ is any sub-group of $G$, the representation of $G$ as a transitive group of permutations in which the sub-group which leaves one symbol unchanged is $g$ is denoted by $G_g$. In this way, by taking for $g$ any one from each of the sets of conjugate sub-groups of $G$ in turn, all possible representations of $G$ as a transitive group of permutations arise.

In the representation $\Gamma_k$, $g$ may or may not be reducible. In any case the symbol $\mu_g^k$ is used to denote the number of linearly independent linear functions of the symbols transformed by $\Gamma_k$ which remain unchanged by every operation of $g$. Then the theorem to be proved is the following:—

In the completely reduced form of the transitive permutation-group $G_g$ the irreducible component $\Gamma_k$ occurs $\mu_g^k$ times; *i.e.*, the complete reduction of $G_g$ is given by the relation

$$G_g = \sum_k \mu_g^k \Gamma_k.$$

It may be noticed that when $g$ consists of the identical operation only then $\mu_g^k$, from the above definition, is $\chi_1^k$; and the result is equivalent to the known reduction of $G$ expressed as a *regular* permutation-group.

Let
$$x_1,\ x_2,\ \ldots,\ x_m$$

be the symbols permuted by $G_g$, $N/m$ being the order of $g$. Any one of them can be expressed linearly in terms of the new symbols (these will be denoted by Greek letters) in terms of which the complete reduction of $G_g$ is effected. Let these symbols be

$$\begin{array}{l} \sigma, \quad \left(= \dfrac{1}{m}(x_1+x_2+\ldots+x_m)\right), \\ \xi_1,\ \xi_2,\ \xi_3,\ \ldots, \\ \eta_1,\ \eta_2,\ \eta_3,\ \ldots, \\ \ldots \quad \ldots \quad \ldots \end{array}$$

those in each line being transformed irreducibly among themselves by one of the irreducible components of $G_g$. A linear transformation of the $\xi$'s (or $\eta$'s, ...) among themselves does not affect the reduction of $G_g$, but only the final form in which the corresponding irreducible component is expressed.

The equation giving $x_1$ in terms of the new symbols may therefore be written in the form

$$x_1 = \sigma+\xi_i+\eta_j+\ldots,$$

there being one symbol corresponding to each reduced set. In fact, $x_1$ takes $m$ linearly independent values under the operations of the group, while, if there were no $\xi$ in the expression for $x_1$, the right-hand side of the above equation would take less than $m$ linearly independent values.

Now the sub-group $g$, which leaves $x_1$ unchanged, must leave $\xi_i$ unchanged. Hence $\Gamma_k$ can only occur in the completely reduced form of $G_g$ if $\mu_g^k \geqslant 1$.

Conversely, if $\mu_g^k \geqslant 1$, let $X_1$ be a linear function of the symbols transformed by $\Gamma_k$, which is unaltered by every operation of $g$. Then, under the operations of $\Gamma_k$, $X_1$ takes just $m$ values (not necessarily distinct and certainly not linearly independent). If these are

$$X_1,\ X_2,\ \ldots,\ X_m,$$

they are permuted transitively among themselves by every operation of $\Gamma_k$. The transitive permutation-group thus arising is identical, except as regards the symbols in which it is written, with that of the $x$'s; *i.e.*, if the $X$'s are taken in a suitable sequence, they undergo for *each* operation of $G$ the same permutation as the $x$'s. Suppose now the complete reduction of the permutation-group in the $X$'s formally carried out. Since the $X$'s are linearly equivalent to just $\chi_1^k$ independent symbols, some of the irreducible sets so formed must be sets of zeroes; but at least one set must occur which does not reduce to a set of zeroes. Moreover, since the $\chi_1^k$ symbols operated on by $\Gamma_k$ undergo the transformations of an irreducible group, any set which does not reduce the zeroes must be $\chi_1^k$ in number and must undergo an irreducible group of linear substitutions equivalent to $\Gamma_k$.

Hence, if $\mu_g^k \geqslant 1$, $\Gamma_k$ must occur in the completely reduced form of $G_g$. Suppose next that $\Gamma_k$, where $\mu_g^k \geqslant 1$, occurs more than once in the reduced form of $G_g$. (For convenience of notation it is taken as occurring three times, but it will be seen that this does not affect the argument.) Let the corresponding reduced sets of variables be denoted by

$$\begin{gathered}\xi_1,\ \xi_2,\ \ldots,\\ \eta_1,\ \eta_2,\ \ldots,\\ \zeta_1,\ \zeta_2,\ \ldots,\end{gathered}$$

and suppose them chosen so that each set undergoes the same linear transformation for *each* operation of $G$.

In the expression of $x_1$,

$$x_1 = \sigma+\xi_i+\eta_j+\zeta_k+\dots,$$

$j$ cannot be the same as $i$. In fact

$$\xi_1+\eta_1,\quad \xi_2+\eta_2,\quad \xi_3+\eta_3,\quad \dots$$

undergo the same transformations as

$$\xi_1,\quad \xi_2,\quad \xi_3,\quad \dots,$$

and may be taken for one of the irreducible sets. If, then, $j = i$,

$$x_1 = \sigma+(\xi_i+\eta_i)+\zeta_k+\dots$$

would take fewer than $m$ independent values under the operations of the group, which is impossible. Similarly any such relation as

$$\zeta_k = \alpha\zeta_i+\beta\zeta_j,$$

where $\alpha$ and $\beta$ are constants, is impossible for the same reason.

Hence $\xi_i$, $\xi_j$, $\xi_k$ are linearly independent. Now, by supposition, the $\xi$'s, $\eta$'s, and $\zeta$'s undergo the same transformation for every operation of $G$. Hence, since $\eta_j$ and $\zeta_k$ are unchanged by every operation of $g$, so also are $\xi_j$ and $\xi_k$. If $\Gamma_k$ occurs three times in the reduced form of $G_g$, $\mu_g^k \geqslant 3$; and generally, if $\Gamma_k$ occurs $a$ times, $\mu_g^k \geqslant a$. Suppose next that, while $\Gamma_k$ occurs just three times in the reduced form of $G_g$, $\mu_g^k > 3$ if possible. There is then at least one other linear function of the $\xi$'s, linearly independent of $\xi_i$, $\xi_j$, and $\xi_k$, which is unaltered by the operations of $g$. The only linear functions of the $x$'s, besides $x_1$, which are unchanged by $g$, are the sums of the sets of $x$'s, which are transitively permuted by $g$. On the supposition made, one of these sums, say

$$x_2+x_3+\dots+x_{s+1},$$

when expressed in terms of the reduced variables, must contain a linear function of the $\xi$'s, which is linearly independent of $\xi_i$, $\xi_j$, and $\xi_k$, so that

$$x_2+x_3+\dots+x_{s+1} = \frac{s}{m}\sigma+\xi_l+\eta_p+\zeta_q+\dots,$$

where $\xi_l$ is linearly independent of $\xi_i$, $\xi_j$, $\xi_k$.

Now, if every operation of $G$ which changes $x_1$ into $x_i$ also changes

$$x_2+x_3+\dots+x_{s+1}$$

into

$$x_{i_1}+x_{i_2}+\dots+x_{i_s},$$

then

$$x_1' = x_2+x_3+\dots+x_{s+1},$$

$$x_2 = x_{2_1}+x_{2_2}+\dots+x_{2_s},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$x_m' = x_{m_1}+x_{m_2}+\dots+x_{m_s}$$

is permutable with every operation of $G$.* When expressed in terms of the reduced variables, such a substitution can only transform among themselves those reduced variables which undergo the operations of equivalent irreducible components of $G$. Hence the above substitution involves for the $\xi$'s, $\eta$'s, and $\zeta$'s the relation

$$\xi_i'+\eta_j'+\zeta_k' = \xi_l+\eta_p+\zeta_q.$$

On the other hand, the sets

$$\begin{array}{l} \xi_1,\ \xi_2,\ \dots, \\ \eta_2,\ \eta_2,\ \dots, \\ \zeta_1,\ \zeta_2,\ \dots, \end{array}$$

undergoing the same transformations for each operation of $G$, the only transformations affecting these variables, which are permutable with every substitution of $G_g$, must be of the form

$$\left.\begin{array}{l} \xi_i' = a_{11}\xi_i+a_{12}\eta_i+a_{13}\zeta_i \\ \eta_i' = a_{21}\xi_i+a_{22}\eta_i+a_{23}\zeta_i \\ \zeta_i' = a_{31}\xi_i+a_{32}\eta_i+a_{33}\zeta_i \end{array}\right\} \quad (i = 1, 2, \dots).$$

Such a substitution replaces $\xi_i+\eta_j+\zeta_k$ by a linear function of $\xi_i$, $\xi_j$, $\xi_k$, $\eta_i$, $\eta_j$, $\eta_k$, $\zeta_i$, $\zeta_j$, $\zeta_k$; and this is not of the form $\xi_l+\eta_p+\zeta_q$, where $\xi_l$ is linearly independent of $\xi_i$, $\xi_j$, and $\xi_k$. Hence the supposition that $\mu_g^k > 3$, when $\Gamma_k$ occurs just three times in the reduced form of $G_g$, leads to a contradiction. An exactly similar process will show that, if $\Gamma_k$ occurs $a$ times, then $\mu_g^k \not> a$. It having been shown already that, if $\Gamma_k$ occurs $a$ times, then $\mu_g^k \not< a$, the theorem is therefore proved.

## II.

The determination of the arithmetical nature of the coefficients in an irreducible group of linear substitutions of finite order is a question which has hitherto received but little attention considering its undoubted importance. The coefficients may be modified by carrying out linear transformations on the variables, and the question arises as to the simplest possible arithmetical form of the coefficients when suitable transformations are carried out. In the cases of a number of groups of relatively small order, it has been shewn by actually constructing the transformations that the coefficients may be exhibited as cyclotomic functions; *i.e.*, rational functions of roots of unity. The only result of a general nature hitherto obtained is due to Herr Maschke.*

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* *Proc. London Math. Soc.*, Vol. XXXIII., p. 174.

He shews that, if a finite group of linear substitutions has an operation whose multipliers are all different, the group can be exhibited in a form in which all the coefficients in the transformations are cyclotomic functions. The general theory of the reduction of a group of linear substitutions has been greatly developed since the date of Herr Maschke's paper. Using the results of the general theory, a method is given below for actually effecting the complete reduction of a simply transitive regular permutation-group in a form such that the coefficients in the completely reduced groups are rational functions of roots of unity. The process is certainly an effective one for any particular irreducible component $\Gamma$, unless there is an integer $A$ ($> 1$), such that each multiplier of each operation in $\Gamma$ is repeated $A$, or a multiple of $A$, times. If this is the case, the characteristics

$$\chi_1, \chi_2, \ldots, \chi_r$$

for $\Gamma$, regarded as a set of algebraic integers, have a common divisor $A$. It seems to me in the highest degree improbable that such an irreducible group can exist, but I have not succeeded in proving its existence impossible.

When the characteristics are not all rational, it is certain that the group cannot be exhibited in a form in which the coefficients are all rational; but it is possible that they may be rational functions of the characteristics. One of my objects here is to establish a condition which shall be sufficient to ensure that an irreducible group of linear substitutions can be expressed so that the coefficients are rational functions of the characteristics, and to this I now proceed. It will be seen that the condition is satisfied for a very considerable class of cases. The quaternion group in two variables however shews that there are cases in which such a form for the group is not possible.

I suppose the group $G$ to be given by its multiplication table, or, what is the same thing, as a regular permutation-group.

The numbers $h_1, h_2, \ldots, h_r$

and $\chi_1^k, \chi_2^k, \ldots, \chi_r^k \quad (k = 1, 2, \ldots, r)$

can be calculated directly from the multiplication table, and the representation of the group in any one of the possible forms as a transitive permutation-group can be directly set up. In the representation denoted above by $G_g$, let

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* *Math. Ann.*, Vol. L., pp. 492–498.

$$x_1' = x_{a_1}, \quad x_2' = x_{a_2}, \quad \dots$$

be any substitution, and

$$\begin{array}{llll} x_1' = x_{b_1}, & x_2' = x_{b_2}, & \dots, \\ x_1' = x_{c_1}, & x_2' = x_{c_2}, & \dots, \\ \dots & \dots & \dots & \dots \end{array}$$

its conjugates. If these operations constitute the $i$-th conjugate set, denote

$$\begin{array}{ll} x_{a_1}+x_{b_1}+x_{c_1}+\dots & \text{by } \chi_1^i, \\ x_{a_2}+x_{b_2}+x_{c_2}+\dots & \text{by } \chi_2^i, \\ \dots \quad \dots & \dots \quad \dots \end{array}$$

and consider the set of transformations

$$\left.\begin{array}{l} x_1' = t_1\chi_1^1+t_2\chi_1^2+\dots+t^r\chi_1^r \\ x_2' = t_1\chi_2^1+t_2\chi_2^2+\dots+t_r\chi_2^r \\ \dots \quad \dots \quad \dots \quad \dots \\ x_m' = t_1\chi_m^1+t_2\chi_m^2+\dots+t_r\chi_m \end{array}\right\} . \qquad \text{(A)}$$

Every one of them is permutable with every operation of $G_g$, and when the completely reduced variables are used they take the form

$$\begin{aligned} \sigma' &= (h_1t_1+h_2t_2+\dots+h_rt_r)\,\sigma, \\ \xi_t' &= (h_1\chi_1^kt_1+h_2\chi_2^kt_2+\dots+h_r\chi_r^kt_k)\,\xi_t/\chi_1^k \\ &\qquad\qquad (t = 1, 2, \dots, \chi_1^k), \\ \eta_t' &= (h_1\chi_1^lt_1+h_2\chi_2^lt_2+\dots+h_r\chi_r^lt_r)\,\eta_t/\chi_1^l \\ &\qquad\qquad (t = 1, 2, \dots, \chi_1^l), \\ \dots & \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

if the $\xi$'s undergo the transformations of $\Gamma_k$, the $\eta$'s those of $\Gamma_l$, and so on. The set of transformations therefore form an Abelian group. Moreover. if in the equations (A), which have been actually constructed from the multiplication table, $x_i'$ is replaced by $\lambda x_i$ for each $i$, and if the ratios of the variables are eliminated, an equation (the characteristic equation) of the $m$-th degree results for $\lambda$, of which

$$h_1t_1+h_2t_2+\dots+h_rt_r$$

is an unrepeated root, while the root

$$(h_1\chi_1^k t_1 + h_2\chi_2^k t_2 + \ldots + h_r\chi_r^k t_r)/\chi_1^k$$

occurs $\chi_1^k\mu_g^k$ times. The equation with these roots, being necessarily an Abelian equation, can be solved by rational processes in the domain of the characteristics; so that this step of the process can also be actually carried out.

If, now, in equations (A), $x_i'$ is replaced by

$$(h_1\chi_1^{k'} t_1 + h_2\chi_2^{k'} t_2 + \ldots + h_r\chi_r^{k'} t_r)x_i/\chi_1^{k'} \quad (i = 1, 2, \ldots, m),$$

where $\Gamma_k$ and $\Gamma_{k'}$ are inverse representations, then, since the invariant factors of the characteristic determinant are all linear whatever the $t$'s may be, $\chi_1^k\mu_g^k$ of the equations are satisfied identically as a consequence of the remaining ones, which determine $m-\chi_1^k\mu_g^k$ of the $x$'s in terms of the rest with coefficients which are rational in $\chi_1^k, \chi_2^k, \ldots, \chi_r^k$, the $t$'s disappearing in the process.

If
$$x_i = \sum_{s=1}^{s=\chi_1^k\mu_g^k} A_i^s x_s \quad (i = \chi_1^k\mu_g^k+1, \ldots, m)$$

are these determinations, then each of the linear functions

$$x_j + \sum_{i=\chi_1^k\mu_g^k+1}^{i=m} A_i^j x_i \quad (j = 1, 2, \ldots, \chi_1^k\mu_g^k),$$

and these only, is changed into

$$(h_1\chi_1^k t_1 + h_2\chi_2^k t_2 + \ldots + h_r\chi_r^k t_r)/\chi_1^k$$

times itself by every operation of the set (A). This set of linear functions of the original variables, with coefficients which are rational functions of

$$\chi_1^k, \chi_2^k, \ldots, \chi_r^k,$$

are therefore transformed linearly among themselves by every substitution of $G_g$, and the group of linear transformations so arising is equivalent to the term $\mu_g^k\Gamma_k$ in the formula for the completely reduced form of $G_g$. Moreover the group of linear transformations can be actually set up, by carrying out the permutations of $G_g$, and when it is so set up the coefficients are rational functions of the $A$'s, and therefore rational functions of the $\chi^k$'s. If $\mu_g^k$ is unity, the group of linear substitutions so arrived at is equivalent to $\Gamma_k$, an irreducible representation of $G$. Hence

THEOREM.—If an irreducible finite group of linear substitutions $\Gamma$, for which the characteristics are

$$\chi_1, \chi_2, \ldots, \chi_r,$$

contains any sub-group which has *one* and *only one* linear invariant, it is always possible to exhibit $\Gamma$ in a form in which the coefficients of all its transformations are rational functions of the $\chi$'s.

The process described above may be illustrated by the details of the calculation in a particular case. To avoid unprofitable length a very simple one is chosen; viz., that of the dihedral group of order 10, defined by

$$S^5 = 1, \quad T^2 = 1, \quad TST = S^{-1}.$$

When represented as a permutation-group in five variables, this is

$$S \sim (x_1x_2x_3x_4x_5), \qquad T \sim x_1(x_2x_5)(x_3x_4).$$

The four sets of conjugate operations are

$$1; \quad S, S^4; \quad S^2, S^3; \quad T, TS, TS^2, TS^3, TS^4.$$

The set of operations (A) is

$$\begin{aligned} x_1' &= (t_1+t_4)\,x_1 + (t_2+t_4)(x_2+x_5) + (t_3+t_4)(x_3+x_4), \\ x_2' &= (t_1+t_4)\,x_2 + (t_2+t_4)(x_3+x_1) + (t_3+t_4)(x_4+x_5), \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_5' &= (t_1+t_4)\,x_5 + (t_2+t_4)(x_1+x_4) + (t_3+t_4)(x_2+x_3). \end{aligned}$$

The equation for $\lambda$ is

$$\begin{vmatrix} t_1+t_4-\lambda, & t_2+t_4, & t_3+t_4, & t_3+t_4, & t_2+t_4 \\ t_2+t_4, & t_1+t_4-\lambda, & t_2+t_4, & t_3+t_4, & t_3+t_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t_2+t_4, & t_3+t_4, & t_3+t_4, & t_2+t_4, & t_1+t_4-\lambda \end{vmatrix} = 0,$$

of which the solution is

$$(-\lambda+t_1+2t_2+2t_3+4t_4)(-\lambda+t_1+Pt_2+P't_3)^2(-\lambda+t_1+P't_2+Pt_3)^2 = 0,$$

where

$$P = \omega+\omega^4, \quad P' = \omega^2+\omega^3, \quad \omega^5 = 1.$$

Corresponding to the unrepeated root the ratios of the variables are

$$x_1 = x_2 = x_3 = x_4 = x_5,$$

and the unchanged linear function

$$\sigma = x_1+x_2+x_3+x_4+x_5.$$

Corresponding to the first of the two repeated factors the ratios of the variables are

$$x_3 = -x_1+Px_2, \quad x_4 = -Px_1-Px_2, \quad x_5 = Px_1-x_2,$$

and the linear functions are

$$\xi_1 = x_1-x_3-Px_4+Px_5, \qquad \xi_2 = x_2+Px_3-Px_4-x_5;$$

and corresponding to the second repeated factor the linear functions are

$$\eta_1 = x_1-x_3-P'x_4+P'x_5, \qquad \eta_2 = x_2+P'x_3-P'x_4-x_5.$$

Finally, carrying out the permutations, the transformations corresponding to $S$ and $T$ in the completely reduced form are

$$\begin{aligned} S &\sim \sigma' = \sigma, \quad \xi_1' = P\xi_1+\xi_2, \quad \xi_2' = -\xi_1, \quad \eta_1' = P'\eta_1+\eta_2, \quad \eta_2' = -\eta_1, \\ T &\sim \sigma' = \sigma, \quad \xi_1' = \xi_1+P\xi_2, \quad \xi_2' = -\xi_2, \quad \eta_1' = \eta_1+P'\eta_2, \quad \eta_2' = -\eta_2. \end{aligned}$$

As a further illustration, I call attention to the fact, not hitherto, I believe, noticed, that the irreducible simple groups of orders 60 and 168 in three variables can be expressed so that the coefficients are rational functions of the characteristics. In each of them a sub-group of order 3 has a single linear invariant. The fact may be easily verified in the forms:—

(i)
$$\begin{aligned} S &\sim x' = y, \quad y' = z, \quad z' = x-(1+P)\,y+(1+P)\,z, \\ T &\sim x' = x, \quad y' = (1+P)\,x-y, \quad z' = x-z, \end{aligned}$$

where
$$P = \omega+\omega^4, \quad \omega^5 = 1$$

are such that
$$S^5 = 1, \quad T^2 = 1, \quad (ST)^3 = 1.$$

(ii)
$$\begin{aligned} S &\sim x' = y, \quad y' = z, \quad z' = x+(1+Q)\,y+Qz, \\ T &\sim x' = x, \quad y' = Qx-y, \quad z' = -x-z, \end{aligned}$$

where
$$Q = \omega+\omega^2+\omega^4, \quad \omega^7 = 1$$

are such that
$$S^7 = 1, \quad T^2 = 1, \quad (ST)^3 = 1, \quad (S^4T)^4 = 1.$$

## III.

To avoid interrupting the argument in dealing with the actual reduction of a regular group of permutations, the following lemma, which is necessary to justify a series of steps in the process, is first proved.

LEMMA.—If $\Gamma$ is a finite group of linear substitutions on $a+b$ symbols

$$x_1,\ x_2,\ \ldots,\ x_{a+b},$$

and if

$$y_i = \Sigma\alpha_r^i x_r \quad (i = 1, 2, \ldots, a)$$

are a set of $a$ linearly independent linear functions of the $x$'s, which are transformed among themselves by every operation of $\Gamma$, then a set of $b$ linear functions

$$z = \Sigma\beta_r^j x_r \quad (j = 1, 2, \ldots, b)$$

linearly independent of each other and of the $y$'s can be constructed, so that

(i) The coefficients $\beta$ are rational functions of the $\alpha$'s and the coefficients of the transformations of $\Gamma$, and

(ii) The $z$'s are transformed among themselves, the coefficients of the transformations being rational functions of the $\alpha$'s and of the coefficients of the transformations of $\Gamma$.

The only part of the lemma that requires proof is that concerning the nature of the various coefficients, the possibility of the reduction, when there is no limitation on the coefficients being known.*

The $y$'s being linearly independent, there must be a set of $a$ $x$'s, in respect of which the equations defining the $y$'s can be solved. Suppose these are

$$x_1,\ x_2,\ \ldots,\ x_a.$$

Then each of these can be expressed in terms of

$$y_1,\ y_2,\ \ldots,\ y_a,\ x_{a+1},\ \ldots,\ x_{a+b},$$

with coefficients which are rational in the $\alpha$'s.

If the group $\Gamma$ be expressed in terms of these symbols, then

(i) The coefficients in the transformations are rational functions of the original coefficients and of the $\alpha$'s; and

(ii) The $y$'s are transformed among themselves.

The transformations will be of the form

$$y_i' = \Sigma c_{ijk} y_j \quad (i, j = 1, 2, \ldots, a),$$

$$x_{a+i}' = \sum_{r=1}^{r=a} d_{irk} y_r + \Sigma e_{ijk} x_{a+j} \quad (i, j = 1, 2, \ldots, b).$$

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* The most recent proof of this result is in a paper by the author "On the Reduction of a Group of Homogeneous Linear Substitutions of Finite Order," *Acta Mathematica*, Vol. XXVIII., pp. 369–387.

Now replace $x_{a+i}$ by $z_i$, where

$$z_i = x_{a+i} + \Sigma A_s^i y_s \quad (i = 1, 2, \ldots, b).$$

The transformations become $\quad y_i' = \Sigma c_{ijk} y_j \quad (i, j = 1, 2, \ldots, a),$

$$z_i' = \sum_r d_{irk}' y_r + \Sigma e_{ijk} z_j \quad (i, j = 1, 2, \ldots, b),$$

where

$$d_{irk}' = d_{irk} + \sum_s A_s^i c_{srk} - \sum_j e_{ijk} A_r^j.$$

Now it is known that the $A$'s can be chosen so that all the $d'$'s vanish. But the equations which they have to satisfy are linear with coefficients which are rational in the $a$'s and the original coefficients. Hence the $A$'s themselves are rational in the $a$'s and the original coefficients; and the lemma is proved.

If the process described in II is carried out on $G$ expressed as a regular permutation group in $N$ variables, a set of $(\chi_1^k)^2$ linear functions of the variables, with coefficients which are rational functions of the characteristics of the irreducible representation $\Gamma_k$, are obtained. These undergo, corresponding to the permutations of $G$, a group of linear transformations among themselves, the coefficients again being rational functions of the characteristics of $\Gamma_k$, and this group is equivalent to that denoted by $\chi_1^k \Gamma_k$ in the reduced form of the regular permutation-group. Moreover, the same set of linear functions are transformed among themselves, when the $N$ symbols undergo the permutations of the regular permutation-group $G'$, every one of whose operations is permutable with every operation of $G$; and this group is equivalent to that denoted by $\chi_1^k \Gamma_k'$. Suppose now that $S'$ is an operation of $G'$, and that, in the representation $\Gamma_k'$, the multipliers of $S'$ are $\omega_a$ repeated $a$ times, $\omega_b$ repeated $b$ times, .... If the permutation $S'$ is carried out in the original $N$ symbols, the $(\chi_1^k)^2$ linear functions of them that have been formed are transformed among themselves with coefficients which are rational functions of the characteristics.

Let

$$\xi_i' = \Sigma a_{ij} \xi_j \quad [i, j = 1, 2, \ldots, (\chi_1^k)^2]$$

be the transformation. Then, replacing $\xi_i'$ by $\omega_a \xi_i$ for each $i$, $a\chi_1^k$ sets of ratios of the $\xi$'s, and thence an equal number of linearly independent linear functions of them, are determined, the coefficients in which are rational functions of $\omega_a$ and the characteristics, such that each of these linear functions is replaced by $\omega_a$ times itself by the transformation $S'$.

Hence from the $(\chi_1^k)^2$ $\xi$'s there can be formed

$a\chi_1^k$ linear functions each of which is multiplied by $\omega_a$ for the operation $S'$,
$b\chi_1^k$ " " " " " $\omega_b$ " "
... ... ... ... ... ... ...

where $a+b+\ldots = \chi_1^k$; and these functions are linearly independent while the coefficients occurring in them are rational functions of roots of unity. Now every operation of $G$ is permutable with $S'$. Hence, every operation of the group $\chi_1^k\Gamma_k$, which represents $G$ so far as these symbols are concerned, must transform among themselves each of these sets of functions. A partial reduction is thus effected from a set of $(\chi_1^k)^2$ symbols to a series of sets containing respectively $a\chi_1^k$, $b\chi_1^k$, ... linearly independent functions of the symbols; and the coefficients involved are rational functions of roots of unity.

Take now another operation $T'$ of $G'$, and effect a similar partial reduction in respect of it. From the $(\chi_1^k)^2$ symbols, a series of sets containing respectively $\alpha\chi_1^k$, $\beta\chi_1^k$, ... linearly independent functions of them will be formed, such that the functions of each set are transformed among themselves by $\chi_1^k\Gamma_k$. Next express the $\alpha\chi_1^k$ functions of one of these sets linearly in terms of the functions of the previous sets that arise in respect of $S'$. This can be done with coefficients which are rational functions of roots of unity. Suppose that in the expression functions belonging to the set $a\chi_1^k$ in number occur. There are then these two possibilities: either the whole of the $a\chi_1^k$ linearly independent functions occur in the expression of the $\alpha\chi_1^k$ functions of the new set, or only a part of them occur. In the latter case, those of the $a\chi_1^k$ functions which occur in the expression of the $\alpha\chi_1^k$ functions are transformed among themselves by every operation of $\chi_1^k\Gamma_k$. Hence, by the lemma, from the $a\chi_1^k$ functions two sets of functions can be formed such that those of each set are transformed among themselves by the operations of $\chi_1^k\Gamma_k$; the coefficients entering in them are rational functions of roots of unity, while the two sets together are linearly equivalent to the original set of $a\chi_1^k$ functions. Moreover the number in each set must be a multiple of $\chi_1^k$; for when the complete reduction is carried out, as it certainly can be in an arbitrary domain of rationality, the number of functions operated on by each of the completely reduced groups is $\chi_1^k$. The partially reduced set of $a\chi_1^k$ functions is thus further reduced to two sets of $a_1\chi_1^k$ and $a_2\chi_1^k$ $(a = a_1+a_2)$ functions, and the coefficients are still rational in the domain of the roots of unity.

By using each of the new sets of $\alpha\chi_1^k$, $\beta\chi_1^k$, ... functions which arise in connection with $T'$, and the similar ones which arise in connection with all the other operations of $G'$, let the reduction of the sets of $a\chi_1^k$, $b\chi_1^k$, ... functions be carried as far as possible. We thus arrive at a series of sets of $A\chi_1^k$, $B\chi_1^k$, $C\chi_1^k$, ...

$$A+B+C+\ldots=\chi_1^k$$

functions, such that those of each set are transformed among themselves by $\chi_1^k\Gamma_k$, while no further reduction can be effected in the manner described. Moreover the coefficients entering in their expression are rational functions of roots of unity. Carry out now on the original variables as they appear in the expression of the $A\chi_1^k$ functions any operation of $G'$. The set of $A\chi_1^k$ functions will be replaced by another set of $A\chi_1^k$ functions which are transformed among themselves by the operations of $\chi_1^k\Gamma_k$, and which can therefore be expressed linearly in terms of the functions of the sets that have been arrived at.

Suppose that the $A\chi_1^k$ equations so expressing them contain functions from the set $B\chi_1^k$ in number. If the set of $B\chi_1^k$ is not exhausted in these equations, it can be further reduced. If the set is exhausted and $A > B$, the $B\chi_1^k$ functions can be eliminated from the equations, and the new set of functions $A\chi_1^k$ in number is reduced, so that the original set from which it has been derived can be reduced. Hence, if no further reduction is possible, then $A = B$. The same conclusions hold for any other set which enters in the expression of the $A\chi_1^k$ new functions.

Hence, when any operation of $G'$ is carried out on the original variables in the expression of the $A\chi_1^k$ functions, and the new functions so obtained are expressed in terms of those of the reduced sets, functions can only enter from these sets which contain the same number $A\chi_1^k$ of functions. If the sets do not all contain the same number of functions, then, when all the operations of $G'$ are effected on the $A\chi_1^k$ functions, fewer than $(\chi_1^k)^2$ linearly independent functions arise. That this is impossible is obvious by considering the completely reduced form of the group. Hence* the reduction can be carried further unless

$$A=B=C=\ldots.$$

Now return to the set of $a\chi_1^k$ functions which have a common multiplier for $T'$, and express them in terms of functions of the reduced sets. Unless $a = A$, from the $a\chi_1^k$ equations expressing them the $A\chi_1^k$ functions of one set can be eliminated. In the process other sets may disappear, but, if $a > A$, some must remain. From the $(a-A)\chi_1^k$ remaining equations eliminate the functions of another set of $A\chi_1^k$ functions. At each stage the number of equations is diminished by $A$, and therefore, if $a$ is not a multiple of $A$, the sets are further reducible. Hence $A$ can only be greater than unity (though even then it is not necessarily so) if each $a$ for each $T'$ is $A$ or a multiple of $A$. In other words, in $\Gamma'_k$, and therefore also in $\Gamma_k$, every operation must have each of its multipliers repeated a multiple of $A$ times.

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* From this it follows that, if $\chi_1^k$ is prime, then $\Gamma_k$ can certainly be expressed in a form in which the coefficients are cyclotomic.

Finally, then, the result may be expressd in the following form :—

THEOREM.—A finite irreducible group of linear substitutions can always be exhibited in a form in which the coefficients are rational functions of some root of unity, unless there is a number $A$ ($>1$) such that every multiplier of every substitution of the group is repeated $A$, or a multiple of $A$, times.

This exceptional case does not, in fact, occur for any group for which the characteristics have hitherto been calculated.

# THE MACLAURIN SUM-FORMULA

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[Received and Read, February 9th, 1905.]

1. The formula

$$\Sigma\phi(x) = C + \int^x \phi(x)\,dx - \tfrac{1}{2}\phi(x) + \frac{B_1}{2!}\frac{d\phi(x)}{dx} + \dots \tag{1}$$

was originally obtained in 1732 by Euler,* and independently rediscovered before 1742 by Maclaurin.† A full discussion of the question of priority appears in Cantor, *Geschichte der Mathematik*, Bd. III. (1898), p. 663. Then Plana,‡ in 1820, and Abel,§ in 1823, obtained the formula from which it can be developed:

$$\Sigma\phi(x) = \int^x \phi(x)\,dx - \tfrac{1}{2}\phi(x) + 2\int_0^\infty \frac{\phi(x+it)-\phi(x-it)}{2\iota}\,\frac{dt}{e^{2\pi t}-1}. \tag{2}$$

A generalisation of this formula, which it is interesting to compare with my own generalisation with which this paper concludes, was given by Abel‖ in 1825. In 1889 Kronecker¶ discussed the formula (2) by Cauchy's theory of residues, and another discussion appeared in 1898 in a text-book of Petersen.** More recently Lindelöf†† has given certain applications, and I have myself‡‡ generalised the analysis and considered the question from the point of view of the theory of asymptotic series in 1903.

I now propose to take up the question again, and, without the intervention of such a theory, to obtain a form for the remainder different from that obtained by other investigators, and to give a fresh demonstration of the conditions under which my extensions of the formula (1) are valid.§§

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* Euler, *Commentarii Acad. Sci. Imp. Petropolitanæ*, T. VI. (1732 and 1733), pp. 68–97.

† Maclaurin, *Treatise on Fluxions*, p. 672, Edinburgh, 1742.

‡ Plana, *Mem. Accad. Torino*, T. XXV.

§ Abel, *Œuvres Complètes* (1881), T. I., pp. 21–25; T. II., p. 77.

‖ Abel, *ibid.*, T. I., pp. 34–39.

¶ Kronecker, *Crelle*, T. CV., pp. 157–159 and pp. 345–354.

** Petersen, *Vorlesungen über Functionentheorie*, §§ 78–80.

†† Lindelöf, *Acta Soc. Sci. Fennicæ*, T. XXXI.; *Acta Mathematica*, T. XXVII., pp. 305–311.

‡‡ Barnes, *Quarterly Journal of Mathematics*, Vol. XXXV., pp. 175–188.

§§ References to other investigations will be found in Boole, *Finite Differences*, Third Edition, p. 163.

2. The discussion of the question is based upon the properties of the generalised Riemann $\zeta$ function, which is defined as follows:—Let $s$, $a$, and $\omega$ be any complex quantities, and let $S_n(a\,|\,\omega)$, or more briefly $S_n(a)$, be the $n$-th Bernoullian function defined by the expansion, valid when $|x| < |2\pi\iota/\omega|$,

$$\frac{xe^{-ax}}{1-e^{-\omega x}} = \sum_{n=0}^{\infty} \frac{S'_n(a)}{n!}(-x)^n,$$

the accent denoting differentiation with regard to $x$, coupled with the condition $S_n(0) = 0$.

Let $B_n(\omega)$ be the $n$-th Bernoullian number of parameter $\omega$ given by

$$B_n(\omega) = S'_n(0\,|\,\omega)/n.$$

Construct the function

$$S_{s,l}(a) = \sum_{m=0}^{l+1} \frac{S'_m(0\,|\,\omega)}{m!}\,\frac{d^m}{da^m}\,\frac{a^{1+s}}{1+s}$$

where the many-valued functions with $s$ as index have their principal values with respect to the axis of $-\omega$, *i.e.*, with respect to a line drawn from the origin to $-\omega$ and produced to infinity. Thus $a^s = \exp[s \log a]$, where $\log a$ has a cross-cut along the axis of $-\omega$, and is real when $a$ is real and positive.

[We assume for convenience that $\omega$ is not real and negative: in this case further definition is necessary.]

Then, if $R(s) > -(l+1)$, where $l$ is a finite positive integer, the series

$$-S_{-s,l}(a) + \sum_{n=0}^{\infty}\left[(a+n\omega)^{-s} - S_{-s,l}\{a+(n+1)\omega\} + S_{-s,l}(a+n\omega)\right]$$

is absolutely convergent except when

$$s = 1 \quad \text{or} \quad a = -n\omega \qquad (n = 0, 1, 2, \ldots, \infty),$$

and is denoted by $\zeta(s, a)$ or by $\zeta(s, a\,|\,\omega)$.

When $R(s) > 1$,

$$\zeta(s, a) = \sum_{n=0}^{\infty} (a+n\omega)^{-s},$$

and, when $a = \omega = 1$, we have Riemann's $\zeta$ function.*

When $R(a/\omega)$ is positive, it can be deduced from the previous definition that, for all values of $s$ and $\omega$,

$$\zeta(s, a) = \frac{\iota\Gamma(1-s)}{2\pi}\int \frac{e^{-az}}{1-e^{-\omega z}}(-z)^{s-1}\,dz,$$

the integral being taken along a contour embracing the axis of $1/\omega$, starting from $+\infty/\omega$, enclosing the origin but no other singularity of the subject of integration, and returning on the other side of the axis of $1/\omega$ to $+\infty/\omega$. The expression $(-z)^{s-1} = \exp\{(s-1)\log(-z)\}$ has its principal value with respect to the axis of $1/\omega$, which is a cross-cut for the logarithm.

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* Riemann, *Gesammelte Werke*, pp. 136–144.

The definition of the function $\zeta(s, a)$ is a development of that due to Mellin*: it is an application of the well-known theory of Mittag-Leffler.† For a proof of the convergency of the series I may refer to my "Theory of the Double Gamma Function"‡ (p. 341), and for the properties of $\zeta(s, a)$ to my "Theory of the Gamma Function"§ (Part III.).

8. The general result obtained in the present paper is as follows:—The Maclaurin sum-formula for a single parameter may, with the greatest generality, be written

$$\sum_{n=0}^{m-1} \phi(a+m\omega) = Y(a)+\sum_{n=0}^{l+1}\frac{S_n'(a)}{n!}\left[\frac{d^n}{dx^n}\int^x \phi(x)\,dx\right]_{x=m\omega}+J_l, \qquad \text{(A)}$$

where $Y(a)$ is a definite function of $a$ and of the coefficients in the expansion of $\phi(x)$, and where $\phi(x)$ is to be expanded by Laurent's series and the lower limit so chosen in the integral that the corresponding term vanishes at this limit.

When $\phi(x)$ is a (possibly non-uniform) function which has no singularities outside a circle centre the origin and finite radius outside which $a, a+\omega, \ldots, a+m\omega$, and $m\omega$ all lie, $J_l$ is a quantity which, when $m$ is large, has its modulus at most of order $1/m^{l+1}$.‖

When $\phi(x)$ is an integral function of order¶ less than unity, $J_l$ tends to zero as $l$ tends to infinity for all values of $m$, however large, and the Maclaurin series is absolutely convergent. In general, when $\phi(x)$ is an integral function of order $\geqslant 1$, the Maclaurin sum-formula has no meaning unless we apply the theory of asymptotic series to evaluate both the series for $Y(a)$ and the series which succeeds it in the enunciation.

The above formula can be generalised for any number of parameters, and corresponding propositions hold good.

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* Mellin, *Acta Soc. Sci. Fennicæ*, T. xxiv., No. 10 (1899).

† Mittag-Leffler, *Acta Mathematica*, T. iv., pp. 1–79.

‡ Barnes, *Phil. Trans. Roy. Soc.* (A), Vol. cxcvi., pp. 265–387.

§ Barnes, *Messenger of Mathematics*, Vol. xxix., pp. 64–128. This paper will be referred to for convenience under the initials *G.F.*, and the previous one as *D.G.F.*

‖ [*Note added April 5th*, 1905.]—By this statement we mean that $|J_l m^{l+1-\epsilon}|$, where $\epsilon$ is an arbitrarily small positive quantity, can be made as small as we please by taking $m$ sufficiently large.

¶ As originally defined by Borel, *Acta Mathematica*, T. xx., p. 360.

4.* THEOREM I.—*If* $s = u + \iota v$ *where* $u$ *is finite, then* $e^{-(\pi-\epsilon')|v|}|\omega^s\zeta(s, a)|$ *tends to zero as* $|v|$ *tends to infinity, if* $0 \leqslant \epsilon' < \epsilon \leqslant \pi$ *and*

$$|\arg(a/\omega)| = \pi - \epsilon.$$

We have, if $R(s) > -(l+1)$,

$$\zeta(s, a) = -S_{-s,l}(a) + \sum_{n=0}^{\infty}\left\{\frac{1}{(a+n\omega)^s} - S_{-s,l}\{a+(n+1)\omega\} + S_{-s,l}(a+n\omega)\right\},$$

where $$S_{-s,l}(a) = \sum_{m=0}^{l+1}\frac{S'_m(0)}{m!}\frac{d^m}{da^m}\frac{a^{1-s}}{1-s}.$$

Now [*G.F.*, p. 80] $$S_n\left(\frac{a}{\omega}\middle|1\right) = \frac{1}{\omega^n}S_n(a\,|\,\omega).$$

Also $(a+n\omega)^s$, when the many-valued function has its principal value with respect to the axis of $-\omega$, is equal to $(a/\omega+n)^s\omega^s$, where each function has its principal value defined as usual with respect to the axis of $-1$.

Hence $$\omega^s S_{-s,l}(a) = \sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega}$$

and

$$\omega^s\zeta(s, a) = -\sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega}$$
$$+\sum_{n=0}^{\infty}\left\{\frac{1}{(a/\omega+n)^s} - \sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega+n+1}\right.$$
$$\left.+\sum_{m=0}^{l+1}\frac{S'_m(0\,|\,1)}{m!}\left[\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right]_{x=a/\omega+n}\right\},$$

principal values of the many-valued functions with respect to the axis of $-1$ being taken.

We can take a finite number $N$, such that, if $n > N$,

$$R(a/\omega)+n > R(a/\omega)+N = \eta > 1,$$

and, if $N$ be sufficiently large, the modulus of $\arg(a/\omega+n) = \theta_n$ may be made as small as we please. Also, if $n > N$, $|a/\omega+n| > R(a/\omega+n) > \eta$.

The expansion $$\frac{1}{(x+1)^s} = \sum_{t=0}^{\infty}(-)^t\frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t+1)x^{s+t}}$$

is valid provided $|x| > 1$, for all finite values of $|s|$ however large.

Now it has been seen that,† if $|x/\omega| > 1$ and we expand

$$\{S_{-s,l}(x+\omega) - S_{-s,l}(x) - x^{-s}\}\,\omega^s$$

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* *April 5th*, 1905.—This paragraph has been modified since the paper was communicated. The reader may compare Mellin, *Acta Soc. Sci. Fennicæ*, T. XXIX., No. 4, pp. 47–48.

† *D.G.F.*, p. 341.

in ascending powers of $1/x$, the initial $(l+1)$ terms vanish, and we have

$$\sum_{r=2}^{\infty} \frac{P_{l+r}(s)}{x^{s+l+r}}\, \omega^{s+l+r},$$

where $$P_{l+r}(s) = (-)^{r+l} \frac{\Gamma(s+l+r)}{\Gamma(s)\,\Gamma(r+1)} \sum_{m=0}^{l+1} \frac{S'_m(0\,|\,1)\, r!}{m!\,(l+r+1-m)!}.$$

Thus $$|P_{l+r}(s)| < \kappa \frac{\Gamma(|s|+l+r)}{\Gamma(|s|)\,\Gamma(r+1)},$$

where $\kappa$ is a definite finite positive quantity independent of $r$, $N$, $n$, and $v$. Now we have

$$|\omega^s \zeta(s, a)| \leqslant \left| -S_{-s,\,l}(a)\,\omega^s - \sum_{n=0}^{N-1} \omega^s \left[S_{-s,\,l}(x+\omega) - S_{-s,\,l}(x) - x^{-s}\right]_{x=a+n\omega} \right|$$
$$+ \sum_{n=N}^{\infty} \sum_{r=2}^{\infty} \frac{|P_{l+r}(s)|}{|(a/\omega+n)^{s+l+r}|}$$

The last series is less than

$$\kappa \sum_{n=N}^{\infty} \frac{1}{|(a/\omega+n)^{s+l+2}|} \sum_{r=0}^{\infty} \frac{\Gamma(|s|+l+r+2)}{\Gamma(r+3)\,\Gamma(|s|)\,\eta^r}$$
$$< \kappa \sum_{n=0}^{\infty} \frac{e^{|\theta_N v|}}{(\eta+n)^{R(s)+l+2}} \frac{\Gamma(|s|+l)}{\Gamma(|s|)} \frac{\eta^2}{(1-1/\eta)^{|s|+l}}.$$

Suppose now that $|v|$ becomes very large. Choose $N$ so that $N : |v| : \eta$ tends to a ratio of equality. Then $(1-1/\eta)^{|s|+l}$ tends to a finite limit. The series $\sum_{n=0}^{\infty} \frac{1}{(\eta+n)^{R(s)+l+2}}$ tends to zero. The product $\kappa e^{|\theta_N v|} \frac{\Gamma(|s|+l)}{\Gamma(|s|)} \eta^2$ tends to zero when multiplied by $e^{-(\pi-\epsilon)|v|}$. Hence the double series tends to zero when multiplied by this expression.

We have now to consider the first series

$$\left| -S_{-s,\,l}(a)\,\omega^s - \sum_{n=0}^{N-1} \left[S_{-s,\,l}(a/\omega+n+1\,|\,1) - S_{-s,\,l}(a/\omega+n\,|\,1) - (a/\omega+n)^{-s}\right] \right|$$

The number of terms in the series ultimately bears a ratio of equality to $|s|$. Therefore, if each tends to zero when multiplied by $e^{-(\pi-\epsilon)|v|}$, the same will be true of the sum.

Now, if we put $a/\omega+n = r_n e^{\iota\theta_n}$, we have $|\theta_n| \leqslant \pi-\epsilon$ and $|(a/\omega+n)^{s+r}|$, where $r$ is an integer, $= r_n^{u+r} \exp\{-\theta_n v\}$.

Hence $\exp\{-(\pi-\epsilon')|v|\}/|(a/\omega+n)^{s+r}|$ tends to zero as $|v|$ tends to infinity, at least like $\exp\{(\epsilon'-\epsilon)|v|\}$. This is true for all finite values of $r$. And the same thing will be true if $(a/\omega+n)^{s+r}$ be multiplied by any algebraical polynomial or quotient of such polynomials in $s$.

Hence, finally, $e^{-(\pi-\epsilon')|v|}\,|\omega^s\zeta(s, a)|$ tends to zero as $|v|$ tends to infinity.

5. THEOREM II.—*If $k$ be any complex quantity of finite modulus, if $|\arg(a/\omega)| = \pi-\epsilon$, where $0 < \epsilon < \pi$, and if $|\arg(t/\omega)| = \epsilon'$, where $0 < \epsilon' < \epsilon$, the integral*

$$\int \zeta(s, a)\, t^{s-k}\,\Gamma(s)\,\Gamma(k-s)\,ds$$

*vanishes when taken along any part of the great circle at infinity for which $u$ is finite and $R(s) > -(l+1)$, $t^{s-k}$ having its principal value with respect to the axis of $-\omega$.*

By the asymptotic formula for $\Gamma(z)$, where $|z|$ is large and not in the vicinity of the negative half of the real axis, we know that $\Gamma(s)\,\Gamma(k-s)$ behaves like

$$2\pi \exp\left[(s-\tfrac{1}{2})\log s - s + (k-s-\tfrac{1}{2})\log(k-s) - k + s\right]$$

$$= 2\pi \exp\left\{-(u+\iota v)\left[\log(k-s)-\log s\right] - \tfrac{1}{2}\log s + (k-\tfrac{1}{2})\log(k-s) - k\right\}$$

the principal values of the logarithms being taken

$$= 2\pi \exp\left[-\pi\,|v| + \text{terms of lower order}\right],$$

for, when $v$ is positive and large, $\log(k-s)-\log s = -\pi\iota$ approximately; and, when $v$ is negative and large, $\log(k-s)-\log s = \pi\iota$ approximately.

The modulus of the subject of integration behaves approximately like

$$\left|\left(\frac{t}{\omega}\right)^{s-k}\right|\; |\omega^s\zeta(s, a)|\;\; |\Gamma(s)\,\Gamma(k-s)|,$$

where $(t/\omega)^{s-k}$ and $\omega^s$ have their principal values with respect to the axis of $-1$. And this expression by Theorem I. behaves at most like

$$\exp\{(\epsilon'-\epsilon)|v|\}.$$

Thus the integral along the part of the great circle specified vanishes.

COR.—The same integral is finite when taken along any line, drawn in the finite part of the plane parallel to the imaginary axis, which does not pass through the finite singularities of the subject of integration.

6. THEOREM III.—*The previous integral vanishes when taken along that part of the great circle at infinity for which $u$ is large and positive ($v$ having any real value), provided*

(1) $|\arg(a/\omega)| = \pi-\epsilon$, *where* $0<\epsilon\leqslant\pi$;

(2) $|\arg(t/\omega)| = \epsilon'$, *where* $0\leqslant\epsilon'<\epsilon$;

(3) $|t| < |a+n\omega|$, $n = 0, 1, 2, \ldots, \infty$;

(4) *the great circle passes between the points* $k+n$.

For the values of $s$ considered $t^s\zeta(s, a) = \sum\limits_{n=0}^{\infty} t^s/(a+n\omega)^s$.

Let $\log\dfrac{t/\omega}{a/\omega+n} = p_n+\iota q_n$, and let $s = Re^{\iota\phi}$; then

$$|t^s\zeta(s, a)| < \sum_{n=0}^{\infty}\exp\{R(p_n\cos\phi-q_n\sin\phi)\}.$$

Now, when $|s|$ is large,

$$\Gamma(s)\Gamma(k-s) = \frac{\pi\Gamma(s)}{\sin\pi(k-s)\Gamma(1-k+s)}$$

behaves like
$$\frac{\pi}{\sin\pi(k-s)}\exp\{(k-1)\log s+1-k\}.$$

Hence $|\zeta(s, a)t^s\Gamma(s)\Gamma(k-s)|$, under the restriction (4), behaves at most like $\sum\limits_{n=0}^{\infty}\exp\{R[p_n\cos\phi-q_n\sin\phi-\pi|\sin\phi|]\}$, and this will tend to zero like $\exp\{-\eta R\}$, where $\eta > 0$, provided $p_n$ is negative and $|q_n| < \pi$.

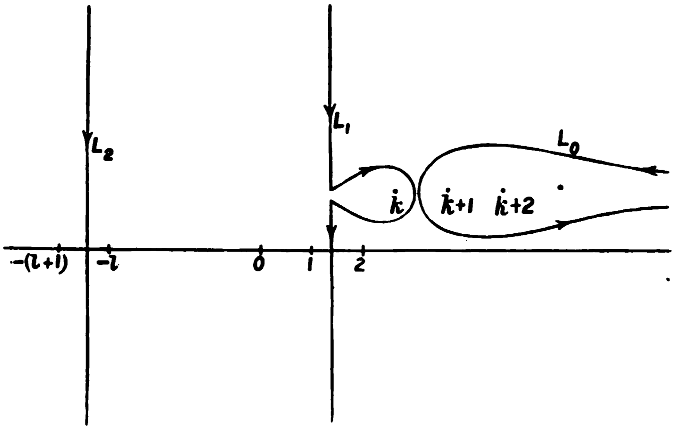
If $p_n$ be negative, we have the condition (3). If $|q_n| < \pi$, we have $\arg(t/\omega)-\arg(a/\omega+n)| < \pi$, and therefore we have the conditions (1) and (2).

We therefore have the theorem stated.

7. THEOREM IV.—*If $k$, $a$, and $t$ be any complex quantities of finite moduli subject to the conditions* (1), (2), (3), *and* (4) *of* § 6, *we have*

$$\zeta(k, a)-\zeta(k, a+t) = \frac{1}{2\pi\iota}\int_{L_0}\zeta(s, a)t^{s-k}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}ds,$$

*where $t^{s-k}$ has its principal value with respect to the axis of $-\omega$, and* *where the contour $L_0$ encloses the points $k+1$, $k+2$, ... as in the figure, but no other singularities of the subject of integration.*



We have, by Taylor's theorem,

$$\zeta(k, a+t) = \zeta(k, a) + \sum_{r=1}^{\infty} \frac{t^r}{r!} \zeta^{(r)}(k, a)$$

$$= \zeta(k, a) + \sum_{r=1}^{\infty} \frac{(-t)^r}{r!} \frac{\Gamma(r+k)}{\Gamma(k)} \zeta(r+k, a),$$

provided $|t| < |a+n\omega|$ for $n = 0, 1, 2, \ldots, \infty$. But

$$\frac{(-t)^r}{r!} \frac{\Gamma(r+k)}{\Gamma(k)} \zeta(r+k, a)$$

is the residue of the function

$$-\zeta(s, a)\, t^{s-k} \frac{\Gamma(s)\, \Gamma(k-s)}{\Gamma(k)}$$

at its pole $s = r+k$, since, if $s = r+k+\epsilon$,

$$\Gamma(k-s) = \frac{\Gamma(-\epsilon)}{(-\epsilon-1)(-\epsilon-2)\ldots(-\epsilon-r)} = \frac{(-)^{r-1}}{r!\,\epsilon} + \ldots.$$

Hence, under the conditions (1), (2), (3), and (4),

$$\zeta(k, a) - \zeta(k, a+t) = \frac{1}{2\pi\iota} \int_{L_0} \zeta(k, a)\, t^{s-k} \frac{\Gamma(s)\, \Gamma(k-s)}{\Gamma(k)}\, ds.$$

8. THEOREM V.—*Let $L_1$ denote a contour, as in the figure, parallel to the imaginary axis, cutting the real axis between $s = 1$ and $s = 2$, and with a loop to ensure that $k$ is to the left and $k+1$ to the right of the contour; then, under the conditions* (1) *and* (2) *solely,*

$$\zeta(k, a) - \zeta(k, a+t) = \frac{1}{2\pi\iota} \int_{L_1} \zeta(s, a)\, t^{s-k} \frac{\Gamma(s)\, \Gamma(k-s)}{\Gamma(k)}\, ds.$$

The integral vanishes by Theorem III. when taken round an infinite contour to the right of the imaginary axis. The subject of integration is one-valued with the prescription assigned to $t^{s-k}$. When we deform the contour $L_0$ into the contour $L_1$, we pass over no poles of the subject of integration. Hence, by Cauchy's theorem, the required equality is valid under the four conditions of § 6. But each side of the equality is, by Theorem II., a uniform analytic continuous function of $t$ even though the conditions (3) and (4) no longer hold. We may therefore eliminate these conditions and obtain the given theorem.

9. THEOREM VI.—*If* $R(k) > -(l+1)$, *and* $|\arg(a/\omega)| < \pi$,

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + \sum_{n=0}^{l+1} \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \int_{\infty, 0}^{x} \frac{dx}{x^k}\right]_{x=m\omega}$$

$$+ \frac{1}{2\pi\iota} \int_{L_2} \zeta(s, a)\, (m\omega)^{s-k}\, \frac{\Gamma(s)\, \Gamma(k-s)}{\Gamma(k)}\, ds,$$

*where the many-valued functions have their principal values with respect to the axis of* $-\omega$, *the lower limit of the integral is* $\infty$ *if* $R(k) > 1$ *and* 0 *if* $R(k) < 1$, *and where* $L_2$ *is a contour, as in the figure, parallel to the imaginary axis, cutting the real axis between* $s = -l$ *and* $s = -(l+1)$, *and such that the point* $k$ *is on its positive side.*

We have [*G.F.*, p. 89]

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) - \zeta(k, a+m\omega).$$

Hence, putting $t = m\omega$ in the previous theorem, we have, if $|\arg(a/\omega)| < \pi$,

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \frac{1}{2\pi\iota} \int_{L_1} \zeta(s, a)\, (m\omega)^{s-k}\, \frac{\Gamma(s)\, \Gamma(k-s)}{\Gamma(k)}\, ds.$$

By Theorem II. we may apply Cauchy's theorem and modify the contour of the integral till it assumes the position of the line $L_2$. We must take account of the poles of the subject of integration which we pass over in the deformation. These poles are at the points

$$s = 1,\ 0,\ -1,\ \ldots,\ -l \qquad \text{and} \qquad s = k.$$

The residue at $s = k$ is $-\zeta(k, a)$. The residue at $s = 1$ is [*G.F.*, p. 95]

$$\frac{(m\omega)^{-k+1}}{\omega}\, \frac{\Gamma(k-1)}{\Gamma(k)}.$$

The residue at $s = -n$ $(n = 0, 1, \ldots, l)$ is [*G.F.*, p. 97]

$$\frac{(-)^{n-1} S'_{n+1}(a)}{n+1}\, \frac{(m\omega)^{-k-n}}{n!}\, \frac{\Gamma(k+n)}{\Gamma(k)} = -\frac{S'_{n+1}(a)}{(n+1)!} \left[\frac{d^n}{dx^n}\, x^{-k}\right]_{x=m\omega}.$$

Hence

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + S_0'(a) \frac{(m\omega)^{1-k}}{1-k} + \sum_{n=0}^{l} \frac{S_{n+1}'(a)}{(n+1)!} \left[\frac{d^n}{dx^n} x^{-k}\right]_{x=m\omega}$$

$$+ \frac{1}{2\pi \iota} \int_{L_2} \zeta(s, a) (m\omega)^{s-k} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds.$$

If, now, $\int_{\infty, 0}^{x} \frac{dx}{x^k}$ has a lower limit $\infty$ if $R(k) > 1$, 0 if $R(k) < 1$, the last series may be written*

$$\sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \int_{\infty, 0}^{x} \frac{dx}{x^k}\right]_{x=m\omega}.$$

We thus have the theorem stated.

10. The form of the previous theorem obviously requires modification when $k$ has any value which makes some of the sets of points

$$\left.\begin{array}{l} k,\ k+1,\ \ldots \\ 1,\ 0,\ -1,\ \ldots,\ -l \end{array}\right\}$$

coincide. In these cases we may appeal to the principle of continuity to establish the final result. For example, when $k = 1$, the points $s = k$ and $s = 1$ coincide. In this case the residue of the function

$$\zeta(s, a) (m\omega)^{s-1} \Gamma(s) \Gamma(1-s)$$

at the point $s = 1$ is the coefficient of $1/\epsilon$ in the expansion of

$$-\frac{\pi}{\sin \epsilon\pi} \left[\frac{S_0'(a)}{\epsilon} - \frac{d}{da} \log \Gamma_1(a) \ldots\right] [1 + \epsilon \log m\omega + \ldots],$$

and is therefore $-S_0'(a) \log m\omega + \psi_1^{(1)}(a)$ where $\psi_1^{(1)}(a) = \frac{d}{da} \log \Gamma_1(a)$. This expression is evidently [*G.F.*, p. 95] the limit when $k = 1$ of

$$-\left[\zeta(k, a) + S_0'(a) \frac{x^{1-k}}{1-k}\right]_{x=m\omega}.$$

The previous theorem is thus true in general if limiting values be taken when infinite terms arise.

11. THEOREM VII.—*If* $R(k) > -(l+1)$ *and* $|\arg(a/\omega)| < \pi$,

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \int^{x} \frac{dx}{x^k}\right]_{x=m\omega} + J_l,$$

*where, when m is very large,* $|J_l|$ *is of lower order than* $m^{-l-R(k)}$.

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* When $R(k) = 1$ and $k$ is not real, such lower limit must be chosen as makes the integral vanish there.

We have $$J_l = \frac{1}{2\pi\iota}\int_{L_2} \zeta(k, a)\, m^{s-k} \omega^{s-k} \frac{\Gamma(s)\,\Gamma(k-s)}{\Gamma(k)}\, ds.$$

Hence $$|J_l| < \frac{1}{2\pi}\int_{L_2} \left|\frac{m^{s-k}}{\omega^k \Gamma(k)}\right| \left|\zeta(s, a)\, \omega^s\, \Gamma(s)\, \Gamma(k-s)\, ds\right|$$

$$< \frac{|m^{-l-k}|}{2\pi\,|\omega^k \Gamma(k)|}\int_{L_2} \left|\zeta(s, a)\, \omega^s\, \Gamma(s)\, \Gamma(k-s)\right| |ds|$$

$$< \frac{C}{m^{l+R(k)}},$$

where $C$ is a finite quantity.

We have thus, provided $|\arg(a/\omega)| < \pi$, established the asymptotic expansion
$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^k} = \zeta(k, a) + \sum_{n=0}^{\infty} \frac{S'_n(a)}{n!}\left[\frac{d^n}{dx^n}\,\frac{x^{1-k}}{1-k}\right]_{x=m\omega}$$

for all finite values of $|k|$ except $k = 1$. We have shown that the expansion is truly asymptotic in that, when $m$ is large, the error committed by stopping at any term of the series has a modulus less than that of the last term retained. This expansion, the use of which was justified by the theory of divergent series, was made fundamental in my "Theory of the Gamma Function."

In the exceptional case $k = 1$, we have

$$\sum_{n=0}^{m-1} \frac{1}{a+n\omega} = -\psi_1^{(1)}(a) + \sum_{n=0}^{\infty} \frac{S'_n(a)}{n!}\left[\frac{d^n}{dx^n} \log x\right]_{x=m\omega}$$

12. THEOREM VIII.—*If* $\phi(x)$ *is a (possibly non-uniform) function which admits outside a circle of finite radius* $\rho$ *the expansion* $\sum_{r=2}^{\infty} c_r/x^r$, *and if the points* $a, a+\omega, \ldots, a+m\omega$, *and* $m\omega$ *all lie outside this circle, we have, when* $|\arg(a/\omega)| < \pi$, *the Maclaurin sum-formula*

$$\sum_{n=0}^{m-1} \phi(a+n\omega) = \sum_{r=2}^{\infty} c_r\, \zeta(r, a) + \sum_{n=0}^{l+1} \frac{S'_n(a)}{n!}\left[\frac{d^n}{dx^n}\int_{\infty}^{x} \phi(x)\, dx\right]_{x=m\omega} + J_l,$$

*where* $|J_l|$ *is, when* $m$ *is large, at most of order* $1/m^{l+2}$.

By Theorem VI. we have

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^r} = \zeta(r, a) + \sum_{n=0}^{l+1} \frac{S'_n(a)}{n!}\left[\frac{d^n}{dx^n}\int \frac{dx}{x^r}\right]_{x=m\omega}$$
$$+ \frac{1}{2\pi\iota}\int_{L_2} \zeta(s, a)\, (m\omega)^{s-r}\, \frac{\Gamma(s)\,\Gamma(r-s)}{\Gamma(r)}\, ds.$$

Hence

$$\sum_{n=0}^{m-1} \phi(a+n\omega) = \sum_{n=0}^{m-1} \sum_{r=2}^{l+2} \frac{c_r}{(a+n\omega)^r} + \sum_{n=0}^{m-1} \sum_{r=l+3}^{\infty} \frac{c_r}{(a+n\omega)^r}$$

$$= \sum_{r=2}^{l+2} c_r \zeta(r, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \sum_{r=2}^{l+2} \int_{\infty}^{x} \frac{c_r}{x^r}\, dx\right]_{x=m\omega} + \sum_{n=0}^{m-1} \sum_{r=l+3}^{\infty} \frac{c_r}{(a+n\omega)^r}$$

$$+ \frac{1}{2\pi\iota} \int_{L_2} \zeta(s, a)(m\omega)^s \Gamma(s) \sum_{r=2}^{l+2} \frac{\Gamma(r-s)}{\Gamma(r)} \frac{c_r}{(m\omega)^r}\, ds$$

$$= \sum_{r=2}^{\infty} c_r \zeta(r, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \int_{\infty}^{x} \phi(x)\, dx\right]_{x=m\omega}$$

$$+ \sum_{r=l+3}^{\infty} c_r \left\{\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^r} - \zeta(r, a)\right\} - \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \sum_{r=l+3}^{\infty} \int_{\infty}^{x} \frac{c_r\, dx}{x^r}\right]_{x=m\omega}$$

$$+ \frac{1}{2\pi\iota} \int_{L_2} \zeta(s, a)(m\omega)^s \Gamma(s) \sum_{r=2}^{l+2} \frac{\Gamma(r-s)\, c_r}{\Gamma(r)(m\omega)^r}\, ds.$$

By hypothesis, when $r$ is large, $|c_r| = \rho^r$ approximately. Again, if $|x| < |a+n\omega|$ ($n = 0, 1, 2, \ldots, \infty$), $\log \Gamma_1(a+x)$ admits the expansion

$$\sum_{r=0}^{\infty} \frac{x^r}{r!} \frac{d^r}{da^r} \log \Gamma_1(a) = \sum_{r=0}^{\infty} \zeta(r, a) \frac{(-x)^r}{r} \quad [G.F., \text{ p. } 95].$$

Hence $|r^{-1}\zeta(r, a)|$, when $r > R$, is less than $(1+\epsilon)/\mu^r$, where $\mu$ is the minimum value of $|a+n\omega|$ and $\epsilon > 0$. This minimum value is not zero, since $|\arg(a/\omega)| < \pi$.

By taking $R$ sufficiently large, we may make $\epsilon$ as small as we please. Hence $|c_r \zeta(r, a)|$ is less than $\{\rho(1+\epsilon)/\mu\}^r$, and therefore $\sum_{r=2}^{\infty} c_r \zeta(r, a)$ is convergent if $a+n\omega$, $n = 0, 1, \ldots, \infty$, lies outside the circle of convergence of $\phi(x)$.

Again,

$$\sum_{r=l+3}^{\infty} c_r \left\{\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^r} - \zeta(r, a)\right\} = \sum_{r=l+3}^{\infty} c_r \sum_{n=m}^{\infty} \frac{1}{(a+n\omega)^r} = \sum_{r=l+3}^{\infty} c_r \zeta(r, a+m\omega).$$

This series is a finite quantity which, when $m$ is large, is at most of order $1/m^{l+3}$.

In the next place

$$\sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \sum_{r=l+3}^{\infty} \int_{\infty}^{x} \frac{c_r}{x^r}\, dx\right]_{x=m\omega}$$

$$= \sum_{n=0}^{l+1} S_n'(a) \int_{\infty}^{x} \sum_{r=l+3}^{\infty} \left[(-)^{n+1} \frac{c_r\, r(r+1)\ldots(r+n)}{n!\, x^{r+n+1}}\, dx\right]_{x=m\omega}.$$

Now the series $\sum_{r=l+3}^{\infty} \frac{r(r+1)\dots(r+n)}{n!} \frac{c_r}{x^{r+n+1}}$ is obviously convergent when $|x| > \rho$, and when $m$ is large is at most of order $1/m^{l+n+4}$. Hence, when $|m\omega| > \rho$ this group of series is finite, and when $m$ is very large at most of order $1/m^{l+3}$. Again, when $r = 2, 3, \dots, l+2$, the integrals

$$\frac{1}{2\pi\iota}\int_{L_2} \zeta(s, a)(m\omega)^s \Gamma(s) \frac{\Gamma(r-s)}{\Gamma(r)} \frac{c_r}{(m\omega)^r} ds$$

are each finite, and each, when $m$ is large, is of order less than $1/m^{l+2}$. Thus we see that, under the conditions enunciated, we have

$$\sum_{n=0}^{m-1} \phi(a+m\omega) = \sum_{r=2}^{\infty} c_r \zeta(r, a) + \sum_{n=0}^{l+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n}\int_{\infty}^{x} \phi(x)\,dx\right]_{x=m\omega} + J_l,$$

where $|J_l|$, when $m$ is large, is at most of order $1/m^{l+2}$.

We have excluded the first term $c_1/x$ from the expansion of $\phi(x)$, since, when $k = 1$, it has been seen that a slight modification of the fundamental formula is necessary. The general theory remains unimpaired. When $\phi(x) = \sum_{r=1}^{\infty} c_r/x^r$, $|J_l|$ is at most of order $1/m^{l+1}$.

Further slight generalisations as to the nature of $\phi(x)$ may be made. We may state the general theorem—*The Maclaurin sum-formula* [§ 8 (A)] *is, when* $|\arg(a/\omega)| < \pi$, *valid for any* (*not necessarily uniform*) *function which has no singularities outside a finite circle outside which* $a$, $a+\omega$, …, $a+m\omega$, *and* $m\omega$ *all lie.*

18. THEOREM IX.—*If* $\phi(x)$ *is an integral function of order less than unity and* $|\arg a/\omega| < \pi$, *the Maclaurin sum-series is absolutely convergent, and we have the equality*

$$\sum_{n=0}^{m-1} \phi(a+n\omega) = \sum_{r=0}^{\infty} c_r \zeta(-r, a) + \sum_{n=0}^{\infty} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n}\int_0^x \phi(x)\,dx\right]_{x=m\omega},$$

*where* $$\phi(x) = \sum_{r=0}^{\infty} c_r x^r.$$

If we put $k = -r$ in Theorem VII., we have

$$\sum_{n=0}^{m-1} (a+n\omega)^r = \zeta(-r, a) + \sum_{n=0}^{r+1} \frac{S_n'(a)}{n!} \left[\frac{d^n}{dx^n} \frac{x^{r+1}}{r+1}\right]_{x=m\omega},$$

for, when $n = r+2, r+3, \dots, l+1$, the terms of the series vanish, and, when $k = -r$, $1/\Gamma(k) = 0$, and therefore the integral $J_l$ vanishes.

If, now, $\phi_k(x) = \sum_{r=0}^{k} c_r x^r$, we have

$$\sum_{n=0}^{m-1} \phi_k(a+n\omega) = \sum_{r=0}^{k} c_r \zeta(-r, a) + \sum_{r=0}^{k} \sum_{n=0}^{r+1} c_r \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \frac{x^{r+1}}{r+1}\right]_{x=m\omega}$$

$$= \sum_{r=0}^{k} c_r \zeta(-r, a) + \sum_{n=0}^{k+1} \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \sum_{r=n-1}^{k} \frac{c_r x^{r+1}}{r+1}\right]_{x=m\omega},$$

wherein, when $n = 0$ in the second series, $r$ ranges from 0 to $k$,

$$= \sum_{r=0}^{k} c_r \zeta(-r, a) + \sum_{n=0}^{k+1} \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \int_0^x \phi_k(x)\, dx\right]_{x=m\omega} \qquad (1)$$

Now [*G.F.*, p. 97] $\zeta(-r, a) = -S'_{r+1}(a)/(r+1)$.

Again, $S'_{r+1}(a)/(r+1)!$ is the coefficient of $(-z)^{r+1}$ in the expansion of $\dfrac{ze^{-az}}{1-e^{-\omega z}}$ in ascending powers of $z$, and this expansion has a radius of convergence equal to $2\pi/|\omega|$.

Hence, when $r$ is large, $|c_r \zeta(-r, a)|$ behaves like $|c_r|\, r!\, \{|\omega|/2\pi\}^r$. Therefore, when $k$ tends to infinity, $\sum_{r=0}^{k} c_r \zeta(-r, a)$ is convergent if $|c_r|$ behaves, when $r$ is large, like $\{r!\}^{-1-\epsilon}$, where $\epsilon > 0$; that is to say, if the order of $\phi(x)$ is less than unity.

When $\phi(x)$ is of order greater than unity, $\sum_{r=0}^{\infty} c_r \zeta(-r, a)$ is divergent: it is only convergent in particular cases when $\phi(x)$ is of order unity.

In the next place, when $k$ tends to infinity, the second series in the equality (1) becomes

$$\sum_{n=0}^{\infty} \frac{S'_n(a)}{n!} \left[\frac{d^n}{dx^n} \int_0^x \phi(x)\, dx\right]_{x=m\omega}$$

This series is absolutely convergent if $\phi(x)$ is of order less than unity. For its general term is $\dfrac{S'_n(a)}{n!}\, \phi^{(n-1)}(m\omega)$. Now $\dfrac{\phi^{(n-1)}(m\omega)}{(n-1)!}$ is the coefficient of $a^{n-1}$ in the expansion of $\phi(a+m\omega)$ in powers of $a$, and therefore $\left|\dfrac{\phi^{(n-1)}(m\omega)}{(n-1)!}\right|$ behaves, when $n$ is large, like $\dfrac{1}{\{(n-1)!\}^{1+\epsilon}}$, where $\epsilon > 0$. Hence $\left|\dfrac{S'_n(a)}{n!}\, \phi^{(n-1)}(m\omega)\right|$ behaves like $\left\{\dfrac{|\omega|}{2\pi}\right\}^n \dfrac{1}{\{(n-1)!\}^{\epsilon}}$, and therefore the series is absolutely convergent.

We thus have the theorem stated.

14. We have now established the results stated in § 8 for the case of a single parameter. We see that it is hopeless to expect, when $\phi(x)$ is an integral function of order $> 1$, to apply the Maclaurin sum-formula, for in such cases $\sum_{r=0}^{\infty} c_r \zeta(-r, a)$ becomes divergent. I have, however, shewn in my previous paper on this subject that in such cases the theory of divergent series will often enable us to interpret such a formula. This mode of interpretation is, however, foreign to the range of ideas of the present investigation.

15. I will now indicate briefly the generalisation of the previous theory to the case of any number of parameters.*

Let ${}_rS_n(a \mid \omega_1, \ldots, \omega_r)$, or briefly ${}_rS_n(a)$, be the $n$-th $r$-ple Bernoullian function defined by the expression [*M.G.F.*, § 3]

$$\frac{(-)^r z e^{-az}}{\prod_{k=1}^{r} (1-e^{-\omega_k z})} = \sum_{s=1}^{r} \frac{(-)^s {}_rS_1^{(s+1)}(a)}{z^{s-1}} + \sum_{n=1}^{\infty} \frac{(-)^{n-1} {}_rS_n'(a) z^n}{n!},$$

coupled with the condition ${}_rS_n(0) = 0$. The expansion is valid when $|z|$ is less than the least of the quantities $|2\pi\iota/\omega_k|$ $(k = 1, 2, \ldots, r)$.

Take

$${}_rS_{-s,\,l}(x) = \sum_{m=0}^{l+r} \frac{{}_rS_m^{(r)}(0)}{m!} \frac{d^m}{dx^m} \left\{ \frac{x^{r-s}}{(r-s) \ldots (1-s)} \right\}.$$

As in my previous theory [*M.G.F.*, §§ 12 and 17], it is necessary to introduce a symbolic notation to simplify the cumbrous expressions to which the algebra otherwise gives rise.

Let $F_r$ be a symbolic operator which is such that

$$F_r[\phi(x)]_{x=m\omega} = \phi(m_1\omega_1 + \ldots + m_r\omega_r) - \sum_{s=1}^{r} \phi(m_1\omega_1 + \ldots + {}^* + \ldots + m_r\omega_r)$$
$$+ \sum_{s=1}^{r} \sum_{s=1}^{r} \phi(m_1\omega_1 + \ldots {}^* + \ldots {}^* + \ldots + m_r\omega_r) - \ldots + (-)^{r-1} \sum_{k=1}^{r} \phi(m_k\omega_k).$$

In the first summation the star denotes that one of the $\omega$'s is to be omitted: in the second summation every two different pairs of $\omega$'s must be successively omitted, and so on.

We assume that in the Argand diagram the points $\omega_1, \ldots, \omega_r$ all lie on the same side of some straight line $P$ through the origin. Let $1/L$ denote a line perpendicular to $P$ drawn from the origin into the region in which the

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* This theory is based on my researches in the domain of multiple gamma, Bernoullian, and Riemann $\zeta$ functions. An account will be found in "The Theory of the Multiple Gamma Function," *Cambridge Phil. Trans.*, Vol. XIX., pp. 374-425. This paper will be referred to as *M.G.F.*

$\omega$'s lie, and let $L$ be the line conjugate to this line with respect to the real axis.

Further, let $(a+\Omega)^{-s} = \exp[-s\log(a+\Omega)]$, where

$$\Omega = n_1\omega_1+\ldots+n_r\omega_r,$$

the logarithm being rendered one-valued by a cross-cut along the axis of $-1/L$ (*i.e.*, the negative direction of the axis of $1/L$) and $\log(a+\Omega)$ being such that it is real when $a+\Omega$ is a positive quantity.

Then, if $R(s) > -(l+1)$, where $l$ is a positive integer, the $r$-ple Riemann $\zeta$ function $\zeta_r(s, a\,|\,\omega_1, \ldots, \omega_r)$, or more briefly $\zeta_r(s, a)$, is defined by the equality [*M.G.F.*, § 20]

$$\zeta_r(s, a)$$

$$= -\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\left\{F_r[{}_rS_{-s,l}(a+\Omega+x)]_{x=\infty}+(-)^r\,{}_rS_{-s,l}(a+\Omega)-\frac{1}{(a+\Omega)^s}\right\}$$

$$+(-)^r\,{}_rS_{-s,l}(a).$$

When $R(s) > r$, we obtain

$$\zeta_r(s, a) = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\frac{1}{(a+n_1\omega_1+\ldots+n_r\omega_r)^s}.$$

Let $L$ denote a contour, embracing the axis $L$, similar to the contour defined in § 2. Further, let $(-z)^{s-1} = \exp\{(s-1)\log(-z)\}$ where the logarithm is rendered one-valued by a cross-cut along the axis $L$, and where $\log(-z)$ is such that it is real when $z$ is real and negative. Then, provided $a$ lies on the same side of the line $P$ as the $\omega$'s, or, as we shall say, provided $a$ is positive with respect to the $\omega$'s, we have

$$\zeta_r(s, a) = \frac{\iota\Gamma(1-s)}{2\pi}\int_L \frac{e^{-az}(-z)^{s-1}\,dz}{\prod_{k=1}^{r}(1-e^{-\omega_k z})},$$

where the contour $L$ encloses no poles of the subject of integration except the origin.

16. We have [*M.G.F.*, p. 401], if $k$ be any complex quantity,

$$\zeta_r(k, a)-\zeta_r(k, a+m_1\omega_1) = \sum_{n_1=0}^{m_1-1}\zeta_{r-1}(k, a+n_1\omega_1\,|\,\omega_2, \ldots, \omega_r).$$

Hence

$$\sum_{n_1=0}^{m_1-1}\cdots\sum_{n_r=0}^{m_r-1}\frac{1}{(a+n_1\omega_1+\ldots+n_r\omega_r)^k} = \zeta\,(k, a)+(-)^r F_r[\zeta_r(k, a+x)]_{x=m\omega},$$

or, since [*M.G.F.*, § 22] $F_r[1]_{x=m\omega} = (-)^{r-1}$,

$$\sum_{n_1=0}^{m_1-1} \cdots \sum_{n_r=1}^{m_r-1} \frac{(-1)^r}{(a+n_1\omega_1+\ldots+n_r\omega_r)^k} = F_r[\zeta_r(k, a+x)-\zeta_r(k, a)]_{x=m\omega}.$$

Again, it has been shewn [*M.G.F.*, § 56] that, if $R(s) > -(l+1)$ and $s = u+\iota v$, the quantity

$$|\zeta_r(s, a)x^s| e^{-\pi|v|},$$

where $x^s$ has its principal value with respect to the axis of $-1/L$, tends exponentially to zero as $|s|$ tends to infinity, provided

$$\frac{x}{a+\Omega} = r_n e^{\iota\psi_n}, \qquad \Omega = n_1\omega_1+\ldots+n_r\omega_r,$$

and $r_n < 1$, $|\psi_n| < \pi$.

We now define the contours $L_0$, ${}_rL_1$, and $L_2$ to be the same as those introduced in the diagram in § 7, except that ${}_rL_1$ cuts the real axis between $r$ and $(r+1)$.

In the same way as formerly, we may now prove that

$$\frac{1}{2\pi\iota}\int \zeta_r(s, a)x^{s-k}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}ds$$

taken along the contours $L_0$, ${}_rL_1$, and $L_2$ is finite provided $r_n < 1$ and $|\psi_n| < \pi$, and that the integrals along the last two contours are finite provided $|\psi_n| < \pi$ for all values of $n_1, \ldots, n_r$, whatever be the value of $r_n$.

Further, the integral along the contour $L_0$ is equal to that along the contour ${}_rL_1$.

Now the integral along the contour $L_0$

= the sum of the residues of the subject of integration at $k+1, k+2, \ldots$

$$= \sum_{p=1}^{\infty} \zeta_r(p+k, a)\,x^p \frac{\Gamma(p+k)}{\Gamma(k)}\frac{(-)^{p-1}}{\Gamma(p+1)}$$

$$= -\sum_{p=1}^{\infty} \frac{x^p}{p!}\frac{\partial^p}{\partial a^p}\zeta_r(k, a)$$

$$\left[\text{since } \frac{\partial^p}{\partial a^p}\zeta_r(k, a) = \frac{(-)^p\Gamma(k+p)}{\Gamma(k)}\zeta_r(p+k, a)\right]$$

$$= \zeta_r(k, a)-\zeta_r(k, a+x).$$

Hence, under the sole set of conditions $|\psi_n| < \pi$, we have

$$\zeta_r(k, a+x)-\zeta_r(k, a) = -\frac{1}{2\pi\iota}\int_{{}_rL_1} \zeta_r(s, a)\,x^{s-k}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}ds.$$

Hence $$\sum_{n_1=0}^{m_1-1} \dots \sum_{n_r=0}^{m_r-1} \frac{1}{(a+n_1\omega_1+\dots+n_r\omega_r)^k}$$
$$= (-)^r F_r[\zeta_r(k, a+x)-\zeta_r(k, a)]_{x=m\omega}$$
$$= \frac{(-)^{r-1}}{2\pi\iota} \int_{{}_rL_1} \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds,$$

$k$ being any complex quantity, and the many-valued functions having their principal values with respect to the axis of $-1/L$.

17. We may now show that, *provided $a$ lie on the same side of $P$ as the $\omega$'s,*

$$\sum_{n_1=0}^{m_1-1} \dots \sum_{n_r=0}^{m_r-1} \frac{1}{(a+n_1\omega_1+\dots+n_r\omega_r)^k}$$
$$= \sum_{n=0}^{l+r} \frac{{}_rS_n^{(r)}(a)}{n!} F_r\left[\frac{d^n}{dx^n}\int^x \dots \int^x \frac{dx^r}{x^k}\right]_{x=m\omega}$$
$$+\zeta_r(k, a)+\frac{(-)^{r-1}}{2\pi\iota} \int_{L_2} \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds,$$

*where the integration in $\int^x \dots \int^x$ is $r$ times repeated, and such lower limits are taken that successive integrals vanish at them.*

For, by Cauchy's theorem,

$$\frac{(-)^{r-1}}{2\pi\iota} \int_{{}_rL_1} \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} ds$$
$$= \frac{(-)^{r-1}}{2\pi\iota} \int_{L_2} + \text{the sum of the residues of}$$
$$(-)^r \zeta_r(s, a) F_r[x^{s-k}]_{x=m\omega} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)},$$

at the poles $s = k$ and $s = r, r-1, \dots, 2, 1, 0, -1, \dots, -l$.

The equality is limited by the condition

$$\left| \arg \left( \frac{m_1\omega_1+\dots+m_r\omega_r}{a+n_1\omega_1+\dots+n_r\omega_r} \right) \right| < \pi,$$

which must hold for all positive integral values of the $n$'s, and when any but not all of the $m$'s are zero. The inequality is equivalent to saying that $a$ must not be such that

$$a+n_1\omega_1+\dots+n_r\omega_r = -\theta(m_1\omega_1+\dots+m_r\omega_r),$$

where $\theta$ is a real positive quantity. It is satisfied provided $a$ lie on the same side of the line $P$ as the $\omega$'s. In particular we may modify the formula so as to take $a = 0$.

Now the residue of the function to be considered at $s = r-n$, where $n = 0, 1, 2, \ldots, r-1$, is [*M.G.F.*, § 31]

$$(-)^{r-n} {}_rS_1^{(r-n+1)}(a) F_r[x^{r-n-k}]_{x=m\omega} \frac{\Gamma(k-r+n)}{\Gamma(k)}$$

$$= \frac{{}_rS_n^{(r)}(a)}{n!} F_r\left[\frac{d^n}{dx^n}\int^x \cdots \int^x \frac{dx^r}{x^k}\right]_{x=m\omega},$$

the integration being $r$ times repeated and the lower limit each time being so chosen that the corresponding integral vanishes at it. The residue at $s = -n$, where $n = 0, 1, \ldots, l$, is [*M.G.F.*, § 31]

$$\frac{{}_rS'_{n+1}(a)}{n+1} F_r[x^{-k-n}]_{x=m\omega} \frac{\Gamma(n+k)}{\Gamma(k)} \frac{(-)^n}{\Gamma(n+1)}$$

$$= \frac{{}_rS_{n+r}^{(r)}(a)}{(n+r)!} F_r\left[\frac{d^{n+r}}{dx^{n+r}}\int^x \cdots \int^x \frac{dx^r}{x^k}\right]_{x=m\omega} \quad [M.G.F., \S\, 6].$$

The residue at $s = k$ is

$$-(-)^r \zeta_r(k, a) F_r[1]_{x=m\omega} = \zeta_r(k, a).$$

We thus obtain the theorem stated.

It is evident that exceptional cases which must be treated by the calculus of limits arise when $k = 1, 2, \ldots, r$. [Cf. *M.G.F.*, §§ 24 and 28.]

18. Substituting $p_1 m$ for $m_1, \ldots, p_r m$ for $m_r$, where the $p$'s are finite positive integers, we now have the important asymptotic equality

$$\sum_{n_1=0}^{p_1 m-1} \cdots \sum_{n_r=0}^{p_r m-1} (a+n_1\omega_1+\ldots+n_r\omega_r)^{-k}$$

$$= \zeta_r(k, a) + \sum_{n=0}^{l+r} \frac{{}_rS_n^{(r)}(a)}{n!} F_r\left[\frac{d^n}{dx^n} \frac{x^{-k+r}}{(1-k)(2-k)\ldots(r-k)}\right]_{x=pm\omega} + J_l,$$

where $|J_l|$ is, when $m$ is very large, at most of order less than $\frac{1}{m^{l+R(k)}}$. The modulus of the last term of the series is of order $m^{-l-R(k)}$, and hence $|J_l|$ is of order less than this last term. The expansion is therefore truly asymptotic. It is the expansion obtained previously [*M.G.F.*, § 18], where its use was justified by the theory of divergent series.

19. Suppose now that $a$ is positive with respect to the $\omega$'s, and that $\phi(x)$ admits outside a circle of finite radius $\rho$ the expansion $\sum_{k=r}^{\infty} \frac{c_k}{x^k}$; then, if the points $a+\Omega$ all lie outside this circle, we have, when $m$ is large,

the equality

$$\sum_{n_1=0}^{p_1 m-1} \cdots \sum_{n_r=0}^{p_r m-1} \phi(a+\Omega)$$

$$= \sum_{k=r}^{\infty} c_k \zeta_r(k, a) + \sum_{n=0}^{l+r} \frac{{}_rS_n^{(r)}(a)}{n!} F_r \left[ \frac{d^n}{dx^n} \int_\infty^x \cdots \int_\infty^x \phi(x)\, dx^r \right]_{x=pm\omega} + J_l,$$

where $| J_l |$ is at most of order $m^{-l-r}$.

The first $r$ terms of the series for $\phi(x)$ have been omitted because of the modification of the fundamental formula thereby introduced. The general theory remains unimpaired: when $\phi(x) = \sum_{k=1}^{\infty} c_k/x^k$, $J_l$ is at most of order $m^{-l-1}$.

The proof proceeds as for the case of a single parameter, and the general theory stated for a single parameter holds good.

20. If $a$ is positive with respect to the $\omega$'s and $\phi(x)$ is an integral function of order less than unity, which admits the expansion $\sum_{k=0}^{\infty} c_k x^k$, we have the absolute equality

$$\sum_{n_1=0}^{p_1 m-1} \cdots \sum_{n_r=0}^{p_r m-1} \phi(a+\Omega)$$

$$= \sum_{k=0}^{\infty} c_k \zeta_r(-k, a) + \sum_{n=0}^{\infty} \frac{{}_rS_n^{(r)}(a)}{n!} F_r \left[ \frac{d^n}{dx^n} \int_0^x \cdots \int_0^x \phi(x)\, dx^r \right]_{x=pm\omega}.$$

In other cases the series $\sum_{k=0}^{\infty} c_k \zeta_r(-k, a)$ is, in general, divergent.

# THE ASYMPTOTIC EXPANSION OF INTEGRAL FUNCTIONS OF FINITE NON-ZERO ORDER

*By* E. W. BARNES.

[Received and Read February 9th, 1905.]

1. In three memoirs* which have been recently published I have investigated the asymptotic expansions of the logarithms of integral functions of finite order, and suggested that such investigations may be regarded as preliminary to a classification of integral functions. The expansions were obtained for functions of simple and multiple† linear sequence, and it was shewn that expansions for similar functions with certain types of repeated sequence could be deduced: such deductions were made in certain cases.

The investigation was based entirely on the theory of divergent series: in the first memoir I attempted to develop the theory of Borel for this purpose. Throughout the investigation no attempt was made to determine remainders for the asymptotic expansions. The fundamental procedure consisted in applying the asymptotic expansions of the Maclaurin sum-formula to a transformation by logarithmic expansion of the function investigated. The terms of the double series which arose in this way were rearranged, and were then summed by an application of Fourier's series. In order to make the application, it was assumed that $|z|$, when $|z|$ is large, was of a limited type of number, and a further assumption was made that this limitation could not affect the validity of the result; that, in fact, the form of the asymptotic expansion did not depend on the arithmetic nature of $|z|$. This assumption is valid in the case of functions of finite (non-zero) order.

It seems, however, advisable to undertake the investigation from another point of view. The theory of divergent series is but little known:

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* "A Memoir on Integral Functions," *Phil. Trans. Roy. Soc.* (A), Vol. 199, pp. 411–500; "The Classification of Integral Functions," *Camb. Phil. Trans.*, Vol. XIX., pp. 322–355; "The Asymptotic Expansion of Integral Functions of Multiple Linear Sequence," *Camb. Phil. Trans.*, Vol. XIX., pp. 426–439.

† For the definition of these terms see § 2 of the present memoir.

parts of the theory are still obscure: it is desirable to place an important series of expansions on a basis which will appeal to mathematicians accustomed to the older methods of analysis. More than this, at the time when the former memoirs were written, I had not developed the theory of Maclaurin expansions, and it was impossible always to assign definitely the range of validity of the results. After the investigation of the previous memoir this can now be done.

The procedure employed in the present paper is that which I have previously used[*] to obtain the asymptotic expansions of the simple and multiple gamma functions. It is an application of Cauchy's theory of residues suggested by a noteworthy investigation of Mellin,[†] and afterwards applied by him to the case of the simple gamma function in a memoir[‡] which has priority to my own, but of the existence of which I was ignorant when my results were being obtained. Mellin has subsequently considered[§] some of the problems of the present paper: the reader may with advantage compare his investigations with my own.

2. It is convenient to repeat at the outset certain definitions which I have introduced in connection with the classification of integral functions.

A *simple* integral function is one which may be expressed as a single Weierstrassian product whose $n$-th zero $a_n$ depends solely upon $n$ and definite constants, and which is such that the law of dependence of $a_n$ upon $n$ is the same for all but a finite number of zeros. The function is called a *non-repeated* function if the $n$-th primary factor of Weierstrass's product does not correspond to a zero of order depending upon $n$. If there is such dependence, it is called a *repeated* simple integral function. The zero is said to be *algebraically* repeated if the number which expresses the repetition is a polynomial in $n$.

Functions of *multiple linear* sequence are functions whose general zero is of the type $f(a+n_1\omega_1+\ldots+n_r\omega_r)$, $a$ and the $\omega$'s being constants and the $n$'s being the integers which define the particular zero.

The *order* of a simple non-repeated integral function whose $n$-th zero is $a_n$ is the number $\rho$ such that $\sum_n \frac{1}{|a_n|^{\rho-\epsilon}}$ is divergent, and $\sum_n \frac{1}{|a_n|^{\rho+\epsilon}}$ is convergent, however small the real positive quantity $\epsilon$ may be. If $a_n = n^\rho$, the order is $1/\rho$.

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[*] *Messenger of Mathematics*, Vol. xxix., Part 4; *Phil. Trans. Roy. Soc.* (A), Vol. 196, Part 5; *Camb. Phil. Trans.*, Vol. xix., §§ 55–57.

[†] *Acta Soc. Sci. Fennicæ*, T. xx., No. 12.

[‡] *Ibid.*, T. xxiv., No. 10.

[§] *Ibid.*, T. xxix., No. 4.

The *order* of the $r$-ple non-repeated function whose general zero is $f(a+\Omega)$, where $\Omega = n_1\omega_1 + \ldots + n_r\omega_r$, is similarly a number $\rho$ such that $\sum_{n_1, \ldots, n_r} \frac{1}{|f(a+\Omega)|^{\rho-\epsilon}}$ is divergent and $\sum_{n_1, \ldots, n_r} \frac{1}{|f(a+\Omega)|^{\rho+\epsilon}}$ is convergent. When $f(x) = x^k$, the order is $r/k$.

3. In the present paper I consider only integral functions of finite non-zero order. We consider first simple non-repeated functions. The three standard functions of this type are

$$ {}_1P_\rho(z) = \prod_{n=0}^{\infty} \left\{ 1 + \frac{z}{(a+n\omega)^\rho} \right\}, $$

where $\rho > 1$;

$$ {}_1Q_\rho(z) = \prod_{n=0}^{\infty} \left\{ \left[ 1 + \frac{z}{(a+n\omega)^\rho} \right] \exp\left[ \sum_{s=1}^{p} \frac{(-z)^s}{s(a+n\omega)^{s\rho}} \right] \right\}, $$

where $\rho < 1$ and $p+1 > 1/\rho > p$;

$$ {}_1R_\rho(z) = \prod_{n=0}^{\infty} \left\{ \left[ 1 + \frac{z}{(a+n\omega)^\rho} \right] \exp\left[ \sum_{s=1}^{1/\rho} \frac{(-z)^s}{s(a+n\omega)^{s\rho}} \right] \right\}, $$

where $1/\rho$ is an integer $\geqslant 1$.

The first function is of order $1/\rho$ less than unity. The second is of non-integral order $1/\rho$ greater than unity. And the third is of integral order $1/\rho$ equal to or greater than unity. These functions are the prototypes of general simple non-repeated integral functions of finite non-zero order.

I proceed in the first place to obtain asymptotic expansions of their logarithms and to establish the conditions under which such expansions are valid.

4. THEOREM I.—*If $\rho$ be real and positive, and if $z^s$ has its principal value with respect to the quantity $-\omega^\rho$, the integral*

$$ \frac{1}{2\pi\iota} \int \frac{\zeta(\rho s, a)}{s} \frac{\pi z^s}{\sin \pi s} \, ds $$

*vanishes when taken along any part of the great circle at infinity for which $R(s)$ is finite, provided*

(1) $|\arg (a/\omega)^\rho| = \pi - \epsilon$, *where* $0 < \epsilon \leqslant \pi$.

(2) $|\arg (z/\omega^\rho)| = \epsilon'$, *where* $0 \leqslant \epsilon' < \epsilon$.

*It will also vanish along that part of the same circle for which $R(s)$ is positive and very large, provided in addition*

(3) $$|z/(a+n\omega)^{\rho}| < 1 \quad (n = 0, 1, \ldots, \infty),$$

(4) *the circle pass between the points $n$, $n+1, \ldots$, $n$ being a large positive integer.*

The integral may be written

$$\frac{1}{2\pi\iota}\int \frac{\omega^{\rho s}\,\zeta(\rho s, a)}{s}\,\frac{\pi}{\sin \pi s}\left(\frac{z}{\omega^{\rho}}\right)^{s} ds,$$

and the proof of the theorem follows the lines of Theorems I., II., and III. of the previous paper.

By saying that $z^s$ has its principal value with respect to the quantity $-\omega^{\rho}$ we mean that

$$z^s = \exp\{s \log z\} = \exp\{s \log|z| + s \arg z\},$$

where $\arg z$ lies in value between $\rho \arg \omega \pm \pi$, so that

$$-\pi < \arg(z/\omega^{\rho}) < \pi.$$

We do not necessarily mean that $z^s$ has its principal value with respect to the axis to the point which represents $-\omega^{\rho}$ in the Argand diagram.

5. Let $L_0$ be a contour embracing the positive half of the real axis and cutting it between $s = 1/\rho$ and $s = 1$, and let $L_1$ be a contour parallel to the imaginary axis and cutting the real axis in the same point as the contour $L_0$. Further, let $L_2$ be a contour parallel to $L_1$ cutting the real axis between $s = -l$ and $s = -(l+1)$. These contours may be compared with those drawn in § 7 of the previous paper.

Theorem II.—*Provided the conditions* (1) *and* (2) *of* § 4 *hold,*

$$\log {}_1P_{\rho}(z) = -\frac{1}{2\pi\iota}\int_{L_1} \frac{\zeta(\rho s, a)}{s}\,\frac{\pi z^s}{\sin \pi s}\, ds.$$

By Cauchy's theorem, provided all four conditions of § 4 hold good,

$$I_{L_1} = I_{L_0},$$

where $I$ denotes the integral under consideration.

The residue of $I_{L_0}$ at $s = k$ is $(-)^k k^{-1} \zeta(\rho k, a) z^k$.

Hence $$-I_{L_0} = -\sum_{k=1}^{\infty} \frac{(-z)^k}{k} \sum_{n=0}^{\infty} \frac{1}{(a+n\omega)^{\rho k}} \quad (\text{since } \rho > 1)$$

$$= -\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-z)^k}{k\,(a+n\omega)^{\rho k}}.$$

the inversion of the double series being legitimate under the condition (3).

Thus $$-I_{L_1} = -I_{L_0} = \log \prod_{n=0}^{\infty} \left\{1 + \frac{z}{(a+n\omega)^{\rho}}\right\},$$

or $$-I_{L_1} = \log {}_1P_{\rho}(z),$$

provided the conditions (1), (2), (3), and (4) are satisfied.

But the condition (4) obviously cannot affect this equality; neither, since each expression is one-valued, continuous, and analytic for all values of $|z|$, can the condition (3).

We thus have the theorem stated.

6. THEOREM III.—*Provided the conditions* (1) *and* (2) *of* § 4 *hold good,*

$$\log {}_1P_{\rho}(z) = -I_{L_3} + \frac{\pi z^{1/\rho}}{\omega \sin \pi/\rho} + \zeta(0, a) \log z + \rho \zeta'(0, a)$$

$$+ \sum_{n=1}^{l} \frac{(-)^{n-1}}{n z^n} \zeta(-\rho n, a),$$

*where* $\zeta'(0, a)$ *denotes* $\left[\frac{\partial}{\partial s} \zeta(s, a)\right]_{s=0}$ *and is equal to* [*G.F.*, p. 102]

$$\log \frac{\Gamma_1(a)}{\rho_1(\omega)} = \log \left\{\frac{\Gamma_1(a)}{\sqrt{2\pi/\omega}}\right\}$$

*in the notation of the simple gamma function of parameter* $\omega$.

By Cauchy's theorem coupled with Theorem I., we see that

$$-I_{L_1} = -I_{L_3}$$

together with the sum of the residues of the subject of integration at $1/\rho, 0, -1, \ldots, -l$.

The residue of the subject of integration at $s = -n$ is

$$(-)^{n-1} \frac{\zeta(-\rho n, a)}{n z^n}$$

The residue at $s = 0$ is the absolute term in the expansion of

$$\frac{\{\zeta(0) + \rho\epsilon\zeta'(0) + \ldots\}\{1 + \epsilon \log z + \ldots\}}{\epsilon\left\{1 - \frac{\pi^2\epsilon^2}{3!} + \ldots\right\}},$$

and is therefore $\zeta(0, a) \log z + \rho \zeta'(0, a)$, the logarithm having its principal value with respect to the quantity $-\omega^{\rho}$ (*vide* § 4).

The residue at $s = 1/\rho$ is the coefficient of $1/\epsilon$ in the expansion of

$$\rho\left\{\frac{1}{\rho\epsilon\omega}-\psi_1'(a)\dots\right\}\frac{\pi z^{1/\rho}}{\sin\pi/\rho}=\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}. \qquad [G.F., \text{ p. } 95.]$$

Combining these results with Theorem II., we have the proposition stated.

7. We have now proved that when

$$(1) \quad |\arg(a/\omega)^\rho| = \pi-\epsilon, \quad \text{where } 0<\epsilon<\pi;$$

$$(2) \quad |\arg(z/\omega^\rho)| = \epsilon', \quad \text{where } 0<\epsilon'<\epsilon,$$

we have

$$\log{}_1P_\rho(z) = \zeta(0,a)\log z+\rho\zeta'(0,a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n,a)$$

$$-\frac{1}{2\pi\iota}\int_{L_2}\frac{\zeta(\rho s,a)}{s}\frac{\pi z^s}{\sin\pi s}ds;$$

and we have to investigate how far the conditions limit the asymptotic expansion to which this equality gives rise.

For this purpose we need the following :—

THEOREM IV.—*Under the conditions* (1) *and* (2) *coupled with*

$$|(a+k\omega)^\rho/z|<1,$$

*we have*

$$\frac{1}{2\pi\iota}\int_{L_2}\frac{z^s}{s(a+k\omega)^{\rho s}}\frac{\pi}{\sin\pi s}ds=\sum_{n=l+1}\frac{(-)^n(a+k\omega)^{\rho n}}{nz^n}.$$

As in § 5, we prove that, under the conditions (1) and (2),

$$I_{L_1}=-\log\left\{1+\frac{z}{(a+k\omega)^\rho}\right\}.$$

Hence, as in § 6, under these same conditions,

$$I_{L_2}=I_{L_1}+\sum_{n=1}^{l}\frac{(-)^{n-1}(a+k\omega)^{n\rho}}{nz^n}+\log z-\rho\log(a+k\omega)$$

$$=-\log\left\{1+\frac{(a+k\omega)^\rho}{z}\right\}+\sum_{n=1}^{l}(-)^{n-1}\frac{(a+k\omega)^{n\rho}}{nz^n}.$$

Therefore, provided we have the original condition $|(a+k\omega)^\rho/z|<1$,

$$I_{L_2}=\sum_{n=l+1}^{\infty}\frac{(-)^n(a+k\omega)^{\rho n}}{nz^n},$$

which is the theorem required.

Now we know that $\zeta(s, a+\omega)-\zeta(s, a) = -a^{-s}$,

so that $$\zeta(s, a+k\omega) = \zeta(s, a)-\sum_{n=0}^{k-1}(a+n\omega)^{-s}.$$

Therefore, under the conditions (1) and (2),

$$\log {}_1P_\rho(z) = \zeta(0, a)\log z+\rho\zeta'(0, a)+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}$$

$$-\frac{1}{2\pi\iota}\int_{L_3}\frac{\left\{\zeta(\rho s, a+k\omega)+\sum\limits_{m=0}^{k-1}\dfrac{1}{(a+m\omega)^{\rho s}}\right\}}{s}\frac{\pi z^s}{\sin\pi s}\,ds.$$

And therefore, provided in addition $|(a+m\omega)^\rho/z| < 1$, $m = 0, 1, \ldots, k-1$, we have

$$\log {}_1P_\rho(z) = \zeta(0, a)\log z+\rho\zeta'(0, a)+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}$$

$$+\sum_{n=l+1}^{\infty}\frac{(-)^{n-1}}{n}\sum_{m=0}^{k-1}\frac{(a+m\omega)^{\rho n}}{z^n}-\frac{1}{2\pi\iota}\int_{L_3}\frac{\zeta(\rho s, a+k\omega)}{s}\frac{\pi z^s}{\sin\pi s}\,ds.$$

In this formula the condition (1) may be modified; for, in order that the integral last written may be convergent, it is merely necessary that $|\arg\{(a+k\omega)/\omega)^\rho\}| = \pi-\epsilon$, where $0 < \epsilon \leqslant \pi$, and that then

$$|\arg(z/\omega^\rho)| = \epsilon',$$

where $0 \leqslant \epsilon' < \epsilon$. But we may, by taking $k$ large but finite, make $\arg(a/\omega+k)^\rho$ as small as we please, and then $|\arg(z/\omega^\rho)|$ may have any value $< \pi$.*

The series $\sum\limits_{n=l+1}^{\infty}\dfrac{(-)^{n-1}}{n}\sum\limits_{m=0}^{k-1}\dfrac{(a+m\omega)^{\rho m}}{z^n}$ will then be the sum of a large but finite number of remainders after $l$ terms of convergent logarithmic expansions (provided $|z|$ be very large), and will therefore be of order less than the order $|z|^{-l}$.

Again $\dfrac{1}{2\pi\iota}\displaystyle\int_{L_3}\frac{\zeta(\rho s, a+k\omega)}{s}\frac{\pi z^s}{\sin\pi s}\,ds$ is, when $|z|$ is large, of order $1/|z|^{l+t}$ $(0 < t < 1)$.

*If, then, $z$ is not in the immediate vicinity of the zeros of ${}_1P_\rho(z)$, and if $a$ is not such as to make any zero $-(a+n\omega)^\rho$ vanish, we have, when $|z|$ is large, the asymptotic expansion*

$$\log {}_1P_\rho(z) = \zeta(0, a)\log z+\rho\zeta'(0, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n, a)+J_l,$$

*where $|J_l|$ is of order less than $|z|^{-l}$.*

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* There is the obvious restriction that $a+n\omega$ must not vanish for positive integral values of $n$.

This result is a slight generalisation of that obtained by me previously by the theory of divergent series [*I.F.*, § 52]. In the present instance we have determined the remainder in the asymptotic expansion of the standard function of simple non-repeated sequence of order less than unity. The result accords with my general theorem that the asymptotic expansion is valid for all large values of $|z|$ which are not in the immediate vicinity of zeros of the function [*I.F.*, § 44].

8. It should be carefully noticed that the previous formula has only been proved under the restriction that $\rho$ is real. In this case the zeros $-(a+n\omega)^\rho$ ultimately lie along a single line tending to infinity. But, when $\rho$ is complex, the zeros in general cover the whole plane near infinity, for they ultimately behave like

$$(n\omega)^\rho = \exp\{\rho \log n + \rho \log \omega\},$$

and therefore each sucessive zero has a different argument. We expect, for this reason, that no asymptotic expansion will exist; and, in fact, it is easy to shew that the foregoing proof breaks down.

9. Few modifications are necessary to establish the asymptotic expansions for

$$\log {}_1Q_\rho(z) = \log \prod_{n=0}^{\infty} \left[\left\{1+\frac{z}{(a+n\omega)^\rho}\right\} \exp\left\{\sum_{s=1}^{p} \frac{(-z)^s}{s(a+n\omega)^{\rho s}}\right\}\right],$$

where $p+1 > 1/\rho > p$.

We take the contours $L_0$ and $L_1$ to cut the real axis between $s = 1/\rho$ and $s = p+1$. Then, provided the conditions (1), (2), (3), and (4) of § 4 hold good,

$$\begin{aligned} -\frac{1}{2\pi i}\int_{L_0} \frac{\zeta(\rho s, a)}{s} \frac{\pi z^s}{\sin \pi s}\, ds &= -\sum_{s=p+1}^{\infty} \frac{(-z)^s}{s} \zeta(\rho s, a) \\ &= -\sum_{s=p+1}^{\infty} \frac{(-z)^s}{s} \sum_{n=0}^{\infty} \frac{1}{(a+n\omega)^{\rho s}} \\ &= -\sum_{n=0}^{\infty} \sum_{s=p+1}^{\infty} \frac{(-z)^s}{s(a+n\omega)^{\rho s}} \\ &= \log {}_1Q_\rho(z). \end{aligned}$$

Hence, by the former argument, under the conditions (1) and (2) alone,

$$-I_{L_1} = \log {}_1Q_\rho(z);$$

and therefore, by Cauchy's theorem,

$$\log {}_1Q_\rho(z) = \zeta(0)\log z + \rho\zeta'(0, a) + \frac{\pi z^{1/\rho}}{\omega \sin \pi/\rho} + \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n}\zeta(-\rho n, a)$$

$$-\frac{1}{2\pi\iota}\int_{L_2} \frac{\zeta(\rho s, a)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds,$$

where the accent in the summation denotes that the term corresponding to $n = 0$ is to be omitted.

From this formula we deduce, exactly as in § 7 and under the same conditions, the asymptotic expansion of $\log {}_1Q_\rho(z)$.

10. Consider next the function

$${}_1R_\rho(z) = \prod_{n=0}^{\infty}\left[\left\{1+\frac{z}{(a+n\omega)^\rho}\right\}\exp\left\{\sum_{s=1}^{1/\rho}\frac{(-z)^s}{s(a+n\omega)^{s\rho}}\right\}\right],$$

where $1/\rho$ is an integer $> 1$.

In the case of this function we take the contours $L_0$ and $L_1$ to cut the real axis between $1/\rho$ and $1/\rho+1$.

We have, under the conditions (1) and (2) of § 4,

$$\log {}_1R_\rho(z) = -\frac{1}{2\pi\iota}\int_{L_1}\frac{\zeta(\rho s, a)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds,$$

and thence, by considering the residues at the poles of the subject of integration between $L_1$ and $L_2$, we deduce

$$\log {}_1R_\rho(z) = \rho\zeta'(0, a) + \zeta(0, a)\log z + \rho(-)^{1/\rho+1}z^{1/\rho}\psi_1^{(1)}(a)$$

$$+\rho(-)^{1/\rho}z^{1/\rho}S_1^{(2)}(a)\left\{\frac{1}{\rho}\log z - 1\right\} + \sum_{n=-1/\rho+1}^{l}{}' \frac{(-)^{n-1}}{nz^n}\zeta(-n\rho, a)$$

$$-\frac{1}{2\pi\iota}\int_{L_2}\frac{\zeta(\rho s, a)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds.$$

This is the result of making $1/\rho = p+\epsilon$ and then putting $\epsilon = 0$ in the formula for $\log {}_1Q_\rho(z)$.

For the two terms in that formula which become infinite are

$$\frac{\pi z^{p+\epsilon}}{\omega \sin \pi(p+\epsilon)} + \frac{(-)^p z^p}{p}\zeta(1-\rho\epsilon, a)$$

$$= \frac{(-z)^p}{\omega\epsilon}\{1+\epsilon\log z+\dots\}\left\{1+\frac{\pi^2\epsilon^2}{3!}+\dots\right\} + \frac{(-z)^p}{p}\left\{-\frac{1}{\omega\rho\epsilon} - \psi_1^{(1)}(a)+\dots\right\}$$

$$= \frac{(-z)^p}{\omega}\log z - \frac{(-z)^p\rho}{\omega} + \frac{(-)^{p-1}z^p}{p}\psi_1^{(1)}(a). \qquad [G.F., \text{p. } 95.]$$

Since $S_1^{(2)}(a) = 1/\omega$ [*G.F.*, p. 80], this agrees with the former result. The calculus of limits can always be employed in this way in the more complicated formulæ to which we now proceed, when infinite terms arise for particular values of the constants involved.

11. We now proceed to consider a very general type of simple non-repeated integral function of finite (non-zero) order, and its asymptotic expansion.

*Let* $\phi(x)$ *be a (not necessarily one-valued) function of* $x$ *which is such that neither* $\phi(x)$ *nor its reciprocal has any singularities outside a circle of finite radius* $k$ *and centre the origin, outside which** $a, a+\omega, \ldots, a+n\omega, \ldots$ *all lie. Further, let* $\underset{|x|=\infty}{\mathrm{Lt}}\ \phi(x) = 1$ *and let* $\rho$ *be a positive quantity. Then the function which, when* $R(s) > 1/\rho$, *is represented by*

$$\sum_{n=0}^{\infty} \frac{1}{[\phi(a+n\omega)(a+n\omega)^{\rho}]^{s}},$$

*the many-valued functions having their principal values with respect to the axis of* $-\omega$, *is a function of* $s$ *which has no singularities in the finite part of the plane except at the points*

$$s = -(m-1)/\rho, \qquad m = 0, 1, 2, \ldots, \infty.$$

Outside the circle of finite radius $k$ we have the expansion

$$\phi(x) = 1 + \frac{c_1}{x} + \ldots + \frac{c_m}{x^m} + \ldots,$$

and, by the conditions attached to $\phi(x)$, we have within the same region, by Abel's investigation of the binomial series,†

$$[x^{\rho}\phi(x)]^{-s} = x^{-\rho s}\left\{1 + \frac{f_1(s)}{x} + \ldots + \frac{f_m(s)}{x^m} + \ldots\right\}.$$

By the Cauchy-Hadamard theorem, when $m$ is large, $|f_r(s)| = k^m$ to a first approximation.

We see now that, when $R(s) > 1/\rho$,

$$\sum_{m=0}^{\infty} \frac{1}{[\phi(a+n\omega)(a+n\omega)^{\rho}]^{s}} = \sum_{m=0}^{\infty} f_m(s)\,\zeta(\rho s+m, a),$$

where $f_0(s) = 1$, provided the series on the right-hand side is convergent. But, when $m$ is large, $\zeta(\rho s+m)$ behaves like

$$\zeta(m) = \frac{(-)^m}{(m-1)!}\,\frac{d^m}{da^m}\log\Gamma_1(a);$$

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* In the sequel (§ 17) this need only be true when $n$ is large, if $a+n\omega$ never vanish and be not a singularity of $\phi(x)$ or its reciprocal.

† Abel, *Œuvres Complètes* (1881), T. I., pp. 219–238.

and, as we see by the Taylor's expansion of $\log \Gamma_1(a+t)$, the modulus of this quantity behaves like $1/\mu^m$; where $\mu$ is the minimum value of

$$|a+n\omega| \quad (n = 0,\, 1,\, 2,\, \ldots,\, \infty).$$

The series $\sum_{m=0}^{\infty} f_m(s)\,\zeta(\rho s+m)$ is therefore convergent with $\sum_m \left(\frac{k}{\mu}\right)^m$, that is to say, if $\mu > k$.

Thus, if $a+n\omega$ for all positive integral (including zero) values of $n$ lies outside the circle of radius $k$, the function

$$Z(s,\, a) = \sum_{m=0}^{\infty} f_m(s)\,\zeta(\rho s+m,\, a)$$

is convergent and represents the continuation of the function which, when $R(s) > 1/\rho$, is represented by $\sum_{m=0}^{\infty} \frac{1}{[x\phi(x)]^s_{x=a+m\omega}}$. The sole poles of the function are at the points

$$\rho s+m = -1 \qquad \text{or} \qquad s = -(m-1)/\rho.$$

12. THEOREM.—*If* $|\arg (a/\omega)^\rho| = \pi-\epsilon$ *where* $0 < \epsilon \leqslant \pi$, *and if* $s = u+\iota v$, *then* $|Z(s)\,\omega^{\rho s}|\,e^{-(\pi-\epsilon')|v|}$, *where* $0 \leqslant \epsilon' < \epsilon$, *tends exponentially to zero as* $|v|$ *tends to infinity, u being finite.*

We have
$$Z(s) = \sum_{r=0}^{\infty} f_r(s)\,\zeta(\rho s+r).$$

Also $f_r(s)$ is an algebraic polynomial of degree $r$ in $s$. Hence the $(r+1)$-th term of $Z(s)$ behaves like $(1,\, s)^r\,\zeta(\rho s+r)$; and therefore, when $|v|$ tends to infinity, $u$ being finite, this term tends to infinity like

$$|\omega^{-s\rho}|\,e^{(\pi-\epsilon)|v|}.$$

Hence, as the series for $Z(s)$ remains convergent, however large $|s|$ may be, if $u$ be finite we see that

$$|Z(s)\,\omega^{\rho s}|\,e^{-(\pi-\epsilon')|v|},$$

where $0 \leqslant \epsilon' < \epsilon$, tends exponentially to zero as $|v|$ tends to infinity.

COROLLARY.—From the formula

$$Z(s) = \sum_{n=0}^{\infty} \frac{1}{[\phi(a+n\omega)\,(a+n\omega)^\rho]^s}$$

we see that, if $u > 1/\rho$, the same expression tends, under the assigned conditions, exponentially to zero, when $|s|$ tends to infinity.

13. THEOREM.—*The integral* $\frac{1}{2\pi\iota}\int \frac{Z(s)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds$, *in which* $z^s$ *has its principal value with respect to the quantity* $-\omega^\rho$, *is finite when taken along*

*any parallel to the imaginary axis in the finite part of the plane which does not pass through finite poles of the subject of integration, provided*

(1) $$|\arg (a/\omega)^\rho| = \pi - \epsilon \quad \textit{where} \quad 0 < \epsilon \leqslant \pi,$$

(2) $$|\arg (z/\omega)^\rho| = \epsilon' \quad \textit{where} \quad 0 \leqslant \epsilon' < \epsilon.$$

The subject of integration may be written

$$\frac{\pi}{s \sin \pi s}\, \omega^{\rho s} Z(s) \left(\frac{z}{\omega^\rho}\right)^s,$$

and this, under the assigned conditions, tends exponentially to zero as $|v|$ tends to infinity, $u$ being finite.

COROLLARY I.—The same integral vanishes when taken along any part of the great circle at infinity for which $u$ is finite.

COROLLARY II.—The same integral vanishes when taken along the great circle for which $u$ is infinite and positive, provided that, in addition to conditions (1) and (2), we have

(3) $$\left|\frac{z}{(a+n\omega)^\rho}\right| < 1 \quad (n = 0, 1, 2, \ldots, \infty),$$

(4) The circle pass between the zeros of $\sin \pi s$.

14. We now can obtain the asymptotic expansion of the logarithm of the very general simple non-repeated integral function of finite (non-zero) order,

$$F_1(z) = \prod_{n=0}^{\infty} \left[ \left\{ 1 + \frac{z}{x^\rho \phi(x)} \right\} \exp \left\{ \sum_{s=1}^{p} \frac{(-z)^s}{s\,[x^\rho \phi(x)]^s} \right\} \right]_{x=a+n\omega},$$

where $p+1 > 1/\rho > p$.

For, by the method previously employed, we evidently have, under the conditions (1) and (2) of § 13,

$$\log F_1(z) = -\frac{1}{2\pi\iota} \int_{L_1} \frac{Z(s)}{s} \frac{\pi z^s}{\sin \pi s}\, ds,$$

where the integral is taken along a contour $L_1$ parallel to the imaginary axis and cutting the real axis between the points $1/\rho$ and $p+1$. Hence, if $L_2$ be the contour defined in § 5, cutting the real axis in $-(l+1)+\epsilon$, where this point is not a pole of the subject of integration, and $\epsilon$ is small and positive, we have, under the same conditions,

$$\log F_1(z) = -\frac{1}{2\pi\iota} \int_{L_2} \frac{Z(s)}{s} \frac{\pi z^s}{\sin \pi s}\, ds,$$

together with the residues of the subject of integration at its poles $s = 0$, $s = -(r-1)/\rho$, $r = 0, 1, \ldots, m$, where $(m-1)/\rho < l+1 < m/\rho$.

Now the residue at $s = 0$ is the absolute term in the expansion of

$$1/\epsilon\,\{1+\epsilon \log z+\dots\}\,\{Z(0)+\epsilon Z'(0)+\dots\} = Z(0)\log z+Z'(0).$$

And the residue at $s = -(r-1)/\rho$ is[*]

$$\frac{\pi\rho}{1-r}\,\frac{z^{-(r-1)/\rho}}{\sin\pi\left(\dfrac{1-r}{\rho}\right)} \times \text{residue of } \sum_{r=0}^{\infty} f_r(s)\,\zeta(\rho s+r)$$
$$= \frac{\pi}{\omega(1-r)}\,\frac{z^{-(r-1)/\rho}}{\sin\pi\left(\dfrac{1-r}{\rho}\right)}\,f_r\left(-\frac{r-1}{\rho}\right).$$

Therefore, under the conditions (1) and (2),

$$\log F_1(z) = Z(0)\log z+Z'(0)+\sum_{n=-p}^{l}{}' \frac{(-)^{n-1}Z(-n)}{nz^n}$$
$$+\sum_{r=0}^{m}{}'' \frac{\pi z^{-(r-1)/\rho}}{\omega(1-r)\sin\pi\left(\dfrac{1-r}{\rho}\right)}\,f_r\left(\frac{1-r}{\rho}\right)-\frac{1}{2\pi\iota}\int_{L_2}\frac{Z(s)}{s}\,\frac{\pi z^s}{\sin\pi s}\,ds,$$

the double accent denoting that $r = 1$ is to be excluded from the summation.

15. Suppose now that $|z|$ is very large. As in § 7, the conditions (1) and (2) can be replaced by the conditions

(1) That $a+n\omega$ does not vanish for any positive integral value of $n$,

(2) That $|\arg(z/\omega^\rho)| < \pi$.

The modulus of the integral along the contour $L_2$ may be proved to be at most of order $|z|^{-l-1+\epsilon}$.

The series $\displaystyle\sum_{r=0}^{\infty} \frac{\pi z^{(1-r)/\rho} f_r\left(\dfrac{1-r}{\rho}\right)}{(1-r)\sin\pi\left(\dfrac{1-r}{\rho}\right)}$ is absolutely convergent if $|z|$ be sufficiently large and differs from $\displaystyle\sum_{r=0}^{m}$ by a quantity which is at most of order $|z|^{-l-1}$.

For $\displaystyle f_r\left(\frac{1-r}{\rho}\right) = \frac{1}{2\pi\iota}\int\frac{x^{r-1}\,dx}{[\phi(x)]^{(1-r)/\rho}} = \frac{1}{2\pi\iota}\int[x^\rho\phi(x)]^{(r-1)/\rho}\,dx,$

the integral being taken round a circle, centre the origin and radius $k' > k$. If $M$ be the maximum value of $[x^\rho\phi(x)]^{1/\rho}$ on this circle, the integral behaves when $r$ is large like $CM^{r-1}$, where $C$ is finite. Therefore the series is convergent if $M < |z^{1/\rho}|$.

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[*] [*Note added April* 5, 1905.]—We assume, of course, that $(1-r)/\rho$ is not an integer. In such cases limiting forms arise.

We see, then, that, under our new conditions (1) and (2),

$$\log F_1(z) = Z(0)\log z + Z'(0) + \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z(-n)$$

$$+ \sum_{n=0}^{\infty}{}'' \frac{\pi z^{(1-n)/\rho} f_n\left(\frac{1-n}{\rho}\right)}{\omega(1-n)\sin\pi\left(\frac{1-n}{\rho}\right)} + J_l,$$

where $J_l$, when $|z|$ is large, is at most of order $|z|^{-l}$.

16. We may give a slightly more elegant form to this formula, as follows.

Suppose that $y = x^\rho \phi(x) = x^\rho \left\{1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots\right\}$ when $|x| > k$.

As $\arg x$ goes from 0 to $2\pi$, $\arg y$ goes from 0 to $2\pi\rho$.

By reversion of series we obtain an expansion valid for large values of $|y|$

$$x = y^{1/\rho}\left\{1 + \frac{d_1}{y^{1/\rho}} + \frac{d_2}{y^{2/\rho}} + \dots\right\},$$

wherein also, as $\arg y$ goes from 0 to $2\pi\rho$, $\arg x$ goes from 0 to $2\pi$.

Now we have $f_n\left(\frac{1-n}{\rho}\right) = \frac{1}{2\pi\iota}\int x^{n-1}[\phi(x)]^{(n-1)\rho}\,dx$

(the integral being taken round a circle, centre the origin and radius $> k$, from $\arg x = 0$ to $\arg x = 2\pi$)

$$= \frac{1}{2\pi\iota}\int [x^\rho\phi(x)]^{(n-1)/\rho}\,dx$$

$$= -\frac{1}{2\pi\iota}\int y^{(n-1)/\rho}\left\{\sum_{m=0}^{\infty}\frac{d_m\left(\frac{m-1}{\rho}\right)}{y^{[(m-1)/\rho]+1}}\right\}dy,$$

wherein the integral is taken along a contour which is a circular arc from $\arg y = 0$ to $\arg y = 2\pi\rho$.

Now, if the integral be taken along a circular arc from $\arg y = 0$ to $\arg y = 2\pi\rho$,

$$\int y^{-k/\rho-1}\,dy = r^{-k/\rho}\int_0^{2\pi\rho}\iota\,d\theta\,e^{-(k/\rho)\iota\theta} = \rho r^{-k/\rho}\int_0^{2\pi} e^{-k\theta}\iota\,d\theta$$

$$= 0 \text{ if } k \neq 0, \text{ and } = 2\pi\iota\rho \text{ if } k = 0.$$

Hence

$$f_n\left(-\frac{n-1}{\rho}\right) = -\frac{1}{2\pi\iota}\,2\pi\iota\rho\,d_n\left(\frac{n-1}{\rho}\right) = -(n-1)\,d_n.$$

This relation is purely algebraical, and can be otherwise obtained by actual calculation of the quantities $f_n(s/\rho)$ and by establishing the set of relations between these quantities and the $d$'s furnished by the reversion process.

We now see that we have the asymptotic expansion

$$\log F_1(z)$$
$$= Z(0)\log z + Z'(0) + \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z(-n) + \sum_{n=0}^{\infty}{}'' \frac{\pi d_n z^{1/\rho}}{\omega \sin \pi \left(\frac{1-n}{\rho}\right) z^{n/\rho}} + J_l.$$

The quantities $Z(-n)$ which intervene may be called the Maclaurin constants.

For
$$Z(-n) = \sum_{r=0}^{\infty} f_r(-n)\,\zeta(r-\rho n),$$

and, since
$$[x^\rho \phi(x)]^n = x^{\rho n} \sum_{r=0}^{\infty} \frac{f_r(-n)}{x^n}$$

when $x = a+p\omega$ $(p = 0, 1, \ldots, m-1)$, we see that we have the Maclaurin sum-formula

$$\sum_{p=0}^{m-1} [a+p\omega)^\rho \phi(a+p\omega)]^n = Z(-n) + \sum_{p=0}^{\infty} \frac{S_p'(a)}{p!} \left[\frac{d^p}{dx^p} \int^x x^{\rho n} [\phi(x)]^n dx\right]_{x=m\omega}.$$

Again, we have

$$Z(0) = \sum_{r=0}^{\infty} f_r(0)\,\zeta(r) = \zeta(0) = -S_1'(a),$$

for $f_r(0) = 0$, $r \neq 0$, and $f_0(0) = 1$.

Also
$$Z'(0) = \rho\,\zeta'(0) + \sum_{r=0}^{\infty} f_r'(0)\,\zeta(r),$$

and therefore $-Z'(0)$ is the Maclaurin constant corresponding to the application of the Maclaurin sum-formula to the function

$$\sum_{n=0}^{m-1} \log \{x^\rho \phi(x)\}_{x=a+n\omega}.$$

17. The integral function $F_1(z)$ which has been considered is not the most general simple non-repeated function of finite non-zero order. A more general function would be*

$$\prod_{n=0}^{\infty} \left[\left\{1+\frac{z}{x^\rho \phi(x)}\right\} \exp\left\{\sum_{s=1}^{p} \frac{(-z)^s}{s[x^\rho \phi(x)]^s}\right\}\right]_{x=a+n\omega},$$

where $\phi(x)$ admits together with its reciprocal, outside a circle outside which $a+n\omega$ ($n=1, 2, \ldots, \infty$) all lie, an absolutely convergent expansion of the form

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* Verbal alterations have been made in this paragraph [*April 5th*, 1905].

$$1+\frac{c_1}{x^{\epsilon_1}}+\ldots+\frac{c_r}{x^{\epsilon_r}}+\ldots,$$

where $0 < \epsilon_1 < \epsilon_2 \ldots$ and the $\epsilon$'s tend to infinity or are finite in number. The asymptotic expansion of the logarithm of this function can be developed in the same way. The function $F_1(z)$ which has been considered is obtained by putting $\epsilon^r = r$ ($r = 1, 2, \ldots, \infty$). By a slight modification of the previous theory we may see that, provided $1/\rho$ is not an integer, provided $a+n\omega$ and $\phi(a+n\omega)$ and its reciprocal vanish for no positive integral value of $n$, and provided $|\arg(z/\omega^\rho)| < \pi$, we have, when $|z|$ is large, the asymptotic expansion

$$\log F_1(z)$$
$$=-S_1'(a)\log z+Z'(0)+\sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z(-n)+\sum_{n=0}^{\infty}{}'' \frac{\pi z^{1/\rho} d_n}{\omega \sin\pi\left(\dfrac{1-n}{\rho}\right) z^{n/\rho}}+J_l,$$

where $p < 1/\rho < p+1$, and where $|J_l|$ is at most of order $|z|^{-l}$.

When $1/\rho$ is an integer $p$, we may obtain the corresponding expansion by the calculus of limits, just as the expansion for ${}_1R_\rho(z)$ was deduced from ${}_1Q_\rho(z)$.

No limitation is involved in the assumption that $\underset{|x|=\infty}{\mathrm{Lt}}\ \phi(x) = 1$, for we may always ensure that this shall be the case by making the substitution $z = c_0\zeta$ in $F_1(z)$.

The result which has been obtained is, in the main, in accordance with that obtained previously for integral functions of finite non-zero order by the theory of asymptotic series [*I.F.*, §§ 53–59]. We have, however, given greater precision to the conditions under which the expansion is valid than was possible before the enumeration of the conditions under which it is legitimate to apply the Maclaurin sum-formula.

The generality of the form of the function $\phi(x)$ which we have taken is very great. We may note among special cases that $\phi(x)$ may be

(1) a rational integral function of $1/x$ which does not vanish when $1/x = 0$, $R(1/x)$ say;

(2) a similar rational integral function of negative fractional powers of $x$;

(3) of the form $R(1/x)e^{G(1/x)}$, where $G(x)$ is an integral function of $x$;

(4) of the the form $R(1/x)\mu(1/x)$, where $\mu(x)$ is a meromorphic function with no poles or zeros within a circle of finite radius surrounding the origin.

We may state the final result as follows:—

The logarithm of the specified general type of integral function of finite non-zero order with a single sequence of non-repeated zeros admits, when $|z|$ is large, an asymptotic expansion valid everywhere but in the neighbourhood of the zeros of the function; and all the coefficients of this expansion can be built up from the simple Riemann $\zeta$ function $\zeta(s, a \mid \omega)$.

If the function is of order $1/\rho$ where $p+1 > 1/\rho > p$, the dominant term of the asymptotic expansion is of the order of magnitude of $z^{1/\rho}$, and, if $1/\rho$ is an integer $p$, is of the order of magnitude of $z^p \log z$.

18. I will now briefly indicate the extension of the previous theory to non-repeated functions of multiple linear sequence. We base the investigation on the properties of the multiple Riemann $\zeta$ function $\zeta_r(s, a \mid \omega_1, \ldots, \omega_r)$, defined in § 15 of the previous paper, the theory of which has been developed in the author's memoir on the multiple gamma function.

Let $\phi(x)$ be the function of $x$ defined in § 11 which is such that neither $\phi(x)$ nor its reciprocal has any singularities outside a circle of finite radius $k$ and centre the origin, outside which $(a+\Omega)$, where $\Omega = n_1\omega_1+\ldots+n_r\omega_r$, lies when $n_1, \ldots, n_r$ have any positive integral values (zero included). We assume that the $\omega$'s all lie on the same side of some straight line $P$ through the origin. Further, let $\operatorname*{Lt}_{|x|=\infty} \phi(x) = 1$.

Then the function which, when $R(s) > r/\rho$, is represented by

$$\sum_{n=0}^{\infty} \frac{1}{[\phi(a+\Omega)(a+\Omega)^{\rho}]^{s}},$$

where the many-valued functions have their principal values with respect to the axis of $-1/L$ defined in § 15 of the previous paper, is represented for all values of $s$ by

$$Z_r(s) = \sum_{m=0}^{\infty} f_m(s)\, \zeta_r(\rho s+m, a),$$

where $\zeta_r(s, a)$ is the $r$-ple Riemann $\zeta$ function, and where

$$[\phi(x)]^{-s} = \sum_{m=0}^{\infty} \frac{f_m(s)}{x^m}.$$

The sole singularities of the function, *qua* function of $s$, in the finite part of the plane are at the points

$$\rho s+t = q, \qquad \text{where } \begin{cases} q = 1, 2, \ldots, r, \\ t = 0, 1, \ldots, \infty, \end{cases}$$

or

$$s = (q-t)/\rho,$$

For [*M.G.F.*, § 31] the sole singularities of $\zeta_r(s, a)$ are at the points $s = q$.

19. We now assume, for otherwise the points $(a+\Omega)^\rho$ will cover the whole region of the plane near infinity, that all the points $\Omega^\rho$ lie within an angle $\theta\,(< 2\pi)$ whose vertex is the origin. This necessitates that, when $\rho \geqslant 2$, the points $\omega$ lie within an angle $\theta/\rho\,(< 2\pi/\rho)$. In this latter case we take the fundamental line $P$ (see § 15 of the previous paper) to be the external bisector of this angle.

We shall obtain the system of points $\Omega^\rho$ within the angle $\theta$ by rotating each point $\Omega$ until its argument is $\rho$ times its former value. Let $1/L_\rho$ be the line which is obtained from $1/L$ in this way. Then, if $\phi$ be the argument of the line $1/L$, that of the *quantity* $1/L_\rho$ is $\rho\phi$.

Let $z^s$ have its principal value with respect to the *quantity* $-1/L_\rho$. (1)

By this we mean that $\rho\phi-\pi < \arg z < \rho\phi+\pi$.

Further, assume that $z$ does not lie within the region of the points $-\Omega^\rho$; (2)
so that therefore

$$|\arg z-\rho\phi| = \eta, \quad \text{where} \quad 0 \leqslant \eta < \begin{cases} \pi-\dfrac{\rho\pi}{2}\ (\rho < 2), \\ \pi-\tfrac{1}{2}\theta\ \ (\rho \geqslant 2). \end{cases}$$

In the expression for $\zeta_r(s, a)$ terms of the type $(a+\Omega)^{\rho s}$ occur, which have their principal values with respect to $-1/L$. By taking $a$ to be positive with respect to the $\omega$'s (3), we ensure that $\arg(a+\Omega)$ differs from $\phi$ by a quantity $< \begin{cases} \tfrac{1}{2}\pi & (\rho < 2), \\ \tfrac{1}{2}\theta/\rho & (\rho \geqslant 2); \end{cases}$ and therefore that

$$|\arg(a+\Omega)^\rho-\rho\phi| = \epsilon, \quad \text{where} \quad 0 \leqslant \epsilon < \begin{cases} \tfrac{1}{2}\rho\pi & (\rho < 2), \\ \tfrac{1}{2}\theta & (\rho \geqslant 2). \end{cases}$$

We may therefore put under the conditions (1), (2), and (3)

$$\frac{z^s}{(a+\Omega)^{\rho s}} = (r_n e^{\iota\psi_n})^s,$$

and we shall have $\quad 0 \leqslant |\psi_n| \leqslant \epsilon+\eta < \pi.$

If, now, we have the further condition $r_n < 1$ (4), we have the proposition *that*

$$Z_r(s) z^s e^{-\pi|v|},$$

*where* $R(s) > -(l+1)$ *and* $s = u+\iota v$, *tends exponentially to zero as* $|s|$ *tends to infinity*. This may be proved by the same methods as those already employed.

Suppose now that $p+1 > r/\rho > p$. Let ${}_rL_0$ and ${}_rL_1$ be contours, similar to those employed in § 5, cutting the real axis between $r/\rho$ and $(p+1)$. Then, by the proposition just stated, the integral

$$\frac{1}{2\pi\iota}\int \frac{Z_r(s)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds.$$

is finite when taken along the contours ${}_rL_0$, ${}_rL_1$, and $L_2$, and the two former integrals are equal to one another.

20. Let $F_r(z)$ denote the integral function of multiple linear sequence and order $r/\rho$

$$\prod_{n_1=0}^{\infty}\cdots\prod_{n_r=0}^{\infty}\left[\left\{1+\frac{z}{x^\rho\,\phi(x)}\right\}\exp\left\{\sum_{s=1}^{p}\frac{(-z)^s}{s\{x^\rho\,\phi(x)\}^s}\right\}\right]_{x=a+\Omega},$$

$p$ being such that $p+1 > r/\rho > p$.

Then, as previously, we see that, provided $r_n < 1$ and $|\psi_n| < \pi$,

$$\log F_r(z) = -\frac{1}{2\pi\iota}\int_{{}_rL_1}\frac{Z_r(s)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds.$$

But both sides of the equality are one-valued continuous analytic functions for all values of $r_n$. Therefore we may dispense with the condition (4) of § 19.

Now apply Cauchy's theorem and change the contour of integration from ${}_rL_1$ to $L_2$. We get

$$F_r(z) = -\frac{1}{2\pi\iota}\int_{L_2}\frac{Z_r(s)\,\pi z^s}{s\sin \pi s}\,ds+\sum_{n=-p}^{l}{}'\,\frac{(-)^{n-1}}{nz^n}\,Z_r(-n)$$

plus the sum of the residues of the subject of integration at the points

$$s = 0,$$

$$s = (q-t)/\rho \qquad (q = 1, 2, \ldots, r;\ \ t = 0, 1, \ldots, m),$$

where $(m+1-q)/\rho > l+1 > (m-q)/\rho$.

Now the residue of $\dfrac{\pi z^s}{s\sin \pi s}\,\zeta_r(\rho s+t)$ at $s = (q-t)/\rho$ is

$$\frac{\pi z^{(q-t)/\rho}}{(q-t)\sin \pi\left(\dfrac{q-t}{\rho}\right)}\,\frac{(-)^{q+r}\,{}_rS_1^{(q+1)}(a)}{(q-1)!} \qquad [M.G.F., \S\ 31].$$

Hence the sum of the residues of the subject of integration at the points $s = (q-t)/\rho$ is the sum of such terms of the infinite series

$$\sum_{q=1}^{r}\frac{(-)^{q+r}\,\pi\,{}_rS_1^{(q+1)}(a)}{(q-1)!}\sum_{t=0}^{\infty}{}''\,f_t\left(\frac{q-t}{\rho}\right)\frac{z^{(q-t)/\rho}}{(q-t)\sin \pi\left(\dfrac{q-t}{\rho}\right)}$$

as are not of order less than $1/|z|^{l+1}$ when $|z|$ is large. The double accent in the summation denotes that the term for which $t = q$ is to be omitted.

The series just written is absolutely convergent for sufficiently large values of $|z|$, since

$$f_t\left(\frac{q-t}{\rho}\right) = \frac{1}{2\pi\iota}\int \frac{x^{t-1}}{[\phi(x)]^{(q-t)/\rho}}\,dx = \frac{1}{2\pi\iota}\int x^{q-1}[x^{\rho}\phi(x)]^{(t-q)/\rho}\,dx,$$

the integral being taken round a circle of radius $k' > k$. Hence, when $t$ is large,

$$\left|f_t\left(\frac{q-t}{\rho}\right)\right| < k'^{q} M^{(t-q)/\rho},$$

where $M$ is the maximum value of $x^{\rho}\phi(x)$ on the circle in question. The series is therefore convergent if $|z| > M$.

Again, the residue of the subject of integration at $s = 0$ is the absolute term in the expansion

$$\sum_{t=0}^{\infty} \frac{\{1+\epsilon \log z+\ldots\}}{\epsilon\left\{1-\frac{\pi^2\epsilon^2}{3!}+\ldots\right\}}\{\zeta_r(t)+\rho\epsilon\zeta_r'(t)+\ldots\}\{f_t(0)+\epsilon f_t'(0)+\ldots\},$$

and is therefore

$$\begin{aligned}\sum_{t=0}^{\infty}\{f_t(0)[\zeta_r(t)\log z+\rho\zeta_r'(t)]+f_t'(0)\,\zeta_r(t)\} &= \zeta_r(0)\log z+\rho\zeta_r'(0)+\sum_{t=0}^{\infty} f_t'(0)\,\zeta_r(t)\\ &= \zeta_r(0)\log z+Z_r'(0).\end{aligned}$$

We note [*M.G.F.*, §§ 22 and 23] that $\zeta_r(0) = (-)^r_r S_1'(a)$ and that

$$\zeta_r'(0) = \left[\frac{\partial}{\partial s}\zeta_r(s, a)\right]_{s=0} = \log\frac{\Gamma_r(a)}{\rho_r(\omega)},$$

where $\Gamma_r(a)$ is the $r$-ple gamma function and $\rho_r(\omega)$ is the $r$-ple Stirling modular form.

We have then

$$\begin{aligned}\log F_r(z) = \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z_r(-n)+Z_r'(0)+\zeta_r(0, a)\log z\\ +\sum_{q=1}^{r}\frac{(-)^{q+r}\pi\,{}_rS_1^{(q+1)}(a)}{(q-1)\,!}\sum_{t=0}^{\infty}{}'' f_t\left(\frac{q-t}{\rho}\right)\frac{z^{(q-t)/\rho}}{(q-t)\sin\pi\left(\frac{q-t}{\rho}\right)}\\ -\frac{1}{2\pi\iota}\int_{L_2}\frac{Z_r(s)\,\pi z^s}{s\sin\pi s}\,ds+J_l,\end{aligned}$$

where $J_l$, when $|Z|$ is large, is of order less than $|z|^{-l}$.

21. In this equality the many-valued functions $z^s$, $z^{(q-t)/\rho}$, and $\log z$ have their principal values with respect to the *quantity* $-1/L_\rho$; that is to

say, in each case the argument of $z$ lies between $\rho\phi \pm \pi$. Further, we have assumed that $z$ does not lie within the region within which the points $-\Omega^{\rho}$ lie. And, finally, we have assumed that $a$ is positive with respect to the $\omega$'s, that is to say, that $a$ lies within the region within which the $\omega$'s lie.

Exactly as in § 7, we may remove the last restriction. And we obtain the theorem:—

*If* $|z|$ *is large and not within that part of the region at infinity within which the points* $-(a+\Omega)^{\rho}$ *lie, and if* $a$ *is of finite modulus, and not such as to make* $a+\Omega$ *vanish identically for any positive integral values of* $n_1, \ldots, n_r$ *(zero included), then*

$$\log F_r(z) = \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z_r(-n) + Z_r'(0) + \zeta_r(0, a)\log z$$

$$+ \sum_{q=1}^{r} \frac{(-)^{q+r}\pi\,{}_rS_1^{(q+1)}(a)}{(q-1)!} \sum_{t=0}^{\infty}{}'' f_t\left(\frac{q-t}{\rho}\right) \frac{z^{(q-t)/\rho}}{(q-t)\sin\pi\left(\frac{q-t}{\rho}\right)} + J_l,$$

*where* $|J_l|$ *is of order less than* $|z|^{-l}$. *In this expression*

$$z^{(q-t)/\rho} = \exp\left[\frac{q-t}{\rho}\log z\right],$$

*and* $\log z$ *is such that its argument lies between* $\rho\phi \pm \pi$.

22. We may give a more elegant form to the series just written.

Let
$$y = x^{\rho}\phi(x) = x^{\rho}\left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots\right),$$

where, as $\arg x$ goes from 0 to $2\pi$, $\arg y$ goes from 0 to $2\pi\rho$, and suppose that, by reversion of series, we obtain

$$x = y^{1/\rho}\left\{1 + \frac{{}_1d_1}{y^{1/\rho}} + \ldots + \frac{{}_1d_t}{y^{t/\rho}} + \ldots\right\}$$

and
$$x^q = y^{q/\rho}\left\{1 + \frac{{}_qd_1}{y^{1/\rho}} + \ldots + \frac{{}_qd_t}{y^{t/\rho}} + \ldots\right\},$$

where $q = 1, 2, \ldots, r$.

Then, as formerly (§ 16), we have

$$f_t\left(\frac{q-t}{\rho}\right) = \frac{1}{2\pi\iota}\int y^{(t-q)/\rho}\,\frac{1}{q}\,d\left\{y^{q/\rho} + \ldots + \frac{{}_qd_t}{y^{(t-q)/\rho}} + \ldots\right\}$$

taken along a circular arc from $\arg y = 0$ to $\arg y = 2\pi\rho$.

Thus
$$f_t\left(\frac{q-t}{\rho}\right) = \frac{\rho}{2\pi\iota q}\int\left(\frac{q-t}{\rho}\right){}_qd_t\,\frac{dy}{y}$$

taken round a circle enclosing the origin,

$$= \frac{q-t}{q}\, {}_q d_t.$$

Hence the previous formula may be written

$$\log F_r(z) = \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z_r(-n) + (-)^r {}_rS_1'(a) \log z + Z_r'(0)$$

$$+ \sum_{q=1}^{r} \frac{(-)^{q+r} \pi\, {}_rS_1^{(q+1)}(a)\, z^{q/\rho}}{q\,!} \sum_{t=0}^{\infty}{}'' \frac{{}_q d_t}{z^{t/\rho} \sin \pi \left(\frac{q-t}{\rho}\right)} + J_l.$$

If we make $l$ infinite, we obtain the asymptotic expansion of $\log F_r(z)$, which is valid when $|z|$ is very large and not within the domain of the large zeros of $F_r(z)$. If these zeros cover the whole plane at infinity, no asymptotic expansion exists. The formula now obtained is a development of the formula previously adumbrated [*Integral Functions of Multiple Linear Sequence*, §§ 11 and 15]. The asymptotic expansions of the general integral functions* of multiple linear sequence, non-repeated zeros, and finite non-zero order follow in the same way.

In obtaining the previous formula we have assumed that $1/\rho$ is not an integer, and that $(q-t)/\rho$ is not an integer for the values

$$q = 1, 2, \ldots, r; \qquad t = 0, 1, \ldots, \infty \quad (q \neq t).$$

When such exceptional cases arise we can always obtain a definite formula by the use of the calculus of limits.

23. As particular cases of the general result just obtained we may write down the asymptotic expansions of the three standard functions of finite non-zero order and multiple linear sequence:—

$${}_rP_\rho(z) = \prod_{n_1=0}^{\infty} \ldots \prod_{n_r=0}^{\infty} \left\{1 + \frac{z}{(a+\Omega)^\rho}\right\} \quad (\rho > r),$$

$${}_rQ_\rho(z) = \prod_{n_1=0}^{\infty} \ldots \prod_{n_r=0}^{\infty} \left[\left(1+\frac{z}{x^\rho}\right) \exp \sum_{s=1}^{p} \frac{(-z)^s}{sx^{p\rho}}\right]_{x=a+\Omega},$$

where $p$ is an integer such that $p+1 > r/\rho > p$,

$${}_rR_\rho(z) = \prod_{n_1=0}^{\infty} \ldots \prod_{n_r=0}^{\infty} \left[\left(1+\frac{z}{x^\rho}\right) \exp \sum_{s=1}^{p} \frac{(-z)^s}{sx^{p\rho}}\right]_{x=a+\Omega},$$

where $r/\rho$ is an integer $p$.

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* Such as correspond to those mentioned in § 17 [*April* 5, 1905].

We have

$$\log {}_rP_\rho(z) = \sum_{n=1}^{l} \frac{(-)^{n-1}}{nz^n}\zeta_r(-\rho n, a) + \rho \log \frac{\Gamma_r(a)}{\rho_r(\omega)} + (-)^r {}_rS_1'(a) \log z$$
$$+ \sum_{q=1}^{r} \frac{(-)^{q+r}\pi\, {}_rS_1^{(q+1)}(a)}{q!} \frac{z^{q/\rho}}{\sin\dfrac{\pi q}{\rho}} + J_l,$$

agreeing with *I.F.M.L.S.*, § 5.

We have a similar expansion for $\log {}_rQ_\rho(z)$, except that the first series is summed from $n = -p$ to $l$ (zero excluded).

To obtain the asymptotic expansion for $\log {}_rR_\rho(z)$ we apply the calculus of limits to the formula for $\log {}_rQ_\rho(z)$. The work has been carried out [*I.F.M.L.S.*, §§ 16 and 17]. The result is that, if $p$ be not a multiple of $r$,

$$\log {}_rR_\rho(z) = \frac{r}{\rho} \log \frac{\Gamma_r(a)}{\rho_r(\omega)} + (-)^r {}_rS_1'(a) \log z + \sum_{s=-p+1}^{l}{}' \frac{(-)^{s-1}}{sz^s}\zeta_r\left(-\frac{sr}{p}, a\right)$$
$$+ \frac{(-)^{p+r}z^p}{p(r-1)!}\psi_r^{(r)}(a) + \frac{(-z)^p}{(r-1)!}\, {}_rS_1^{(r+1)}(a)\left\{\frac{\log z}{r} - \sum_{k=1}^{r}\frac{1}{kp}\right\}$$
$$+ \sum_{m=1}^{r-1} \frac{(-)^{m+r}\pi z^{mp/r}\, {}_rS_1^{(m+1)}(a)}{m! \sin\dfrac{mp\pi}{r}} + J_l.$$

If $p = kr$, where $k$ is an integer, we have

$$\log {}_rR_\rho(z) = \frac{r}{p} \log \frac{\Gamma_r(a)}{\rho_r(\omega)} + (-)^r {}_rS_1'(a) \log z + \sum_{s=-p}^{l}{}^{*} \frac{(-)^{s-1}}{sz^s}\zeta_r\left(-\frac{sr}{p}, a\right)$$
$$+ \sum_{m=1}^{r} \frac{(-)^{(k+1)m} z^{km}}{m!\,k}\psi_r^{(m)}(a) + \sum_{m=1}^{r} \frac{(-)^{(k+1)m+r}}{m!\,k}\, {}_rS_1^{(m+1)}(a)\left\{k \log z - \sum_{n=1}^{m}\frac{1}{n}\right\} + J_l,$$

the star indicating that the terms which correspond to $s = 0$, $-k$, $-2k$, …, $-rk$ are to be omitted in the summation.

# GENERATIONAL RELATIONS FOR THE ABSTRACT GROUP SIMPLY ISOMORPHIC WITH THE GROUP $LF[2, p^n]$

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[Received December 10th, 1904.—Read January 12th, 1905.]

## Introduction.

1. The object of this paper is the proof of the following six theorems concerning sets of generational relations for the abstract group $G$, simply isomorphic with the group $LF[2, p^n]$ of all linear fractional transformations, on one variable, having determinant unity and coefficients belonging to the $GF(p^n)$.

**Theorem I.***—*The abstract group $G_{\frac{1}{2}p(p^2-1)}$, simply isomorphic with the group $LF[2, p]$, $p > 2$, may be generated by two operators $T$ and $S$, subject to the generational relations*

$$\text{(A)} \qquad S^p = I, \quad T^2 = I, \quad (ST)^3 = I, \quad (S^\tau T S^{2/\tau} T)^2 = I, \quad \tau \neq 0.$$

**Theorem II.**—*The abstract group $G_{\frac{1}{2}p^n(p^{2n}-1)}$, simply isomorphic with the group $LF[2, p^n]$, $p > 2$, $n > 1$, may be generated by $(p^n+1)$ operators $T$ and $S_\lambda$, $\lambda$ running through the marks of the $GF(p^n)$, subject to the generational relations*

$$\text{(B)} \quad \left\{ \begin{array}{ll} (1) & S_0 = I, \quad S_\lambda S_\mu = S_{\lambda+\mu} \quad (\lambda,\ \mu \text{ any marks}), \\ (2) & T^2 = I, \quad (S_1 T)^3 = I, \\ (3) & (S_\tau T S_{2/\tau} T)^2 = I \quad (\tau \text{ any mark} \neq 0), \\ (4) & [1/a,\ a^2],\ [1/a,\ ia^2],\ [i,\ a],\ [1/i,\ a] \quad (a \neq 0), \end{array} \right.$$

*where $i$ is a primitive root of the $GF(p^n)$, and $a$ is any mark subject to a restriction implied in the notation $[\lambda, \mu]$.*

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* For the special cases in which $p^n \leqslant 47$, this theorem has been proved by Prof. Dickson, *Proc. London Math. Soc.*, Vol. xxxv., pp. 292-306; *Bull. Amer. Math. Soc.*, Vol. ix., p. 297.

*Note.*—The symbol $[\lambda, \mu]$ is used to denote the relation[*]

$$S_\lambda T S_\mu T S_{(\lambda-1)/(\lambda\mu-1)} T S_{1-\lambda\mu} T S_{(\mu-1)/(\lambda\mu-1)} T = I,$$

where $\lambda$, $\mu$ are marks such that $\lambda\mu \neq 1$.

THEOREM III.[†]—*For the special cases in which* $p^n = 49, 81, 121$ *relations* (1), (2), (3) *form a set of generational relations for* $G_{\frac{1}{2}p^n(p^n-1)}$, $p > 2$.

THEOREM IV.[‡]—*The abstract group* $G_{2^n(2^{2n}-1)}$, *simply isomorphic with the group* $LF[2, 2^n]$, *may be generated by three operators* $a$, $b$, $c$, *subject to the generational relations*

$$\text{(C)} \qquad \begin{cases} (5) \quad a^{2^n-1} = I, \quad b^2 = I, \quad ba^\xi b = a^\eta b a^\zeta, \\ (6) \quad c^2 = I, \quad (ca)^2 = I, \quad (cb)^3 = I, \end{cases}$$

*where* $\xi = 1, 2, 3, \ldots, (2^n-2)$, *and* $\eta$, $\zeta$ *are determined by the relations* $i^\zeta = 1 + i^\xi$, $\eta \equiv \xi - \zeta$, *mod* $(2^n-1)$, $i$ *being a primitive root of the* $GF(2^n)$.

THEOREM V.—*The abstract group* $G_{2^n(2^{2n}-1)}$, *simply isomorphic with the group* $LF[2, 2^n]$, *may be generated by two operators* $a$ *and* $d$ *subject to the generational relations*

$$\text{(D)} \qquad a^{2^n-1} = I, \quad d^3 = I, \quad (da^\xi d^{-1} a^\xi)^2 = I, \quad (da^\xi da^\zeta)^2 = I,$$

*where* $\xi = 1, 2, 3, \ldots, (2^n-2)$, *and* $\zeta$ *is determined by the relation* $i^\zeta = 1 + i^\xi$, $i$ *being a primitive root of the* $GF(2^n)$.

THEOREM VI.[§]—*In the special cases in which* $n = 2, 3, 4, 5, 6$, *the abstract group* $G_{2^n(2^{2n}-1)}$, *simply isomorphic with the group* $LF[2, 2^n]$,

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[*] Relations (1), $T^2 = I$, $[\lambda, \mu]$, $\lambda$, $\mu$ any marks such that $\lambda\mu \neq 1$, constitute a set of generational relations for $G$. This is a special case of a more general theorem valid for any field due to Moore. See *Proc. London Math. Soc.*, Vol. xxxv., p. 293, and Dickson's *Linear Groups*, p. 300.

Note that, when $\lambda = 0$ or $1$, $[\lambda, \mu]$ reduces to $(S_1 T)^3 = I$, and, when $\lambda = -1$, $[\lambda, \mu]$ reduces to (3).

[†] For the special cases in which $p^n = 9, 25, 27, 125, 243$, Prof. Dickson has proved that (1), (2), (3) constitute a set of generational relations for $G_{\frac{1}{2}p^n(p^{2n}-1)}$, $p > 2$, *loc. cit.* The proofs for the cases in which $p^n = 125, 243$ have not been published.

[‡] This theorem is due to de Seguier, *Journal de Mathématiques*, Tome VIII., p. 253.

[§] The set of generational relations (E) is due to Prof. Dickson. He has proved Theorem VI. for $n = 2, 3, 4$. See *Proc. London Math. Soc.*, Vol. xxxv., p. 306 and p. 443; *Bull. Amer. Math. Soc.*, Vol. IX., pp. 194–204.

For $n = 2$, the set (E) reduces to $A^5 = I$, $B^2 = I$, $(AB)^3 = I$.

*may be generated by two operators $A$ and $B$ subject to the generational relations*

$$\text{(E)} \qquad A^{2^n+1} = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (BA^rBA^s)^2 = I,$$

*where* $r = 1, 2, 3, \ldots, 2^n$, *and the value of $s$ is determined by the relation* $\zeta^s(\zeta^r+i^2+1) = \zeta^r(i^2+1)+1$, *$\zeta$ being defined by the relation* $\zeta^2 = i^2\zeta+1$, *$i$ being a primitive root of the* $GF(2^n)$.

### THE GROUP $G$ SIMPLY ISOMORPHIC WITH $LF[2, p^n]$, $p > 2$.

2. LEMMA.—*The abstract group* $G_{\frac{1}{2}p^n(p^{2n}-1)}$, *simply isomorphic with the group* $LF[2, p^n]$, $p > 2$, *may be generated by* $(p^n+2)$ *operators $R$, $T$, and $S_\lambda$, $\lambda$ running through the marks of* $GF(p^n)$, *subject to relations* (1) *and*

$$\text{(7)} \qquad R^{\frac{1}{2}(p^n-1)} = I,$$

$$\text{(8)} \qquad S_\lambda R^\sigma = R^\sigma S_{\lambda i^{2\sigma}} \qquad (\lambda \textit{ any mark}, \ \sigma = 0 \textit{ or any integer}),$$

$$\text{(9)} \qquad (TR^\sigma)^2 = I \qquad (\sigma = 0 \textit{ or any integer}),$$

$$\text{(10)} \qquad TS_\gamma T = R^\rho S_{-\gamma} T S_{-1/\gamma} \ (\gamma \textit{ any mark} \neq 0, \ i^\rho = -\gamma),$$

*$i$ being a primitive root of the* $GF(p^n)$.

*Proof.*—The group $LF[2, p^n]$, $p > 2$, may be generated by the $p^n+1$ transformations

$$T : z' = \frac{-1}{z}, \qquad S_\lambda : z' = z+\lambda \qquad (\lambda \text{ any mark}),$$

while the sub-group $K$ of the transformations

$$S_{\alpha,\beta} : z' = \frac{\alpha z+\beta}{\alpha^{-1}}$$

may be generated by the transformations

$$S_{1,\lambda} \equiv S_\lambda : z' = z+\lambda, \qquad R : z' = iz/i^{-1},$$

$i$ being a primitive root of the $GF(p^n)$.

These generators of $K$ satisfy relations (1), (7), and (8). Since the group $LF[2, p^n]$, $p > 2$, when represented as a permutation group on $(p^n+1)$ letters, is doubly transitive while the sub-group $K$, being then a permutation group on $p^n$ letters, is simply transitive, it follows from the work of Jordan* that it is possible to determine $\gamma$, $\delta$, $\eta$, $\zeta$, and $\mu$ such that the following relations are true :—

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* *Traité des Substitutions*, p. 32.

$$(11) \qquad TS_{a,\beta}T = S_{\gamma,\delta}TS_{\eta,\zeta} \quad (a, \beta \text{ any marks} \neq 0),$$

$$(12) \qquad (TR^{\sigma})^2 = R^{\mu} \qquad (\sigma = 0 \text{ or any integer}).$$

The concrete expressions for $TS_{a,\beta}T$ and $S_{\gamma,\delta}TS_{\eta,\zeta}$ are respectively

$$z' = \frac{a^{-1}z}{-\beta z + a}, \qquad z' = \frac{\gamma\zeta . z + \delta\zeta - \eta\gamma^{-1}}{\eta^{-1}\gamma . z + \eta^{-1}\delta}.$$

A comparison of these expressions shows that a determination of $\gamma$, $\delta$, $\eta$, $\zeta$ is $\gamma = -\beta$, $\delta = a$, $\eta = 1$, $\zeta = -a^{-1}\beta^{-1}$. The concrete expression for $TR^{\sigma}T$ is $z' = i^{-2\sigma}z$. Therefore $TR^{\sigma}T = R^{-\sigma}$ and a determination of $\mu$ is $\mu = 0$. With these values of $\gamma$, $\delta$, $\eta$, $\zeta$, $\mu$, relations (12) and (11) become, respectively, (9) and

$$(13) \qquad TS_{a,\beta}T = S_{-\beta,a}TS_{1,-a^{-1}\beta^{-1}} \quad (a, \beta \text{ any marks} \neq 0).$$

A comparison of the concrete expressions for $S_{a,\beta}$ and $R^{\lambda}S_{1,a\beta}$ shows that $S_{a,\beta} = R^{\lambda}S_{1,a\beta}$ if $\lambda$ be determined by the relation $i^{\lambda} = a$. Therefore (13) may be written in the form

$$(14) \qquad TR^{\lambda}S_{1,a\beta}T = R^{\mu}S_{1,-a\beta}TS_{1,-a^{-1}\beta^{-1}},$$

where $\lambda$, $\mu$ are determined by the relations $i^{\lambda} = a$, $i^{\mu} = -\beta$.

In view of (9), relation (14) may be written

$$R^{-\lambda}TS_{1,a\beta}T = R^{\mu}S_{1,-a\beta}TS_{1,-a^{-1}\beta^{-1}},$$

or, if we write $a\beta = \gamma$, and use the notation $S_{1,\gamma} \equiv S_{\gamma}$, in the form (10).

This completes the first part of the proof, namely, that the generators $T$, $R$, $S_{\lambda}$ of the group $LF[2, p^n]$, $p > 2$, satisfy relations (1), (7), (8), (9), (10). From this fact it follows that the abstract group $G'$ defined by relations (1), (7), (8), (9), (10) is either the group $G_{\frac{1}{2}p^n(p^{2n}-1)}$ or a larger group. That it cannot be a larger group is seen as follows:—Every element $g'$ of the group $G'$ is a product whose constituents are $T$, $R$, $S_{\lambda}$. Every such product that does not involve $T$ may be reduced by means of (1), (7), (8) to the form $R^{\sigma}S_{\lambda}$, where $\lambda$ is a mark of the $GF(p^n)$, and $\sigma = 1, 2, 3, \ldots,$ or $\frac{1}{2}(p^n - 1)$. The maximum number of distinct products of this type is $\frac{1}{2}p^n(p^n - 1)$. Every product that does involve $T$ may be written in the form

$$g' = R^{\sigma_1}S_{\lambda_1}TR^{\sigma_2}S_{\lambda_2}TR^{\sigma_3}S_{\lambda_3}TR^{\sigma_4}S_{\lambda_4}T\ldots,$$

a product containing $n$ $T$'s. If $\lambda_2 = 0$, then $S_{\lambda_2} = I$ and $TR^{\sigma_2}T = R^{-\sigma_2}$, by (9). In this case $g'$ reduces to

$$g' = R^{\sigma_1}S_{\lambda_1}R^{-\sigma_2}R^{\sigma_3}S_{\lambda_3}TR^{\sigma_4}S_{\lambda_4}T\ldots,$$

which, by (1), (7), (8), may be reduced to

$$g' = R^{\sigma'} S_{\lambda'} T R^{\sigma_4} S_{\lambda_4} T \ldots,$$

a product containing $(n-2)$ $T$'s. If $\lambda_2 \neq 0$, we have

$$T R^{\sigma_2} S_{\lambda_2} T = R^{-\sigma_2} T S_{\lambda_2} T$$

by (9). Also, by (10), $T S_{\lambda_2} T = R^{\rho} S_{-\lambda_2} T S_{-1/\lambda_2}$.

Therefore $g'$ becomes

$$g' = R^{\sigma_1} S_{\lambda_1} R^{-\sigma_2} R^{\rho} S_{-\lambda_2} T S_{-1/\lambda_2} R^{\sigma_3} S_{\lambda_3} T R^{\sigma_4} S_{\lambda_4} T \ldots,$$

which, by means of (1), (7), (8), may be reduced to

$$g' = R^{\sigma''} S_{\lambda''} T R^{\sigma'''} S_{\lambda'''} T R^{\sigma_4} S_{\lambda_4} T \ldots,$$

a product containing $(n-1)$ $T$'s.

The process indicated reduces the number of $T$'s in the expression for $g'$ by one or two at each step. A finite number of steps will reduce the expression for $g'$ to a product containing one $T$ or no $T$'s. In the latter case $g'$ may be reduced to the form $R^{\sigma} S_{\lambda}$, as above. In the former case $g' = R^{\sigma} S_{\lambda} T R^{\sigma'} S_{\lambda'}$. In view of (9) this becomes $g' = R^{\sigma} S_{\lambda} R^{-\sigma'} T S_{\lambda'}$, which may be reduced by means of (1), (7), (8) to the type form $g' = R^{\sigma_1} S_{\lambda_1} T S_{\lambda'}$.

The maximum number of distinct products of this type is $\frac{1}{2} p^{2n} (p^n - 1)$. The maximum order of the group $G'$ is

$$\tfrac{1}{2} p^n (p^n - 1) + \tfrac{1}{2} p^{2n} (p^n - 1) = \tfrac{1}{2} p^n (p^{2n} - 1),$$

which is precisely the order of the group $G_{\frac{1}{2} p^n (p^{2n}-1)}$. Therefore the two groups are identical, and relations (1), (7), (8), (9), (10) constitute a set of generational relations for the group $G_{\frac{1}{2} p^n (p^{2n}-1)}$.

### 3. *Proof of Theorem II.*

Relations (1) and (2) are those of relations (1), (7), (8), (9), (10) which do not involve $R$. $T^2 = I$ comes from (9) when $\sigma = 0$, and $(S_1 T)^3 = I$ comes from (10) when $\gamma = \pm 1$. When $\gamma = -i$, relation (10) becomes

$$R = T S_{-i} T S_{-1/i} T S_{-i}. \tag{15}$$

The rest of relations (10), viz., $R^{\rho} = T S_{\gamma} T S_{1/\gamma} T S_{\gamma}$, $\gamma = -i^{\rho}$, $\rho \neq 0$ or 1, follow from (1), (2), (4), and (15). To prove this it will be sufficient to show that the relation

$$T S_{\gamma} T S_{1/\gamma} T S_{\gamma} T S_{-i} T S_{-1/i} T S_{-i} = T S_{i\gamma} T S_{1/i\gamma} T S_{i\gamma} \tag{16}$$

(where $\gamma = -i^{\rho}$, $\rho \neq 0$ or 1) follows from (1), (2), and (4). By means of (1) and (2) relation (16) may be written

$$(T S_{i\gamma - \gamma} T S_{1/i\gamma} T) S_{i\gamma + i} T S_{1/i} (T S_i T S_{-\gamma} T) S_{-1/\gamma} = I.$$

But, by (4), $TS_i TS_{-\gamma} T = S_{(-\gamma-1)/(i\gamma+1)} TS_{-i\gamma-1} TS_{(i-1)/(i\gamma+1)}$,

and $TS_{i\gamma-\gamma} TS_{1/i\gamma} T = S_{(1-i\gamma)/\gamma} TS_{-1/i} TS_{i^2\gamma-i\gamma-i}$.

Therefore (16) becomes

$$S_{(1-i\gamma)/\gamma} TS_{-1/i} TS_{i^2\gamma-i\gamma-i} S_{i\gamma+i} TS_{1/i} S_{(-\gamma-1)/(i\gamma+1)} TS_{-i\gamma-1} TS_{(i-1)/(i\gamma+1)} S_{-1/\gamma} = I,$$

or, by (1) and (2),

$$TS_{-1/i} TS_{i^2\gamma} TS_{(1-i)/(i+i^2\gamma)} TS_{-i\gamma-1} TS_{(-1-i^2\gamma)/(1+i\gamma)} = I,$$

which is the inverse of $\left[\frac{1}{i}, \frac{1+i^2\gamma}{1+i\gamma}\right]$, one of relations (4).

To complete the proof of the theorem it will be sufficient to show that relations (7), (8), and (9), expressed in terms of $T$ and $S_\lambda$ by means of (10), follow from (1), (2), (3), and (4).

The substitution in (7) of the value of $R^{\frac{1}{2}(p^n-1)}$ given by (10) results in $(S_1 T)^3 = I$, one of relations (2).

The substitution in (9) of the value of $R^\sigma$ given by (10) results in $(S_{-i^\sigma} TS_{-i^{-\sigma}} TS_{-i^\sigma})^2 = I$, which follows from (1) and (3).

Relations (8) are seen to be equivalent to the relations

$$(17) \qquad S_{i^{2k}} = R^{-k} S_1 R^k, \qquad\qquad (18) \qquad S_{i^{2k+1}} = R^{-k} S_i R^k,$$

the range for $k$ being 1, 2, 3, ..., $\frac{1}{2}(p^n-3)$.

The substitution in (17) of the value of $R^k$ given by (10) results in the relations $[-1/i^{2k}, i^{2k}]$ which follow from (4). The same substitution in (18) results in

$$(19) \qquad S_{-ia^2} TS_{-1/a} TS_{-a} TS_i TS_a TS_{1/a} T = I, \qquad a = -i^k.$$

When $ia \neq 1$, relation (19) follows from (1), (2), and $[i, a]$, $[1/a, ia^2]$, as may be seen by substituting in (19) the value of $TS_i TS_a T$ given by $[i, a]$. When $ia = 1$, relation (19) follows from (1), (2), and $[i, -1/i]$, $[1/i, \frac{1}{2}(i+1)]$, as may be seen by substituting in (19) the value of $TS_i TS_{-1/i} T$ given by $[i, -1/i]$. This substitution is made after replacing $a$ by $1/i$ in (19).

This completes the elimination of $R$ from the set of generational relations (1), (7), (8), (9), (10), the result being the set of relations (B).

4. LEMMA.—*The abstract group $G_{\frac{1}{2}p(p-1)}$, simply isomorphic with the group $LF[2, p^n]$, may be generated by two operators $T$ and $S$ subject to the generational relations* (A) *and* (20) $[1/i, i^2]$, *$i$ being a primitive root of $p$.*

*Proof.*—For the group $LF[2, p]$, $p > 2$, only two generators, $T$ and $S$, are necessary, and in relations (1), (7), (8), (9), (10) we may write $S^\lambda$ in place of $S_\lambda$. Some simplifications result from the fact that the marks of the $GI(p)$ are all integers. Relations (17) and (18), which were seen to be equivalent to (8), now become

$$(21) \qquad S^{\iota^{2k}} = R^{-k}SR^{k}, \qquad S^{\iota^{2k+1}} = R^{-k}S^{\iota}R^{k}.$$

Relations (21) all follow from $S_{\iota^2} = R^{-1}SR$, as may be proved easily by induction. Relations (9) all follow from $T^2 = I$ and $(TR)^2 = I$, and the set of relations (1),. (7), (8), (9), (10) becomes the following:—

$$(22) \qquad S^{p} = I, \quad T^2 = 1, \quad (ST)^3 = I;$$

$$(23) \qquad R^{\frac{1}{2}(p-1)} = I;$$

$$(24) \qquad S^{\iota^2} = R^{-1}SR;$$

$$(25) \qquad (TR)^2 = I;$$

$$(26) \qquad R^{\rho} = TS^{\gamma}TS^{1/\gamma}TS^{\gamma}, \quad \text{where} \quad \gamma = -\iota^{\rho}, \ \rho \neq 0.$$

When $\rho = 1$, (26) gives $R$ in terms of $S$ and $T$, viz.,

$$(27) \qquad R = TS^{-\iota}TS^{-1/\iota}TS^{-\iota}.$$

When $\rho = 2k$, an even number; (26) becomes

$$R^{2k} = TS^{-\iota^{2k}}TS^{-\iota^{-2k}}TS^{-\iota^{2k}}.$$

This follows from (22), (24), (25), and (27), for

$$\begin{aligned} TS^{-\iota^{2k}}TS^{-\iota^{-2k}}TS^{-\iota^{2k}} &= T(R^{-k}S^{-1}R^{k})T(R^{k}S^{-1}R^{-k})T(R^{-k}S^{-1}R^{k}), && \text{by (24)} \\ &= TR^{-k}S^{-1}(R^{k}TR^{k})S^{-1}(R^{-k}TR^{-k})S^{-1}R^{k} \\ &= TR^{-k}(S^{-1}TS^{-1}TS^{-1})R^{k}, && \text{by (25),} \\ &= TR^{-k}TR^{k}, && \text{by (22),} \\ &= R^{k}T^{2}R^{k}, && \text{by (25),} \\ &= R^{2k}, && \text{by (22).} \end{aligned}$$

When $\rho = 2k+1$, an odd number, (26) becomes

$$R^{2k+1} = TS^{-\iota^{2k+1}}TS^{-\iota^{2k+1}}TS^{-\iota^{2k+1}}.$$

This follows from (22), (24), (25), and (27), for

$$\begin{aligned} TS^{-\iota^{2k+1}}TS^{-\iota^{-(2k+1)}}TS^{-\iota^{2k+1}} &= T(R^{-k}S^{-\iota}R^{k})T(R^{k+1}S^{-\iota}R^{-k-1})T(R^{-k}S^{-\iota}R^{k}), \\ & && \text{by (24),} \\ &= (TR^{-k})S^{-\iota}(R^{k}TR^{k})RS^{-\iota}R^{-1}(R^{-k}TR^{-k})S^{-\iota}R^{k}, \\ &= R^{k}TS^{-\iota}T(RS^{-\iota}R^{-1})TS^{-\iota}R^{k}, && \text{by (25),} \\ &= R^{k}(TS^{-\iota}TS^{-1/\iota}TS^{-\iota})R^{k}, && \text{by (24),} \\ &= R^{k}RR^{k}, && \text{by (27),} \\ &= R^{2k+1}. \end{aligned}$$

To complete the proof of the theorem it will be sufficient to show that relations (23), (24), and (25), expressed in terms of $S$ and $T$ by means of (26), follow from relations (A) and (20).

The substitution in (23) of the value of $R^{\frac{1}{2}(p-1)}$ given by (26) results in the relation $(ST)^3 = I$. The substitution in (24) of the value of $R$ given by (26) results in a relation which reduces to (20) by means of relations (A). The same substitution in (25) results in the relation $(S^{-i}TS^{-1/i}TS^{-i})^2 = I$, which follows from relations (A).

This completes the elimination of $R$ from the set of generational relations (22), (23), (24), (25), (26), the result being the set of generational relations (A) and (20).

## 5. *Proof of Theorem I.*

The proof of Theorem I. consists in showing that (20) is a consequence of relations (A). (20) may be written in the form

$$(TS^{1/i}TS^{2i})S^{i^2-2i}TS^{-1/i}T^{1-i}TS^{1+i} = I.$$

Replace the expression in parentheses by its inverse, as may be done by (A), and invert. The relation becomes

$$(28) \qquad TS^{i-1}TS^{1/i}TS^{2i-i^2}TS^{1/i}TS^{i-1} = I.$$

Conversely, (20) follows from (A) and (28). Consider the relation

$$(29) \qquad TS^{i-k}TS^{1/(i-k+1)}TS^{-(i-k-1)(i-k+1)}TS^{1/(i-k+1)}TS^{i-k} = I,$$

where $k$ is an odd integer. It may be written

$$TS^{i-k}TS^{1/(i-k+1)}(TS^{-(i-k+1)(i-k-1)}TS^{-2/(i-k+1)(i-k-1)})S^{1/(i-k-1)}TS^{i-k} = I.$$

Replace the expression in parentheses by its inverse, as may be done by (A), and it becomes

$$TS^{-i+k+2}(S^{2i-2k-2}TS^{1/(1-k-1)}T)S^{(i-k-1)(i-k+1)}(TS^{1/(i-k-1)}TS^{2i-2k-2})S^{-i+k+2} = I.$$

Replace the expression in each pair of parentheses by its inverse, as may be done by (A), and then invert. The relation becomes

$$(30) \qquad TS^{i-k'}TS^{1/(i-k'+1)}TS^{-(i-k'-1)(i-k'+1)}TS^{1/(i-k'+1)}TS^{i-k'} = I,$$

where $k' = k+2$.

Relation (30) is a consequence of (A) and (29), and, conversely, (29) is a consequence of (A) and (30). But, for $k = 1$, (29) becomes (28). Therefore, for any value of $k$, $k$ being an odd integer, relation (29) is a consequence of (A) and (28), and, conversely, (28) is a consequence of (A) and (29). But, for $k = i$ or $k = i+1$, according as $i$ is an odd or even integer, relation (29) reduces to $(ST)^3 = I$. Therefore (28) is a consequence of the relations (A).

### 6. *Theorem III.—Outline of Proof.*

By making use of the method used by Prof. Dickson* for other special cases, the writer has made the computations which prove that, for the special cases in which $p^n = 49, 81, 121$, relations (4) are a consequence of relations (1), (2), (3). The method was proved to fail for $p^n = 169$. For this computation it was found convenient to have the marks of the $GF(p^n)$ arranged in two tables. In each table every mark is expressed as a power of a primitive root $i$, and as a polynomial in $i$ of degree $k \leqslant n-1$. The coefficients in this polynomial are integers reduced modulo $p$. The mark $\alpha i^k + \beta i^{k-1} + \gamma i^{k-2} + \ldots + \delta i + \epsilon$ is denoted by the symbol $(\alpha\beta\gamma \ldots \delta\epsilon)$. This symbol is the usual symbol for a positive integer in the notation of the number system whose base is $p$. In the first table the marks are arranged according to ascending powers of $i$. In the second table the marks are arranged so that the symbols $(\alpha\beta\gamma \ldots \delta\epsilon)$ represent the positive integers in their natural order. These two tables make it possible to perform with ease the operations of addition, subtraction, multiplication, and division within the field $GF(p^n)$. For $p^n \leqslant 169$, these tables have been computed by the writer and have been deposited in the mathematical library of the University of Chicago.

## THE GROUP $G$ SIMPLY ISOMORPHIC WITH $LF[2, 2^n]$.

### 7. *Proof of Theorem IV.*

Relations (5) and (6) are found to be satisfied when $a$, $b$, $c$ are identified with the transformations $z' = iz$, $z' = z+1$, $z' = 1/z$ of the group $LF[2, 2^n]$. Therefore the abstract group $G'$ defined by (5) and (6) must be either the group $G_{2^n(2^{2n}-1)}$ or a larger group. That it cannot be a larger group is seen as follows.

Let $H'$ be the sub-group of $G'$ that is defined by (5). Every element $g'$ of the group $G'$ can be reduced by means of (5) and (6) to one of the two type forms $h_1'$, $h_1' c h_2'$, where $h_1'$ and $h_2'$ are elements of the group $H'$. Every element $h'$ of $H'$ can be reduced by means of (5) to one of the two type forms $a^\lambda$, $a^\lambda b a^\mu$. Therefore every element $g'$ of $G'$ can be reduced by means of (5) and (6) to one of the six type forms

$$a^\lambda, \quad a^\lambda b a^\mu, \quad a^\lambda c a^\mu, \quad a^\lambda c a^\mu b a^\nu, \quad a^\lambda b a^\mu c a^\nu, \quad a^\lambda b a^\mu c a^\nu b a^\sigma.$$

The last four of these can be further reduced by means of (5) and (6), so that the six type forms are

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* *Proc. London Math. Soc.*, Vol. xxxv., pp. 292–305.

$$a^\lambda, \quad a^\lambda b a^\mu, \quad a^\lambda c, \quad a^\lambda c b a^\mu, \quad a^\lambda b c a^\mu, \quad a^\lambda b c a^\mu b a^\nu.$$

The maximum numbers of distinct elements of the six types are respectively $(2^n-1)$, $(2^n-1)^2$, $(2^n-1)$, $(2^n-1)^2$, $(2^n-1)^2$, $(2^n-1)^3$. Therefore the maximum number of distinct elements $g'$ is

$$2(2^n-1)+3(2^n-1)^2+(2^n-1)^3 = 2^n(2^{2n}-1),$$

which is precisely the order of the group $G_{2^n(2^{2n}-1)}$.

8. The relations $ba^\xi b = a^\eta b a^\zeta$ are highly redundant. If they be denoted symbolically by $(\xi, \eta, \zeta)$, it appears that the relations $(-\xi, -\zeta, -\eta)$, $(\eta, \xi, -\zeta)$, $(-\eta, \zeta, -\xi)$, $(\zeta, -\eta, \xi)$, $(-\zeta, -\xi, \eta)$, and $(2^k\xi, 2^k\eta, 2^k\zeta)$ follow from $(\xi, \eta, \zeta)$, and the two relations $a^{2^n-1} = I$, $b^2 = I$. In the special case in which $x = 3$, de Seguier has reduced the system (C) to the system (D) of Theorem V. The attempt to do this in general has resulted in Theorem V.

### 9. *Proof of Theorem V.*

The proof consists in expressing relations (5) and (6) in terms of $d = cb$, and in simplifying the set of relations thus obtained. Since $(ca)^2 = I$, and consequently $ca^\zeta c = a^{-\zeta}$, the relation $(-\xi, -\zeta, -\eta)$ may be written $ba^{-\xi} b = ca^\zeta c b a^{-\eta}$, whence

$$b = a^\xi d^{-1} a^\zeta d a^{-\eta}, \tag{31}$$

$$c = d a^\xi d^{-1} a^\zeta d a^{-\eta}. \tag{32}$$

The relation $b^2 = I$, expressed in terms of $a$ and $d$, is, since $\xi - \eta \equiv \zeta$ (mod $2^n-1$),

$$(d^{-1} a^\zeta d a^\zeta)^2 = I. \tag{33}$$

The relation $(ca)^2 = I$ may be replaced by the relation $(ca^\eta)^2 = I$, since the latter includes the former and follows from it. Expressed in terms of $a$ and $d$, this relation is

$$(d^{-1} a^\xi d^{-1} a^\zeta)^2 = I. \tag{34}$$

The relation $c^2 = I$, expressed in terms of $a$ and $d$, is

$$(a^{-\xi} d a^{\xi-\zeta} d^{-1} a^{-\zeta} d)^2 = I$$

or

$$[(a^{-\xi} d a^{\xi-\zeta})\, d^{-1} a^{-\zeta} d a^{-\xi} d a^{\xi-\zeta} (a^{-\xi} d a^{\xi-\zeta})]^2 = I,$$

which is equivalent to $(d^{-1} a^{-\zeta} d a^{-\xi} d a^{\xi-\zeta})^2 = I$.

If $da^{-\xi}da^{-\zeta}$ be replaced by its inverse, as may be done by (34), this becomes

$$(da^{\xi}d^{-1}a^{\xi})^2 = I, \tag{35}$$

which is equivalent to (33).

The relation $ba^{\xi}b = a^{\eta}ba^{\zeta}$, expressed in terms of $a$ and $d$, is

$$(d^{-1}a^{\zeta}da^{\zeta})\,a^{\xi}d^{-1}a^{\zeta}(da^{-\zeta}d^{-1}a^{-\zeta})\,da^{-\eta} = I. \tag{36}$$

Replace each of the expressions in parentheses by its inverse, as may be done by (33), transform by $a^{\zeta}$, and (36) becomes

$$\begin{aligned} & d\,(da^{-\zeta}da^{-\xi})\,(a^{2\xi}d^{-1}a^{2\zeta}d^{-1})\,d^{-1}a^{-\eta} \\ &= d\,(a^{\xi}d^{-1}a^{\zeta}d^{-1})\,(da^{-2\zeta}da^{-2\xi})\,d^{-1}a^{-\eta} \\ &= da^{\xi}d^{-1}a^{-\zeta}da^{-\xi}\,(a^{-\xi}d^{-1}a^{-\eta}d^{-1})\,d \\ &= da^{\xi}d^{-1}\,(a^{-\zeta}da^{-\xi}d)\,a^{\eta}da^{\xi}d \\ &= d\,(a^{\xi}da^{\xi}d^{-1})\,a^{\zeta}a^{\eta}da^{\xi}d \\ &= d^{-1}a^{-\xi}d^{-1}a^{-\xi+\zeta+\eta}da^{\xi}d \\ &= d^{-1}a^{-\xi}d^{-1}da^{\xi}d \qquad\qquad (\text{since } \zeta+\eta-\xi = 0) \\ &= d^3 = I. \end{aligned}$$

The relation $(cb)^3 = I$, expressed in terms of $d$, is $d^3 = I$. Relations (5) and (6) have now been expressed in terms of $a$ and $d$, and the resulting set of relations has been proved to follow from (5) and (6).

10. In proving Theorem VI. for $n = 2, 3, 4$, Prof. Dickson's point of departure was Prof. Moore's set of relations for $G_{2^n(2^{2n}-1)}$ (see note to Theorem II. in the Introduction). He defined $T$ and $S_\lambda$ in terms of $A$ and $B$ by means of certain relations which express $T$ and $S_\lambda$ in terms of $A$ and $B$ when $T$, $S_\lambda$, $A$, $B$ are identified with certain concrete transformations of the group $LF[2, 2^n]$. He then proved that $T$ and $S_\lambda$, thus defined, satisfy Moore's relations in view of relations (E).

The same method of proof was applied successfully by the writer to the case in which $n = 5$, but the computation was so excessive as to render it unadvisable to try it for higher cases. The proof for the case $n = 5$ was much simplified by using the same method with relations (C) as the point of departure. Finally, a simpler proof was made for the cases in which $n = 2, 3, 4, 5, 6$, by applying the same method to the set of relations (D).

11. To obtain relations with which to define $a$ and $d$ of the set (D) in terms of $A$ and $B$, the generators $A$, $B$, $d$, $a$ were identified with the transformations $S_iT$, $S_{1+i}$, $TS_1$, $z' = iz$, where $T$ and $S_\lambda$ denote the transformations $z' = 1/z$ and $z' = z+\lambda$. Among these concrete transformations exist the relations

$$d = A^{-1}B, \qquad a = S_{i^{2^{n-1}}-1}TS_{i^{2^{n-1}}}TS_{i^{2^{n-1}}-1}T.$$

For any particular value of $n$, $T$ and $S_\lambda$ can be expressed in terms of $A$ and $B$, as is indicated by Prof. Dickson,* and therefore $a$ can be expressed in terms of $A$ and $B$. This expression can be reduced by means of relations ($E$), which are satisfied by the concrete transformations $A$ and $B$.

The result obtained in the special cases in which $n = 2, 3, 4, 5, 6$ is

$$a = A^{-1}BA^{2^{n-1}}B = (BA)\,BA^{2^{n-1}+2}(BA)^{-1}. \tag{41}$$

## 12. *Proof of Theorem VI. for $n = 3$.*

The $GF[2^3]$ is defined by the primitive irreducible congruence $i^3 \equiv i+1 \pmod 2$. The pairs of values $(r, s)$ of (E) are (1, 2), (2, 1), (3, 5), (4, 6), (5, 3), (6, 4), (7, 8), (8, 7). The set (D) reduces to

$$\text{(38)}\quad a^7 = I,\ d^3 = I,\ (d^{-1}ada)^2 = I,\ (d^{-1}a^3da^3) = I,\ (d^{-1}ad^{-1}a^3)^2 = I.$$

Define $d = A^{-1}B$ and $a = (BA)BA^6(BA)^{-1}$ and substitute in (38). The relation $a^7 = I$ becomes

$$\begin{aligned}
&BA^6BA^6\,(BA^6BA^4)\,A^2BA^6\,(BA^6BA^4)\,A^2\\
&= BA^6BA^6\,(A^5BA^3B)\,A^2BA^6\,(A^5BA^3B)\,A^2\\
&= BA^6BA^2BA^3\,(BA^3BA)\,ABA^3BA^2\\
&= BA^6BA^2BA^3A^{-1}BA^{-2}\,(BAB)\,A^3BA^2\\
&= BA^6\,(BA^2BA)\,ABA^{-2}\,(A^{-1}BA^{-1})\,A^3BA^2\\
&= BA^6\,(A^{-1}BA^{-2}B)\,ABA^{-3}(BA^2BA)\,A\\
&= BA^5BA^7(A^{-1}BA^{-1})\,A^{-3}(A^{-1}BA^{-2}B)A\\
&= BA^5\,(BA^6BA^4)BA^7BA\\
&= BA^5\,(A^{-4}BA^{-6}B)BA^7BA\\
&= (BA)^3 = I,
\end{aligned}$$

which is one of relations (E).

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* *Proc. London Math. Soc.*, Vol. xxxv., pp. 306 and 443.

The relation $d^3 = I$ becomes $(A^{-1}B)^3 = I$, which is the inverse of one of relations (E).

The relation $(d^{-1}a\,da)^2 = I$ becomes $(A^9B)^2 = I$, which follows from $A^9 = I$ and $B^2 = I$.

The relation $(d^{-1}a^3\,da^3)^2 = I$ becomes

$$\begin{aligned}
&[A\,(A^4BA^6B)\,A^2(BA^6BA^4)\,AB]^2\\
&= [A\,(BA^3BA^5)\,A^2(A^5BA^3B)\,AB]^2\\
&= [ABA^3BA^3BA^3(BAB)]^2\\
&= [ABA^3BA^3BA^2BA^{-1}]^2\\
&= ABA^3(BA^3BA^5)\,BA^3BA^2BA^{-1}\\
&= ABA^3(A^4BA^6B)\,BA^3BA^2BA^{-1}\\
&= ABA^7BA^9BA^2BA^{-1} = I,
\end{aligned}$$

which follows from relations (E).

The relation $(d^{-1}a\,d^{-1}a^3)^2 = I$ becomes

$$\begin{aligned}
&[A^5\,(BAB)\,A^6BA^6BA^5B]^2\\
&= [A^4BA^5(BA^6BA^4)\,AB]^2\\
&= [A^4BA^5(A^5BA^3B)\,AB]^2\\
&= [A^4(BAB)\,A^3(BAB)]^2\\
&= [A^4(A^{-1}BA^{-1})\,A^3(A^{-1}BA^{-1})]^2\\
&= [A^3(BAB)\,A^{-1}]^2\\
&= [A^2BA^{-2}]^2 = I,
\end{aligned}$$

which follows from $B^2 = I$.

This completes the proof of Theorem VI. for $n = 3$.

## 13. *Proof of Theorem VI. for* $n = 4$.

The $GF(2^4)$ is defined by the primitive irreducible congruence $i^4 \equiv i+1 \pmod 2$. The pairs of values $(r, s)$ of (E) are (1, 2), (2, 1), (3, 7), (4, 12), (5, 13), (6, 9), (7, 3), (8, 11), (9, 6), (10, 14), (11, 8), (12, 4), (13, 5), (14, 10), (15, 16), (16, 15).

The set of relations (D) reduces to

$$(39)\qquad \begin{cases} a^{15} = I, \quad d^3 = I, \quad (d^{-1}a^{\xi}\,da^{\xi})^2 = I, \quad \xi = 1,\ 4,\ 5,\ 10,\\ (d^{-1}a\,d^{-1}a^4)^2 = I, \quad (d^{-1}a^5\,d^{-1}a^{10})^2 = I. \end{cases}$$

Define $d = A^{-1}B$, $a = (BA)\,BA^{10}(BA)^{-1}$, and substitute in (39). Expressions for various powers of $BA^{10}$ reduced by means of (E) are as follows:—

$(BA^{10})^5 = A^{12}BA^{14}BA^{13}BA^{12}$, $(BA^{10})^{10} = A^{14}BA^{13}BA^{14}B$, $(BA^{10})^{15} = A^{34}$.

The relation $a^{15} = I$ becomes $A^{34} = I$, which follows from (E).

The relation $(d^{-1}a\,da)^2 = I$ becomes $(A^{17}B)^2 = I$, which follows from $A^{17} = I$ and $B^2 = I$.

The relation $(d^{-1}a^4\,da^4)^2 = I$ becomes

$$[A^{-1}B(BA^{10})^4A^{-1}B(BA^{10})^4A^{-1}B]^2 = I,$$

which reduces, by means of (E), to $(BA^9BA^6)^2 = I$.

The relation $(d^{-1}a^5\,da^5)^2 = I$ becomes

$$[A^{-1}B(BA^{10})^5A^{-1}B(BA^{10})^5A^{-1}B]^2 = I,$$

which reduces, by means of (E), to $(BA^7BA^8)^2 = I$.

The relation $(d^{-1}a^{10}da^{10})^2 = I$ becomes

$$[A^{-1}B(BA^{10})^{10}A^{-1}B(BA^{10})^{10}A^{-1}B]^2 = I,$$

which reduces, by means of (E), to $(BA^6BA^9)^2 = I$.

The relation $(d^{-1}a\,d^{-1}a^4)^2 = I$ becomes

$$[A^8BA^9BA^{10}BA^{10}BA^9B]^2 = I,$$

which reduces, by means of (E), to $A^{17} = I$.

The relation $(d^{-1}a^5d^{-1}a^{10})^2 = I$ becomes

$$[A^{-1}BA^{12}BA^{14}BA^{13}BA^{12}BA^{15}BA^{13}BA^{14}BA^{-1}B]^2 = I,$$

which reduces, by means of (E), to $(BA^5BA^{13})^2 = I$.

This completes the proof of Theorem VI. for $n = 4$.

## 14. *Proof of Theorem VI. for* $n = 5$.

The $GF[2^5]$ is defined by the primitive irreducible congruence $i^5 \equiv i^3+i^2+i+1 \pmod 2$. The pairs of values $(r, s)$ of (E) are (1, 2), (2, 1), (3, 25), (4, 10), (5, 17), (6, 15), (7, 11), (8, 30), (9, 14), (10, 4), (11, 7), (12, 20), (13, 21), (14, 9), (15, 6), (16, 28), (17, 5), (18, 27), (19, 24), (20, 12), (21, 13), (22, 26), (23, 29), (24, 19), (25, 3), (26, 22), (27, 18), (28, 16), (29, 23), (30, 8), (31, 32), (32, 31).

The set of relations (D) reduces to

$$(40)\qquad \begin{cases} a^{31} = I, \quad d^3 = I, \quad (d^{-1}a^{\xi}\,da^{\xi})^2 = I, \quad \xi = 1, 3, 8, 12, \\ (d^{-1}a\,d^{-1}a^{12})^2 = I, \quad (d^{-1}a^8\,d^{-1}a^8)^2 = I. \end{cases}$$

Define $d = A^{-1}B$, $a = (BA)BA^{18}(BA)^{-1}$, and substitute in (40). The relation $a^{31} = I$ becomes $(BA^{18})^{31} = I$. Before proceeding further it will be necessary to simplify the expressions for various powers of $(BA^{18})$ by means of (E).

$$\begin{aligned}
(BA^{18})^7 &= BA^{18}BA^{18}BA^{18}(BA^{18}BA^{27})A^{24}BA^{18}BA^{18}\\
&= BA^{18}BA^{-1}(A^{19}BA^{24}B)A^{15}(BA^{24}BA^{19})A^{-1}BA^{18}\\
&= BA^{18}(ABA)A^{9}BA^{14}A^{15}A^{14}BA^{9}(ABA)A^{18}\\
&= BA^{19}BA^{10}BA^{10}BA^{10}BA^{19}.
\end{aligned}$$

$$\begin{aligned}
(BA^{18})^{15} &= (BA^{18})^4(BA^{18})^7(BA^{18})^4\\
&= (BA^{18})^2(BA^{18}BA^{27})(A^{24}BA^{19}B)A^{10}BA^{10}BA^{10}(BA^{19}BA^{24})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (A^{27}BA^{18}B)A^{18}BA^{18}\\
&= BA^{18}BA\cdot(A^{28}BA^{29}B)A^{19}BA^{10}BA^{19}(BA^{29}BA^{28})ABA^{18}\\
&= BA^{18}(BAB)A^{4}BA^{29}BA^{10}BA^{29}BA^{4}(BAB)A^{18}\\
&= BA^{17}(BA^{8}BA^{25})A^{4}(BA^{10}BA^{4})(A^{25}BA^{8}B)A^{17}\\
&= BA^{25}BA^{17}BA^{25}
\end{aligned}$$

$$\begin{aligned}
(BA^{18})^{31} &= (BA^{18})^{15}(BA^{18})^{15}(BA^{18})\\
&= BA^{25}BA^{17}BA^{25}(BA^{25}BA^{8})A^{14}BA^{25}BA^{18}\\
&= BA^{25}BA^{17}BA^{22}BA^{-1}(A^{9}BA^{14}B)A^{25}BA^{18}\\
&= BA^{25}BA^{17}BA^{28}BA^{20}(BA^{16}BA^{28})A^{28}\\
&= BA^{25}BA^{17}BA^{28}(BA^{25}BA^{8})A^{14}BA^{28}\\
&= BA^{25}BA^{17}BA^{20}BA^{-1}(A^{9}BA^{14}B)A^{28}\\
&= BA^{25}BA^{17}BA^{21}(BA^{20}BA^{18})A^{2}\\
&= BA^{25}BA^{17}(BA^{9}BA^{14})A^{-1}BA^{2}\\
&= (BA^{25}BA^{8})^2.
\end{aligned}$$

The relation $(BA^{18})^{31} = I$ becomes $(BA^{25}BA^{8})^2 = I$, which is one of relations (E).

The relation $(d^{-1}a\,da)^2 = I$ becomes $(A^{33}B^{2})^2 = I$, which follows from $A^{33} = I$ and $B^2 = I$.

The relation $(d^{-1}a^{8}da^{8})^2 = I$ becomes

$$[A^{17}BA^{18}BA^{2}BA^{18}BA^{17}B]^2 = I$$

or

$$\begin{aligned}
& A^{17}BA^{18}BA^{2}BA^{18}(BA^{17}BA^{5})A^{12}BA^{18}BA^{2}(BA^{18}BA^{27})A^{28}B \\
&= A^{17}BA^{18}BA^{2}(BA^{18}BA^{21})A^{28}BA^{12}(BA^{18}BA^{27})A^{14}BA^{15}BA^{28}B \\
&= A^{17}BA^{18}BA^{14}BA^{20}BA(A^{27}BA^{18}B)A^{15}BA^{14}BA^{15}BA^{28}B \\
&= A^{17}BA^{18}BA^{14}BA^{19}BA^{14}(BA^{21}BA^{13})ABA^{15}BA^{28}B \\
&= A^{17}BA^{18}BA^{14}BA^{19}(BAB)A^{12}(BAB)A^{15}BA^{28}B \\
&= A^{17}BA^{18}BA^{14}BA^{18}BA(A^{9}BA^{14}B)A^{28}B \\
&= A^{17}BA^{18}BA^{14}(BA^{17}BA^{5})A^{18}BA^{14}B \\
&= A^{17}BA^{4}(A^{14}BA^{9}B)A^{16}BA^{18}BA^{14}B \\
&= B(BA^{17}BA^{5})A^{-1}BA^{24}BA^{2}BA^{18}BA^{14}B \\
&= BA^{-5}BA^{17}(BA^{25}BA^{8})A^{-1}BA^{18}BA^{14}B \\
&= BA^{-5}(BA^{14}BA^{9})^{2}A^{5}B = I,
\end{aligned}$$

which follows from relations (E).

The relation $(d^{-1}a^{8}da^{8})^{2} = I$ becomes

$$[A^{17}(BA^{18})^{7}A^{17}(BA^{19})^{7}A^{-1}B]^{2} = I$$

or

$$\begin{aligned}
& [A^{17}(BA^{19}BA^{24})A^{19}BA^{10}(BA^{10}BA^{4})A^{-1}BA^{19}(BA^{10}BA^{4})A^{6}BA^{10}BA^{18}B]^{2} \\
&= [A^{26}BA^{14}BA^{19}BA^{6}(BA^{24}BA^{19})A^{30}(BA^{28}BA^{29})A^{10}BA^{10}BA^{18}B]^{2} \\
&= [A^{28}BA^{14}(BA^{19}BA^{24})A^{29}BA^{9}(BAB)A^{10}BA^{10}BA^{10}BA^{18}B]^{2} \\
&= [A^{26}BA^{28}BA^{14}BA^{-1}(A^{30}BA^{8}B)A^{9}BA^{10}BA^{10}BA^{18}B]^{2} \\
&= [A^{26}BA^{28}(BA^{15}BA^{6})(A^{20}BA^{12}B)A^{10}BA^{10}BA^{18}B]^{2} \\
&= [A^{26}(BA^{17}BA^{5})ABA^{28}BA^{10}BA^{18}B]^{2} \\
&= [A^{21}BA^{15}BA^{22}BA^{10}BA^{18}B]^{2} \\
&= A^{21}BA^{15}BA^{22}(BA^{10}BA^{4})A(A^{18}BA^{21}B)A^{15}BA^{22}(BA^{10}BA^{4})A^{14}B \\
&= A^{21}BA^{15}BA^{18}BA^{22}(BA^{11}BA^{7})A(A^{27}BA^{18}B)A^{28}BA^{14}B \\
&= A^{21}BA^{15}BA^{18}BA^{15}(BA^{21}BA^{18})ABA^{29}BA^{14}B \\
&= A^{21}BA^{15}BA^{18}BA^{2}(BA^{11}BA^{7})A^{21}BA^{14}B \\
&= A^{21}BA^{15}(BA^{18}BA^{27})ABA^{22}(BA^{21}BA^{13})AB \\
&= A^{21}BA^{21}BA^{15}(BAB)A^{8}BA^{12}BAB \\
&= A^{21}(BA^{21}BA^{13})ABA^{8}BA^{12}BAB \\
&= A^{8}BA^{12}(BAB)A^{8}BA^{12}(BAB) \\
&= A(A^{7}BA^{11}B)^{2}A^{-1} = I,
\end{aligned}$$

which follows from relations (E).

The relation $(d^{-1}a^{12}da^{12})^2 = I$ becomes

$$[A^{-1}B(BA^{18})^{12}A^{-1}B(BA^{18})^{12}A^{-1}B]^2 = I$$

or

$$\begin{aligned}
&[A^{-1}B\,\{BA^{25}BA^{17}BA(A^6BA^{15}B)A^{16}BA\}^2]^2 \\
&= [A^{-1}B(BA^{25}BA^{16}BA^{17}BA^{10}BA)^2]^2 \\
&= [A^{24}(BA^{16}BA^{28})A^{22}BA^9BA^{24}BA^{16}BA^{17}BA^{10}BA]^2 \\
&= [A^{29}BA^{17}(BA^{22}BA^{26})A^{16}BA^{24}BA^{16}BA^{17}BA^{10}BA]^2 \\
&= [A^{29}BA^{24}BA^{11}(BA^{16}BA^{28})A(A^{28}BA^{16}B)A^{17}BA^{10}BA]^2 \\
&= [A^{29}BA^{24}B(A^{16}BA^{16}BA^{28})A^{21}BA^{22}BA^{10}BA]^2 \\
&= [A^{29}BA^{24}BA^{21}BA^{17}(BA^{21}BA^{18})A^9BA^{10}BA]^2 \\
&= [A^{29}BA^{24}BA^{21}BA^4BA^{12}(BA^9BA^{14})A^{-4}BA]^2 \\
&= [A^{29}BA^{24}BA^{21}BA^4BA^{31}BA(A^{28}BA^{29}B)A]^2 \\
&= [A^{29}BA^{11}(A^{18}BA^{21}B)A^4BA^{30}BA^8BA^{11}]^2 \\
&= [A^{29}(BA^{11}BA^7)A^5BA^{24}BA^{30}BA^8BA^{11}]^2 \\
&= [A^{22}BA^{22}(BA^5BA^{17})A^{-1}(A^8BA^{30}B)A^8BA^{11}]^2 \\
&= [A^{22}BA^5BA^{29}BA^4BA^{28}BA^{11}]^2 = I,
\end{aligned}$$

which follows from relations (E).

The relation $(d^{-1}ad^{-1}a^{12})^2 = I$ becomes

$$[A^{17}BA(BA^{18})^{12}A^{-1}B]^2 = I$$

or

$$\begin{aligned}
&[A^{17}(BAB)A^{25}BA^{17}BA(A^6BA^{15}B)A^{16}BA]^2 \\
&= [A^{16}BA^{24}BA^{16}(BA^{17}B^5A)A^6BA]^2 \\
&= [A^{16}BA^{24}BA^{11}BA^{-1}(A^{17}BA^5B)A]^2 \\
&= [A^{16}BA^{24}BA^{12}BA^{29}BA^{17}]^2 \\
&= (A^{16}BA^{24})(BA^{12}BA^{20})^2(A^{16}BA^{24})^{-1} = I,
\end{aligned}$$

which follows from relations (E).

The relation $(d^{-1}a^8d^{-1}a^8)^2 = I$ becomes

$$[A^{-1}B(BA^{18})^8BA(BA^{18})^8A^{-1}B]^2 = I$$

$$
\begin{aligned}
&[A^{17}BA^{18}BA^{17}(BA^{18}BA^{27})A^{16}BA^{10}BA^{10}(BA^{19}BA^{24})A^{26}B]^2\\
&= [A^{17}BA^{18}BA^{23}BA^{15}(BA^{16}BA^{28})A^{15}BA^{19}(BA^{14}BA^{9})A^{17}B]^2\\
&= [A^{17}BA^{18}BA^{23}(BA^{20}BA^{12})A^{-1}(A^{6}BA^{15}B)A^{10}BA^{19}BA^{17}B]^2\\
&= [A^{17}BA^{18}BA^{11}(BA^{14}BA^{9})(A^{10}BA^{4}B)A^{19}BA^{17}B]^2\\
&= [A^{17}BA^{18}BA^{2}BA(A^{14}BA^{9}B)A^{17}B]^2\\
&= [A^{17}BA^{18}(BAB)A^{23}BA^{8}B]^2\\
&= A^{17}BA^{17}BA^{22}BA^{8}(BA^{17}BA^{5})A^{12}BA^{22}BA^{8}B\\
&= A^{17}BA^{17}(BA^{22}BA^{26})A^{6}BA^{16}BA^{12}BA^{22}BA^{8}B\\
&= A^{17}BA^{24}BA^{11}(BA^{5}BA^{17})A^{-1}BA^{12}BA^{22}BA^{8}B\\
&= A^{17}BA^{24}BA^{27}BA^{29}(BA^{18}BA^{21})ABA^{8}B\\
&= A^{17}BA^{24}BA^{27}BA^{8}(BA^{19}BA^{24})A^{11}B\\
&= A^{17}BA^{24}(BA^{27}BA^{18})A^{-1}BA^{14}BA^{11}B\\
&= A^{17}BA^{6}BA(A^{6}BA^{15}B)A^{11}B\\
&= [A^{17}BA^{5}B]^2 = I,
\end{aligned}
$$

ch follows from relations (E). This completes the proof of orem VI. for $n = 5$.

### 15. *Proof of Theorem VI. for $n = 6$.*

The $GF(2^6)$ is defined by the primitive irreducible congruence $\equiv i+1 \pmod 2$. The pairs of values $(r, s)$ of (E) are (1, 2), (2, 1), 50), (4, 17), (5, 30), (6, 10), (7, 43), (8, 46), (9, 16), (10, 6), (11, 25), , 21), (13, 28), (14, 20), (15, 62), (16, 9), (17, 4), (18, 42), (19, 57), , 14), (21, 12), (22, 58), (23, 47), (24, 32), (25, 11), (26, 38), (27, 39), , 13), (29, 34), (30, 5), (31, 36), (32, 24), (33, 41), (34, 29), (35, 60), , 31), (37, 52), (38, 26), (39, 27), (40, 54), (41, 33), (42, 18), (43, 7), , 53), (45, 51), (46, 8), (47, 23), (48, 61), (49, 56), (50, 3), (51, 45), , 37), (53, 44), (54, 40), (55, 59), (56, 49), (57, 19), (58, 22), (59, 55), , 35), (61, 48), (62, 15), (63, 64), (64, 63).

The set of relations (D) reduces to

$$
) \left\{ \begin{array}{l} a^{63} = I, \quad d^3 = I, \quad (d^{-1}a^{\xi}da^{\xi})^2 = I, \quad \xi = 1, 6, 7, 9, 21, 26, 42, 45,\\ (d^{-1}a^{\xi}d^{-1}a^{\zeta})^2 = I, \text{ the pairs } (\xi, \zeta) \text{ being } (1, 6), (7, 26), (9, 45), (21, 42). \end{array} \right.
$$

Define $d = A^{-1}B$, $a = (BA)BA^{34}(BA)^{-1}$, and substitute in (41). The expressions for various powers of $(BA^{34})$ reduced by means of (E) are

$(BA^{34})^3 = A^{35}BA^{59}BA^4$, $(BA^{34})^9 = A^{20}BA^2BA^{35}BA^{58}BA^{26}BA^{45}$,

$(BA^{34})^{18} = A^{42}BA^{25}BA^{50}BA^9$, $(BA^{34})^{21} = A^{31}BA^{41}BA^{12}BA^{31}BA^{14}$,

$(BA^{34})^{26} = A^{53}BA^{58}BA^{14}BA^{39}BA^{17}$, $(BA^{34})^{42} = A^{63}BA^{60}BA^{46}BA^{19}BA^{22}BA^{24}$,

$(BA^{34})^{45} = A^{63}BA^{52}BA^{42}BA^{40}BA^4$, $(BA^{34})^{63} = A^2(BA^{28}BA^{47})^2A^{-2}$.

The relation $a^{63} = I$ becomes

$$(BA)A^2(BA^{28}BA^{47})^2A^{-2}(BA)^{-1} = I,$$

which follows from relations (E).

The relation $(d^{-1}ada)^2 = I$ becomes

$$(A^{65}B)^2 = I,$$

which follows from $A^{65} = I$ and $B^2 = I$.

The relation $(d^{-1}a^6da^6)^2 = I$ becomes

$$[A^{-1}B(A^{35}BA^{59}BA^{39}BA^{59}BA^8B)^2]^2 = I,$$

which reduces, by means of (E), to $(BA^{19}BA^{57})^2 = I$.

The relation $(d^{-1}a^7da^7)^2 = I$ becomes

$$[A^8(BA^{59}BA^{39}BA^{59}BA^7)^2A^{61}B]^2 = I,$$

which reduces, by means of (E), to $(BA^{52}BA^{57})^2 = I$.

The relation $(d^{-1}a^9da^9)^2 = I$ becomes

$$[A^{-1}B(A^{20}BA^2BA^{35}BA^{58}BA^{26}BA^{44}B)^2]^2 = I,$$

which reduces, by means of (E), to $(BA^{13}BA^{28})^2 = I$.

The relation $(d^{-1}a^{21}da^{21})^2 = I$ becomes

$$[A^{-1}B(A^{31}BA^{41}BA^{12}BA^{31}BA^{13}B)^2]^2 = I,$$

which reduces, by means of (E), to $(BA^{18}BA^{42})^2 = I$.

The relation $(d^{-1}a^{26}da^{26})^2 = I$ becomes

$$[A^{-1}B(A^{53}BA^{58}BA^{14}BA^{39}BA^{16}B)^2]^2 = I,$$

which reduces, by means of (E), to $(BA^4BA^{17})^2 = I$.

The relation $(d^{-1}a^{42}da^{42})^2 = I$ becomes

$$[A^{-1}B(A^{63}BA^{60}BA^{46}BA^{19}BA^{22}BA^{23}B)^2]^2 = I,$$

which reduces, by means of (E), to $(BA^{22}BA^{58})^2 = I$.

The relation $(d^{-1}a^{45}da^{45})^2 = I$ becomes

$$[A^{-1}B(A^{63}BA^{52}BA^{42}BA^{40}BA^{8}B)^2]^2 = I,$$

which reduces, by means of (E), to $(BA^{8}BA^{50})^2 = I$.

The relation $(d^{-1}a\,d^{-1}a^{6})^2 = I$ becomes

$$[A^{33}BA^{36}BA^{59}BA^{30}BA^{59}BA^{8}B]^2 = I,$$

which reduces, by means of (E), to $(BA^{64}BA^{63})^2 = I$.

The relation $(d^{-1}a^{7}d^{-1}a^{28})^2 = I$ becomes

$$[A^{8}BA^{59}BA^{30}BA^{59}BA^{4}BA^{54}BA^{58}BA^{14}BA^{30}BA^{16}B]^2 = I,$$

which reduces, by means of (E), to $(BA^{24}BA^{32})^2 = I$.

The relation $(d^{-1}a^{9}d^{-1}a^{45})^2 = I$ becomes

$$[A^{-1}BA^{20}BA^{2}BA^{35}BA^{58}BA^{26}BA^{45}BA^{-1}BA^{52}BA^{42}BA^{40}BA^{8}B]^2 = I,$$

which reduces, by means of (E), to $(BA^{17}BA^{4})^2 = I$.

The relation $(d^{-1}a^{21}d^{-1}a^{42})^2 = I$ becomes

$$[A^{-1}BA^{31}BA^{41}BA^{12}BA^{31}BA^{14}BA^{-1}BA^{60}BA^{46}BA^{19}BA^{22}BA^{23}B]^2 = I,$$

which reduces, by means of (E), to $(BA^{32}BA^{24})^2 = I$. This completes the proof of Theorem VI. for $n = 6$.

# ON THE REDUCIBILITY OF COVARIANTS OF BINARY QUANTICS OF INFINITE ORDER

## PART II.

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[Received January 22nd, 1905.—Read February 9th, 1905.]

### *Introduction* (§§ 1, 2).

1. The first part of this paper contained a preliminary investigation of the reducibility of covariants of unit degree in the coefficients of each of any number of binary quantics of infinite order, a reducible covariant being defined as a covariant which can be expressed as a sum of products of covariants of lower total degrees. It was there shown that the necessary and sufficient condition, that the covariant

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$$

of total degree $(\delta+1)$ is reducible, is that the expression $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_\delta^{\lambda_\delta}$ should be annihilated by the successive application of the $(2^\delta - 1)$ operators like

$$\frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \dots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}},$$

$r^{(1)}, r^{(2)}, \dots, r^{(\epsilon)}$ being any $\epsilon$ of the suffixes $1, 2, 3, \dots, \delta$ such that $r^{(1)} < r^{(2)} < \dots < r^{(\epsilon)}$. In general, therefore, the determination of the reducibility of any covariant of degree $(\delta+1)$ requires the evaluation of the product of operators

$$P_\delta \equiv \prod_{(2^\delta-1)} \left\{ \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \dots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}} \right\};$$

for the covariant $\qquad (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$

is clearly reducible if, and only if, $P_\delta$ contains *no* term

$$\left(\frac{\partial}{\partial x_1}\right)^{\mu_1} \left(\frac{\partial}{\partial x_2}\right)^{\mu_2} \dots \left(\frac{\partial}{\partial x_\delta}\right)^{\mu_\delta}$$

such that $\qquad \lambda_1 \geqslant \mu_1, \; \lambda_2 \geqslant \mu_2, \; \dots, \; \lambda_\delta \geqslant \mu_\delta.$

It is important for our purpose to determine what terms in the product $P_\delta$ have zero coefficients.

2. Two general classes of reducible covariants were determined in the first part of this paper; it was, in fact, there proved that the covariant

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$$

is reducible if

(1) The covariant

$$(b_1 b_2)^{\lambda_{r^{(1)}}} (b_2 b_3)^{\lambda_{r^{(2)}}} \dots (b_\epsilon b_{\epsilon+1})^{\lambda_{r^{(\epsilon)}}}$$

is reducible; $r^{(1)}, r^{(2)}, \dots, r^{(\epsilon)}$ being any $\epsilon$ of the suffixes $1, 2, 3, \dots, \delta$ such that $r^{(1)} < r^{(2)} < \dots < r^{(\epsilon)}$ (included among such covariants are those for which, for any value of $\kappa$, the sum of $\kappa$ of the indices $\lambda_1, \lambda_2, \dots, \lambda_\delta$ is less than $2^\kappa - 1$);

(2) The sum of $\kappa$ of the indices $\lambda_1, \lambda_2, \dots, \lambda_\delta$ is equal to $2^\kappa - 1$, and the sum of the indices is odd in any of the groups into which the $\delta$ indices are divided by the removal of those $\kappa$ indices whose sum is $2^\kappa - 1$.

The present paper does not seek for any other *general* classes of reducible covariants, but gives a complete investigation of the reducibility of covariants

(i.) $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4}$, of degree 5;

(ii.) $\left((a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4},\ a_6\right)$: Jacobians of degree 6.*

### *Covariants of Degree* 5 (§§ 3–5).

3. Putting $z_s = \dfrac{\partial}{\partial x_s}$ $(s = 1, 2, 3, 4)$, we have

$$P_4 \equiv \prod_{15} \left( \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \dots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}} \right)$$

$$\equiv z_1 z_2 z_3 z_4 (z_1 - z_2)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)(z_1 - - z_2 + z_3)$$

$$\times (z_1 - z_2 + z_4)(z_1 - z_3 + z_4)(z_2 - z_3 + z_4)(z_1 - z_2 + z_3 - z_4).$$

The expanded form of this product of 15 linear factors follows immediately.

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* See Note *ad fin.*

*Expansion of*

$$z_1 z_2 z_3 z_4 (z_1-z_2)(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)(z_3-z_4)(z_1-z_2+z_3)$$
$$\times (z_1-z_2+z_4)(z_1-z_3+z_4)(z_2-z_3+z_4)(z_1-z_2+z_3-z_4)$$

*of order* 15 *in the Variables* $z_1, z_2, z_3, z_4$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | | | | | | | **8** | | | | | | | | **7** | | | | | | | | **6** | | | | | | | | **5** | | 8 |
| 7 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 7 |
| 6 | | | | | | | | | | | | | | | | | 2 | $\overline{2}$ | | | | | | | | $\overline{6}$ | 6 | | | | | | 6 |
| 5 | | | | | | | | | | | | | | | | | $\overline{6}$ | | 6 | | | | | | | 18 | | $\overline{18}$ | | | | | 5 |
| 4 | $\overline{1}$ | 1 | | | | | | | | 4 | $\overline{4}$ | | | | | | 8 | 6 | $\overline{14}$ | | | | | | | $\overline{24}$ | $\overline{18}$ | 28 | 14 | | | | 4 |
| 3 | 2 | | $\overline{2}$ | | | | | | | $\overline{8}$ | | 8 | | | | | $\overline{6}$ | $\overline{8}$ | 16 | 8 | $\overline{10}$ | | | | | 18 | 24 | $\overline{20}$ | $\overline{24}$ | 2 | | | 3 |
| 2 | $\overline{1}$ | $\overline{2}$ | 2 | 1 | | | | | | 4 | 8 | $\overline{8}$ | $\overline{4}$ | | | | 2 | 6 | $\overline{8}$ | $\overline{16}$ | 10 | 6 | | | | $\overline{6}$ | $\overline{18}$ | 10 | 20 | $\overline{2}$ | $\overline{4}$ | | 2 |
| 1 | | 1 | | 1 | | | | | | | $\overline{4}$ | | 4 | | | | | $\overline{2}$ | | 8 | | $\overline{6}$ | | | | | 6 | | $\overline{10}$ | | 4 | | 1 |
| 8 | $\overline{1}$ | 1 | | | | | **4** | | | 2 | $\overline{2}$ | | | | **3** | | 1 | $\overline{2}$ | | 1 | | | **2** | | | $\overline{1}$ | 2 | $\overline{1}$ | | | **1** | | 8 |
| 7 | 4 | | $\overline{4}$ | | | | | | | $\overline{8}$ | | 8 | | | | | $\overline{4}$ | 4 | 8 | $\overline{4}$ | $\overline{4}$ | | | | | 4 | $\overline{4}$ | $\overline{4}$ | 4 | | | | 7 |
| 6 | $\overline{8}$ | $\overline{4}$ | 12 | | | | | | | 16 | 8 | $\overline{14}$ | $\overline{10}$ | | | | 6 | 2 | $\overline{24}$ | $\overline{2}$ | 12 | 6 | | | | $\overline{6}$ | $\overline{2}$ | 16 | $\overline{2}$ | $\overline{6}$ | | | 6 |
| 5 | 10 | 8 | $\overline{24}$ | $\overline{8}$ | 14 | | | | | $\overline{20}$ | $\overline{16}$ | 18 | 16 | 2 | | | $\overline{4}$ | $\overline{10}$ | 24 | 22 | $\overline{16}$ | $\overline{12}$ | $\overline{4}$ | | | 4 | 10 | $\overline{14}$ | $\overline{14}$ | 10 | 4 | | 5 |
| 4 | $\overline{8}$ | $\overline{10}$ | 32 | 24 | $\overline{22}$ | $\overline{16}$ | | | | 16 | 20 | $\overline{24}$ | $\overline{18}$ | 2 | 4 | | 1 | 8 | $\overline{8}$ | $\overline{28}$ | 8 | 14 | 4 | 1 | | $\overline{1}$ | $\overline{8}$ | | 18 | | $\overline{8}$ | $\overline{1}$ | 4 |
| 3 | 4 | 8 | $\overline{24}$ | $\overline{32}$ | 16 | 24 | 4 | | | $\overline{8}$ | $\overline{16}$ | 18 | 24 | $\overline{8}$ | $\overline{8}$ | $\overline{2}$ | | $\overline{2}$ | | 14 | | $\overline{12}$ | | | | | 2 | 4 | $\overline{6}$ | $\overline{6}$ | 4 | 2 | 3 |
| 2 | $\overline{1}$ | $\overline{4}$ | 8 | 24 | $\overline{8}$ | $\overline{16}$ | $\overline{4}$ | 1 | | 2 | 8 | $\overline{6}$ | $\overline{18}$ | 4 | 8 | 2 | | | | $\overline{4}$ | | 6 | | $\overline{2}$ | | | | $\overline{1}$ | | 2 | | $\overline{1}$ | 2 |
| 1 | | 1 | | $\overline{8}$ | | 8 | | $\overline{1}$ | | | $\overline{2}$ | | 6 | | $\overline{4}$ | | | | | 1 | | $\overline{2}$ | | 1 | | | | | | | | | 1 |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |

The coefficient of the term $z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4}$ $(\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 15)$ is given thus:—In the rectangle which has the large figure $\lambda_1$ in the right-hand top corner, take the number lying in the column which has the figure $\lambda_2$ at its ends in the horizontal borders, and also lying in the row which has the figure $\lambda_3$ at its ends in the vertical borders: this number is the required coefficient. Negative signs are placed *above* the coefficients which they affect.

*E.g.*, the coefficients of $z_1^5 z_2^4 z_3^4 z_4^2$ and $z_1 z_2^4 z_3^4 z_4^6$ are 28 and 0 respectively. It is obvious that no index can be greater than 8.

[*Note.*—The following tests for the accuracy of the expansion are easily applied:—

(1) Since $P_4$ changes its sign, but is otherwise unaltered, when $z_1$ is changed with $z_4$ and $z_2$ with $z_3$ simultaneously, the coefficients of $z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4}$ and $z_1^{\lambda_4} z_2^{\lambda_3} z_3^{\lambda_2} z_4^{\lambda_1}$ are equal and of opposite sign.

(2) Since $P_4$ vanishes identically when we put $z_3 = z_4$, the sum of the coefficients in any column of any one of the eight rectangles is zero.

(3) Since $P_4 \equiv 0$, if we put $z_2 = z_4$, the sum of the coefficients in any row of one of the eight rectangles is zero.

(4) Since $P_4 \equiv 0$, if we put $z_2 = z_3$, the sum of the coefficients in any diagonal from left to right downwards in any rectangle is zero.

(5) Since $P_4 \equiv 0$, if we put $z_1 = z_4$, the sum of the eight coefficients lying in similar positions, one in each of the eight rectangles, is zero.]

4. In the preceding expansion it is easily verified that the only terms $z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4}$ having zero coefficients, and therefore the only reducible covariants $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4}$ of weight 15, are thōse for which

(1) The sum of $\kappa$ indices is less than $2^\kappa - 1$ $(\kappa = 1, 2, 3)$;

(2) (i.) $$\begin{cases} \lambda_2 = 1, \text{ and } \lambda_1 \text{ is odd;} \\ \lambda_3 = 1, \text{ and } \lambda_4 \text{ is odd;} \end{cases}$$

(ii.) $$\begin{cases} \lambda_1 + \lambda_3 = 3, \text{ and } \lambda_2 \text{ is odd;} \\ \lambda_2 + \lambda_4 = 3, \text{ and } \lambda_3 \text{ is odd;} \end{cases}$$

(3) $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are given by the scheme—

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|
| 8 | 2 | 3 | 2 |
| 2 | 8 | 3 | 2 |
| $\lambda_4$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ |

(4) $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are given by the scheme—

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|
| 7 | 3 | 3 | 2 |
| 6 | 4 | 4 | 1 |
| 6 | 2 | 5 | 2 |
| 5 | 3 | 5 | 2 |
| 4 | 4 | 6 | 1 |
| 4 | 2 | 7 | 2 |
| 3 | 3 | 7 | 2 |
| $\lambda_4$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ |

5. Of these covariants of weight 15 those in the sets (1) are included in the first general class of § 2; so also are those of the set (3), for the covariant $(a_1a_2)^2(a_2a_3)^2(a_3a_4)^2$ of degree 4 is known to be reducible. The covariants in the set (2) are included in the second general class of § 2.

Hence the only new reducible covariants $(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}(a_4a_5)^{\lambda_4}$ of weight 15 are those 14 whose indices have the values given by the scheme (4) in § 4.

It is easily seen that the only reducible covariants of degree 5 and of weight greater than 15 are included in the first general class of § 2.

### *Jacobians of Degree* 6 (§§ 6–9).

6. We shall next discuss completely the reducibility of a special class of Jacobians of degree 6, namely, those of the form

$$((a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}(a_4a_5)^{\lambda_4},\ a_6).$$

Jacobians of the other two classes, namely,

$$((a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}.\quad (a_5a_6)^{\lambda_4}),$$

and
$$((a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2},\quad (a_4a_5)^{\lambda_4}(a_5a_6)^{\lambda_5}),$$

may be treated similarly, but the difficulty of evaluating the continued products of the corresponding operators is no less than in the case here considered.*

---

* See Note *ad fin.*

It is easily seen that the Jacobian

$$((a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4}, a_6)$$

is reducible if, and only if, $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} x_4^{\lambda_4} x_5$ is annihilated by $P_5$, that is, by $\{z_5 [P_5/z_5]_{z_5=0}\}$, that is, by $z_5 P_4^2$; hence the Jacobian

$$((a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4}, a_6)$$

is reducible if, and only if, $P_4^2$ contains *no* term $z_1^{\mu_1} z_2^{\mu_2} z_3^{\mu_3} z_4^{\mu_4}$ such that

$$\lambda_1 \geqslant \mu_1,\ \lambda_2 \geqslant \mu_2,\ \lambda_3 \geqslant \mu_3,\ \lambda_4 \geqslant \mu_4.$$

It is necessary, therefore, to expand the expression $P_4^2$ of order 30 in $z_1, z_2, z_3, z_4$, and this expansion is given on the following pages.

*Expansion of*

$$\{z_1 z_2 z_3 z_4 (z_1-z_2)(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)(z_3-z_4)(z_1-z_2+z_3) \\ \times (z_1-z_2+z_4)(z_1-z_3+z_4)(z_1-z_2+z_3-z_4)\}^2,$$

*of order* 30 *in the variables* $z_1, z_2, z_3, z_4$.*

7. The method of tabulating the result is similar to that used for the expansion of $P_4$; the coefficient $z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4}$ $(\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 30)$ is given thus:—In the table headed by the number $\lambda_1$ take the number lying in the column which has the number $\lambda_2$ at its ends in the horizontal borders and also lying in the row which has the number $\lambda_3$ at its ends in the vertical borders. The number thus determined is the required coefficient. Negative signs are placed *above* the coefficients which they affect.

*E.g.*, the coefficients of $z_1^8 z_2^7 z_3^8 z_4^7$ and $z_1^7 z_2^8 z_3^8 z_4^7$ are 21048 and $-23184$ respectively.

It is obvious that no index can be greater than 16 or less than 2.

[*Note.*—As before (§ 3), various tests for the accuracy of the expansion may be applied:—

(1) The coefficients of the terms $z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4}$ and $z_1^{\lambda_4} z_2^{\lambda_3} z_3^{\lambda_2} z_4^{\lambda_1}$ are equal.

(2) The sum of the coefficients in any row or column of each table is zero.

(3) The sum of the coefficients in any diagonal from left to right downwards in each table is zero.

(4) The sum of the fifteen coefficients lying both in the column $\lambda_2$ and the row $\lambda_3$ of each of the fifteen tables is zero.]

---

* For valuable assistance in the calculation and verification of this expansion I am considerably indebted to my brother, Mr. H. Worsley Wood.

## 16

|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 16 |
| 15 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 15 |
| 14 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 14 |
| 13 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 13 |
| 12 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 12 |
| 11 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 11 |
| 10 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 10 |
| 9 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 9 |
| 8 | 1 | $\bar{2}$ | 1 |  |  |  |  |  |  |  |  |  |  |  |  | 8 |
| 7 | $\bar{4}$ | 4 | 4 | $\bar{4}$ |  |  |  |  |  |  |  |  |  |  |  | 7 |
| 6 | 6 | 2 | $\bar{16}$ | 2 | 6 |  |  |  |  |  |  |  |  |  |  | 6 |
| 5 | $\bar{4}$ | $\bar{10}$ | 14 | 14 | $\bar{10}$ | $\bar{4}$ |  |  |  |  |  |  |  |  |  | 5 |
| 4 | 1 | 8 |  | $\bar{18}$ |  | 8 | 1 |  |  |  |  |  |  |  |  | 4 |
| 3 |  | $\bar{2}$ | $\bar{4}$ | 6 | 6 | $\bar{4}$ | $\bar{2}$ |  |  |  |  |  |  |  |  | 3 |
| 2 |  |  | 1 |  | $\bar{2}$ |  | 1 |  |  |  |  |  |  |  |  | 2 |
|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |

## 15

|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 16 |
| 15 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 15 |
| 14 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 14 |
| 13 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 13 |
| 12 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 12 |
| 11 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 11 |
| 10 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 10 |
| 9 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 9 |
| 8 |  | $\bar{8}$ | 16 | $\bar{8}$ |  |  |  |  |  |  |  |  |  |  |  | 8 |
| 7 |  | 32 | $\bar{32}$ | $\bar{32}$ | 32 |  |  |  |  |  |  |  |  |  |  | 7 |
| 6 |  | $\bar{48}$ | $\bar{16}$ | 128 | $\bar{16}$ | $\bar{48}$ |  |  |  |  |  |  |  |  |  | 6 |
| 5 |  | 32 | 80 | $\bar{112}$ | $\bar{112}$ | 80 | 32 |  |  |  |  |  |  |  |  | 5 |
| 4 |  | $\bar{8}$ | $\bar{64}$ |  | 144 |  | $\bar{64}$ | $\bar{8}$ |  |  |  |  |  |  |  | 4 |
| 3 |  |  | 16 | 32 | $\bar{48}$ | $\bar{48}$ | 32 | 16 |  |  |  |  |  |  |  | 3 |
| 2 |  |  |  | $\bar{8}$ |  | 16 |  | $\bar{8}$ |  |  |  |  |  |  |  | 2 |
|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |

## 14

|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 16 |
| 15 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 15 |
| 14 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 14 |
| 13 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 13 |
| 12 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 12 |
| 11 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 11 |
| 10 | $\overline{4}$ | 8 | $\overline{4}$ |  |  |  |  |  |  |  |  |  |  |  |  | 10 |
| 9 | 20 | $\overline{20}$ | $\overline{20}$ | 20 |  |  |  |  |  |  |  |  |  |  |  | 9 |
| 8 | $\overline{44}$ |  | 120 | $\overline{64}$ | $\overline{12}$ |  |  |  |  |  |  |  |  |  |  | 8 |
| 7 | 56 | 56 | $\overline{240}$ | 16 | 184 | $\overline{72}$ |  |  |  |  |  |  |  |  |  | 7 |
| 6 | $\overline{44}$ | $\overline{96}$ | 248 | 232 | $\overline{456}$ | $\overline{32}$ | 148 |  |  |  |  |  |  |  |  | 6 |
| 5 | 20 | 84 | $\overline{120}$ | $\overline{432}$ | 336 | 456 | $\overline{236}$ | $\overline{108}$ |  |  |  |  |  |  |  | 5 |
| 4 | $\overline{4}$ | $\overline{40}$ |  | 304 | 56 | $\overline{528}$ | $\overline{32}$ | 216 | 28 |  |  |  |  |  |  | 4 |
| 3 |  | 8 | 20 | $\overline{76}$ | $\overline{144}$ | 176 | 180 | $\overline{108}$ | $\overline{56}$ |  |  |  |  |  |  | 3 |
| 2 |  |  | $\overline{4}$ |  | 36 |  | $\overline{60}$ |  | 28 |  |  |  |  |  |  | 2 |
|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |

## 13

|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 16 |
| 15 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 15 |
| 14 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 14 |
| 13 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 13 |
| 12 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 12 |
| 11 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 11 |
| 10 |  | 28 | $\overline{56}$ | 28 |  |  |  |  |  |  |  |  |  |  |  | 10 |
| 9 |  | $\overline{140}$ | 140 | 140 | $\overline{140}$ |  |  |  |  |  |  |  |  |  |  | 9 |
| 8 |  | 308 |  | $\overline{700}$ | 168 | 224 |  |  |  |  |  |  |  |  |  | 8 |
| 7 |  | $\overline{392}$ | $\overline{392}$ | 1120 | 448 | $\overline{728}$ | $\overline{56}$ |  |  |  |  |  |  |  |  | 7 |
| 6 |  | 308 | 672 | $\overline{896}$ | $\overline{1344}$ | 952 | 504 | $\overline{196}$ |  |  |  |  |  |  |  | 6 |
| 5 |  | $\overline{140}$ | $\overline{588}$ | 280 | 1624 | $\overline{392}$ | $\overline{1232}$ | 252 | 196 |  |  |  |  |  |  | 5 |
| 4 |  | 28 | 280 | 140 | $\overline{1008}$ | $\overline{392}$ | 1176 | 224 | $\overline{392}$ | $\overline{56}$ |  |  |  |  |  | 4 |
| 3 |  |  | $\overline{56}$ | $\overline{140}$ | 252 | 448 | $\overline{392}$ | $\overline{420}$ | 196 | 112 |  |  |  |  |  | 3 |
| 2 |  |  |  | 28 |  | $\overline{112}$ |  | 140 |  | $\overline{56}$ |  |  |  |  |  | 2 |
|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |

## 12

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | | | | | | | | | | | | | | | | 16 |
| 15 | | | | | | | | | | | | | | | | 15 |
| 14 | | | | | | | | | | | | | | | | 14 |
| 13 | | | | | | | | | | | | | | | | 13 |
| 12 | 6 | $\overline{12}$ | 6 | | | | | | | | | | | | | 12 |
| 11 | $\overline{36}$ | 36 | 36 | $\overline{36}$ | | | | | | | | | | | | 11 |
| 10 | 102 | $\overline{14}$ | $\overline{272}$ | 178 | 6 | | | | | | | | | | | 10 |
| 9 | $\overline{180}$ | $\overline{110}$ | 770 | $\overline{190}$ | $\overline{590}$ | 300 | | | | | | | | | | 9 |
| 8 | 216 | 292 | $\overline{1296}$ | $\overline{536}$ | 2032 | $\overline{28}$ | $\overline{680}$ | | | | | | | | | 8 |
| 7 | $\overline{180}$ | $\overline{400}$ | 1400 | 1868 | $\overline{2804}$ | $\overline{1992}$ | 1584 | 524 | | | | | | | | 7 |
| 6 | 102 | 350 | $\overline{944}$ | $\overline{2636}$ | 1840 | 4020 | $\overline{1104}$ | $\overline{1634}$ | 6 | | | | | | | 6 |
| 5 | $\overline{36}$ | $\overline{202}$ | 330 | 2144 | $\overline{188}$ | $\overline{4028}$ | $\overline{320}$ | 2282 | 214 | $\overline{196}$ | | | | | | 5 |
| 4 | 6 | 72 | | $\overline{990}$ | $\overline{688}$ | 2304 | 1264 | $\overline{1758}$ | $\overline{672}$ | 392 | 70 | | | | | 4 |
| 3 | | $\overline{12}$ | $\overline{36}$ | 198 | 490 | $\overline{576}$ | $\overline{992}$ | 586 | 678 | $\overline{196}$ | $\overline{140}$ | | | | | 3 |
| 2 | | | 6 | | $\overline{98}$ | | 248 | | $\overline{226}$ | | 70 | | | | | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

## 11

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | | | | | | | | | | | | | | | | 16 |
| 15 | | | | | | | | | | | | | | | | 15 |
| 14 | | | | | | | | | | | | | | | | 14 |
| 13 | | | | | | | | | | | | | | | | 13 |
| 12 | | $\overline{36}$ | 72 | $\overline{36}$ | | | | | | | | | | | | 12 |
| 11 | | 216 | $\overline{216}$ | $\overline{216}$ | 216 | | | | | | | | | | | 11 |
| 10 | | $\overline{612}$ | 84 | 1268 | $\overline{340}$ | $\overline{400}$ | | | | | | | | | | 10 |
| 9 | | 1080 | 660 | $\overline{2800}$ | $\overline{680}$ | 1720 | 20 | | | | | | | | | 9 |
| 8 | | $\overline{1296}$ | $\overline{1752}$ | 3772 | 3216 | $\overline{3456}$ | $\overline{1288}$ | 804 | | | | | | | | 8 |
| 7 | | 1080 | 2400 | $\overline{3304}$ | $\overline{6112}$ | 3720 | 4672 | $\overline{1496}$ | $\overline{960}$ | | | | | | | 7 |
| 6 | | $\overline{612}$ | $\overline{2100}$ | 1660 | 7080 | $\overline{1576}$ | $\overline{7376}$ | 72 | 2524 | 328 | | | | | | 6 |
| 5 | | 216 | 1212 | $\overline{160}$ | $\overline{5220}$ | $\overline{1056}$ | 6696 | 1920 | $\overline{2772}$ | $\overline{920}$ | 84 | | | | | 5 |
| 4 | | $\overline{36}$ | $\overline{432}$ | $\overline{364}$ | 2300 | 1944 | $\overline{3632}$ | $\overline{2488}$ | 1812 | 1120 | $\overline{168}$ | $\overline{56}$ | | | | 4 |
| 3 | | | 72 | 216 | $\overline{460}$ | $\overline{1120}$ | 908 | 1584 | $\overline{604}$ | $\overline{792}$ | 84 | 112 | | | | 3 |
| 2 | | | | $\overline{36}$ | | 224 | | $\overline{396}$ | | 264 | | $\overline{56}$ | | | | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

10

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | | | | | | | | | | | | | | | | 16 |
| 15 | | | | | | | | | | | | | | | | 15 |
| 14 | $\overline{4}$ | 8 | $\overline{4}$ | | | | | | | | | | | | | 14 |
| 13 | 28 | $\overline{28}$ | $\overline{28}$ | 28 | | | | | | | | | | | | 13 |
| 12 | $\overline{98}$ | 26 | 248 | $\overline{182}$ | 6 | | | | | | | | | | | 12 |
| 11 | 224 | 68 | $\overline{916}$ | 332 | 692 | $\overline{400}$ | | | | | | | | | | 11 |
| 10 | $\overline{362}$ | $\overline{312}$ | 2144 | 356 | $\overline{3016}$ | 120 | 1070 | | | | | | | | | 10 |
| 9 | 424 | 642 | $\overline{3396}$ | $\overline{2710}$ | 5930 | 3084 | $\overline{2958}$ | $\overline{1016}$ | | | | | | | | 9 |
| 8 | $\overline{362}$ | $\overline{838}$ | 3696 | 5888 | $\overline{6952}$ | $\overline{8560}$ | 3456 | 3722 | $\overline{50}$ | | | | | | | 8 |
| 7 | 224 | 744 | $\overline{2756}$ | $\overline{7484}$ | 4808 | 13064 | $\overline{1820}$ | $\overline{7076}$ | $\overline{456}$ | 752 | | | | | | 7 |
| 6 | $\overline{98}$ | $\overline{460}$ | 1352 | 6230 | $\overline{1160}$ | $\overline{13216}$ | $\overline{1400}$ | 9030 | 2080 | $\overline{1916}$ | $\overline{442}$ | | | | | 6 |
| 5 | 28 | 198 | $\overline{364}$ | $\overline{3468}$ | $\overline{1150}$ | 8884 | 3892 | $\overline{7630}$ | $\overline{3516}$ | 1988 | 1110 | 28 | | | | 5 |
| 4 | $\overline{4}$ | $\overline{56}$ | | 1212 | 1352 | $\overline{3720}$ | $\overline{3688}$ | 3960 | 3328 | $\overline{1236}$ | $\overline{1120}$ | $\overline{56}$ | 28 | | | 4 |
| 3 | | 8 | 28 | $\overline{202}$ | $\overline{612}$ | 744 | 1810 | $\overline{990}$ | $\overline{1848}$ | 412 | 678 | 28 | $\overline{56}$ | | | 3 |
| 2 | | | $\overline{4}$ | | 102 | | $\overline{362}$ | | 462 | | $\overline{226}$ | | 28 | | | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

9

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | | | | | | | | | | | | | | | | 16 |
| 15 | | | | | | | | | | | | | | | | 15 |
| 14 | | 20 | $\overline{40}$ | 20 | | | | | | | | | | | | 14 |
| 13 | | $\overline{140}$ | 140 | 140 | $\overline{140}$ | | | | | | | | | | | 13 |
| 12 | | 490 | $\overline{130}$ | $\overline{910}$ | 250 | 300 | | | | | | | | | | 12 |
| 11 | | $\overline{1120}$ | $\overline{340}$ | 2600 | 320 | $\overline{1480}$ | 20 | | | | | | | | | 11 |
| 10 | | 1810 | 1560 | $\overline{5110}$ | $\overline{2550}$ | 4124 | 1182 | $\overline{1016}$ | | | | | | | | 10 |
| 9 | | $\overline{2120}$ | $\overline{3210}$ | 7080 | 7500 | $\overline{7320}$ | $\overline{5850}$ | 2860 | 1560 | | | | | | | 9 |
| 8 | | 1810 | 4190 | $\overline{6600}$ | $\overline{13380}$ | 7524 | 13320 | $\overline{1330}$ | $\overline{4706}$ | $\overline{828}$ | | | | | | 8 |
| 7 | | $\overline{1120}$ | $\overline{3720}$ | 3880 | 15420 | $\overline{3096}$ | $\overline{18636}$ | $\overline{2120}$ | 7044 | 2456 | $\overline{108}$ | | | | | 7 |
| 6 | | 490 | 2300 | $\overline{1150}$ | $\overline{11900}$ | $\overline{2076}$ | 17196 | 5812 | $\overline{7662}$ | $\overline{3624}$ | 362 | 252 | | | | 6 |
| 5 | | $\overline{140}$ | $\overline{990}$ | $\overline{160}$ | 6230 | 3880 | $\overline{10584}$ | $\overline{6920}$ | 5942 | 3984 | $\overline{546}$ | $\overline{644}$ | $\overline{52}$ | | | 5 |
| 4 | | 20 | 280 | 330 | $\overline{2100}$ | $\overline{2756}$ | 4190 | 4910 | $\overline{2904}$ | $\overline{3176}$ | 438 | 672 | 104 | $\overline{8}$ | | 4 |
| 3 | | | $\overline{40}$ | $\overline{140}$ | 350 | 1080 | $\overline{838}$ | $\overline{2120}$ | 726 | 1584 | $\overline{146}$ | $\overline{420}$ | $\overline{52}$ | 16 | | 3 |
| 2 | | | | 20 | | $\overline{180}$ | | 424 | | $\overline{396}$ | | 140 | | $\overline{8}$ | | 2 |
| | 2 | 3 | | | | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

## 8

|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 1 | $\overline{2}$ | 1 |  |  |  |  |  |  |  |  |  |  |  |  | 16 |
| 15 | $\overline{8}$ | 8 | 8 | $\overline{8}$ |  |  |  |  |  |  |  |  |  |  |  | 15 |
| 14 | 36 | $\overline{20}$ | $\overline{80}$ | 76 | $\overline{12}$ |  |  |  |  |  |  |  |  |  |  | 14 |
| 13 | $\overline{112}$ | 28 | 420 | $\overline{252}$ | $\overline{308}$ | 224 |  |  |  |  |  |  |  |  |  | 13 |
| 12 | 248 | 48 | $\overline{1416}$ | 192 | 1752 | $\overline{144}$ | $\overline{680}$ |  |  |  |  |  |  |  |  | 12 |
| 11 | $\overline{396}$ | $\overline{320}$ | 3112 | 1156 | $\overline{4748}$ | $\overline{1640}$ | 2032 | 804 |  |  |  |  |  |  |  | 11 |
| 10 | 462 | 726 | $\overline{4656}$ | $\overline{4548}$ | 8320 | 6204 | $\overline{3536}$ | $\overline{2922}$ | $\overline{50}$ |  |  |  |  |  |  | 10 |
| 9 | $\overline{396}$ | $\overline{990}$ | 4910 | 8664 | $\overline{9708}$ | $\overline{13836}$ | 4920 | 6990 | 274 | $\overline{828}$ |  |  |  |  |  | 9 |
| 8 | 248 | 908 | $\overline{3688}$ | $\overline{10584}$ | 6960 | 21048 | $\overline{3216}$ | $\overline{13560}$ | $\overline{1608}$ | 2700 | 792 |  |  |  |  | 8 |
| 7 | $\overline{112}$ | $\overline{576}$ | 1944 | 8884 | $\overline{2076}$ | $\overline{21792}$ | $\overline{2496}$ | 17988 | 5028 | $\overline{4280}$ | $\overline{2288}$ | $\overline{224}$ |  |  |  | 7 |
| 6 | 36 | 252 | $\overline{688}$ | $\overline{5220}$ | $\overline{1160}$ | 15420 | 6960 | $\overline{15396}$ | $\overline{8200}$ | 4412 | 3152 | 476 | $\overline{44}$ |  |  | 6 |
| 5 | $\overline{8}$ | $\overline{76}$ | 140 | 2144 | 1660 | $\overline{7484}$ | $\overline{6600}$ | 8664 | 7764 | $\overline{2964}$ | $\overline{3104}$ | $\overline{308}$ | 148 | 24 |  | 5 |
| 4 | 1 | 16 |  | $\overline{588}$ | $\overline{944}$ | 2400 | 3696 | $\overline{3210}$ | $\overline{4656}$ | 1280 | 2192 | 84 | $\overline{224}$ | $\overline{48}$ | 1 | 4 |
| 3 |  | $\overline{2}$ | $\overline{8}$ | 84 | 308 | $\overline{400}$ | $\overline{1296}$ | 642 | 1810 | $\overline{320}$ | $\overline{992}$ | $\overline{28}$ | 180 | 24 | $\overline{2}$ | 3 |
| 2 |  |  | 1 |  | $\overline{44}$ |  | 216 |  | $\overline{362}$ |  | 248 |  | $\overline{60}$ |  | 1 | 2 |
|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |

## 7

|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 |  | $\overline{4}$ | 8 | $\overline{4}$ |  |  |  |  |  |  |  |  |  |  |  | 16 |
| 15 |  | 32 | $\overline{32}$ | $\overline{32}$ | 32 |  |  |  |  |  |  |  |  |  |  | 15 |
| 14 |  | $\overline{144}$ | 80 | 200 | $\overline{64}$ | $\overline{72}$ |  |  |  |  |  |  |  |  |  | 14 |
| 13 |  | 448 | $\overline{112}$ | $\overline{840}$ | 168 | 392 | $\overline{56}$ |  |  |  |  |  |  |  |  | 13 |
| 12 |  | $\overline{992}$ | $\overline{192}$ | 2724 | 12 | $\overline{1944}$ | $\overline{132}$ | 524 |  |  |  |  |  |  |  | 12 |
| 11 |  | 1584 | 1280 | $\overline{5728}$ | $\overline{2584}$ | 5768 | 2264 | $\overline{1624}$ | $\overline{960}$ |  |  |  |  |  |  | 11 |
| 10 |  | $\overline{1848}$ | $\overline{2904}$ | 7764 | 8832 | $\overline{9352}$ | $\overline{8592}$ | 2204 | 3144 | 752 |  |  |  |  |  | 10 |
| 9 |  | 1584 | 3960 | $\overline{6920}$ | $\overline{15396}$ | 8232 | 17604 | $\overline{840}$ | $\overline{6060}$ | $\overline{2056}$ | $\overline{108}$ |  |  |  |  | 9 |
| 8 |  | $\overline{992}$ | $\overline{3632}$ | 3892 | 17196 | $\overline{2496}$ | $\overline{23184}$ | $\overline{3372}$ | 9696 | 3280 | $\overline{164}$ | $\overline{224}$ |  |  |  | 8 |
| 7 |  | 448 | 2304 | $\overline{1056}$ | $\overline{13216}$ | $\overline{3096}$ | 21048 | 8232 | $\overline{11352}$ | $\overline{5368}$ | 1080 | 840 | 136 |  |  | 7 |
| 6 |  | $\overline{144}$ | $\overline{1008}$ | $\overline{188}$ | 7080 | 4808 | $\overline{13380}$ | $\overline{9708}$ | 8832 | 6904 | $\overline{1332}$ | $\overline{1512}$ | $\overline{336}$ | $\overline{16}$ |  | 6 |
| 5 |  | 32 | 304 | 280 | $\overline{2636}$ | $\overline{3304}$ | 5888 | 7080 | $\overline{4548}$ | $\overline{5728}$ | 668 | 1624 | 328 | 16 | $\overline{4}$ | 5 |
| 4 |  | $\overline{4}$ | $\overline{64}$ | $\overline{120}$ | 672 | 1400 | $\overline{1752}$ | $\overline{3396}$ | 1560 | 3112 | $\overline{192}$ | $\overline{1064}$ | $\overline{192}$ | 32 | 8 | 4 |
| 3 |  |  | 8 | 32 | $\overline{96}$ | $\overline{392}$ | 292 | 1080 | $\overline{312}$ | $\overline{1120}$ | 48 | 448 | 64 | $\overline{48}$ | $\overline{4}$ | 3 |
| 2 |  |  |  | $\overline{4}$ |  | 56 |  | $\overline{180}$ |  | 224 |  | $\overline{112}$ |  | 16 |  | 2 |
|  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |

### 6

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | $\bar{2}$ | 6 | | $\bar{10}$ | 6 | | | | | | | | | | | 16 |
| 15 | 16 | $\bar{32}$ | $\bar{48}$ | 80 | 32 | $\bar{48}$ | | | | | | | | | | 15 |
| 14 | $\bar{60}$ | 64 | 328 | $\bar{240}$ | $\bar{312}$ | 72 | 148 | | | | | | | | | 14 |
| 13 | 140 | $\bar{28}$ | $\bar{1064}$ | 224 | 1344 | | $\bar{420}$ | $\bar{196}$ | | | | | | | | 13 |
| 12 | $\bar{226}$ | $\bar{146}$ | 2192 | 668 | $\bar{3760}$ | $\bar{700}$ | 1512 | 454 | 6 | | | | | | | 12 |
| 11 | 264 | 412 | $\bar{3176}$ | $\bar{2964}$ | 6904 | 4192 | $\bar{4508}$ | $\bar{1968}$ | 516 | 328 | | | | | | 11 |
| 10 | $\bar{226}$ | $\bar{604}$ | 3328 | 5942 | $\bar{8200}$ | $\bar{11352}$ | 6544 | 7590 | $\bar{928}$ | $\bar{1652}$ | $\bar{442}$ | | | | | 10 |
| 9 | 140 | 586 | $\bar{2488}$ | $\bar{7630}$ | 5812 | 17988 | $\bar{3372}$ | $\bar{14988}$ | $\bar{1630}$ | 3792 | 1538 | 252 | | | | 9 |
| 8 | $\bar{60}$ | $\bar{392}$ | 1264 | 6696 | $\bar{1400}$ | $\bar{18636}$ | $\bar{3216}$ | 17604 | 6544 | $\bar{4960}$ | $\bar{2840}$ | $\bar{560}$ | $\bar{44}$ | | | 8 |
| 7 | 16 | 176 | $\bar{392}$ | $\bar{4028}$ | $\bar{1576}$ | 13064 | 7524 | $\bar{13836}$ | $\bar{9352}$ | 4192 | 3804 | 448 | $\bar{24}$ | $\bar{16}$ | | 7 |
| 6 | $\bar{2}$ | $\bar{48}$ | 56 | 1624 | 1840 | $\bar{6112}$ | $\bar{6952}$ | 7500 | 8320 | $\bar{2584}$ | $\bar{3760}$ | $\bar{280}$ | 328 | 64 | 6 | 6 |
| 5 | | 6 | | $\bar{432}$ | $\bar{896}$ | 1868 | 3772 | $\bar{2710}$ | $\bar{5110}$ | 1156 | 2724 | 224 | $\bar{480}$ | $\bar{112}$ | $\bar{10}$ | 5 |
| 4 | | | | 80 | 248 | $\bar{392}$ | $\bar{1296}$ | 660 | 2144 | $\bar{340}$ | $\bar{1416}$ | $\bar{112}$ | 328 | 96 | | 4 |
| 3 | | | | $\bar{10}$ | $\bar{48}$ | 56 | 308 | $\bar{110}$ | $\bar{612}$ | 68 | 490 | 28 | $\bar{144}$ | $\bar{32}$ | 6 | 3 |
| 2 | | | | | 6 | | $\bar{44}$ | | 102 | | $\bar{98}$ | | 36 | | $\bar{2}$ | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

### 5

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | | 6 | $\bar{18}$ | 14 | 2 | $\bar{4}$ | | | | | | | | | | 16 |
| 15 | | $\bar{48}$ | 96 | 32 | $\bar{128}$ | 16 | 32 | | | | | | | | | 15 |
| 14 | | 180 | $\bar{192}$ | $\bar{480}$ | 440 | 320 | $\bar{160}$ | $\bar{108}$ | | | | | | | | 14 |
| 13 | | $\bar{420}$ | 84 | 1624 | $\bar{280}$ | $\bar{1680}$ | | 476 | 196 | | | | | | | 13 |
| 12 | | 678 | 438 | $\bar{3104}$ | $\bar{1332}$ | 3804 | 1512 | $\bar{1158}$ | $\bar{642}$ | $\bar{196}$ | | | | | | 12 |
| 11 | | $\bar{792}$ | $\bar{1236}$ | 3984 | 4412 | $\bar{5368}$ | $\bar{4960}$ | 1880 | 1700 | 296 | 84 | | | | | 11 |
| 10 | | 678 | 1812 | $\bar{3516}$ | $\bar{7662}$ | 5028 | 9696 | $\bar{1630}$ | $\bar{4332}$ | $\bar{356}$ | 254 | 28 | | | | 10 |
| 9 | | $\bar{420}$ | $\bar{1758}$ | 1920 | 9080 | $\bar{2120}$ | $\bar{13560}$ | $\bar{840}$ | 7590 | 1880 | $\bar{1250}$ | $\bar{420}$ | $\bar{52}$ | | | 9 |
| 8 | | 180 | 1176 | $\bar{320}$ | $\bar{7376}$ | $\bar{1820}$ | 13320 | 4920 | $\bar{8592}$ | $\bar{4508}$ | 1512 | 1260 | 224 | 24 | | 8 |
| 7 | | $\bar{48}$ | $\bar{528}$ | $\bar{392}$ | 4020 | 3720 | $\bar{8560}$ | $\bar{7320}$ | 6204 | 5768 | $\bar{700}$ | $\bar{1680}$ | $\bar{432}$ | $\bar{48}$ | $\bar{4}$ | 7 |
| 6 | | 6 | 144 | 336 | $\bar{1344}$ | $\bar{2804}$ | 3216 | 5930 | $\bar{2550}$ | $\bar{4748}$ | 12 | 1344 | 440 | 16 | 2 | 6 |
| 5 | | | $\bar{18}$ | $\bar{112}$ | 232 | 1120 | $\bar{536}$ | $\bar{2800}$ | 356 | 2600 | 192 | $\bar{840}$ | $\bar{240}$ | 32 | 14 | 5 |
| 4 | | | | 14 | $\bar{16}$ | $\bar{240}$ | | 770 | 84 | $\bar{916}$ | $\bar{130}$ | 420 | 80 | $\bar{48}$ | $\bar{18}$ | 4 |
| 3 | | | | | 2 | 32 | | $\bar{140}$ | $\bar{14}$ | 216 | 26 | $\bar{140}$ | $\bar{20}$ | 32 | 6 | 3 |
| 2 | | | | | | $\bar{4}$ | | 20 | | $\bar{36}$ | | 28 | | $\bar{8}$ | | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

**4**

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 1 | $\overline{4}$ | | 14 | $\overline{16}$ | 4 | 1 | | | | | | | | | 16 |
| 15 | $\overline{8}$ | 24 | 32 | $\overline{112}$ | 16 | 96 | $\overline{40}$ | $\overline{8}$ | | | | | | | | 15 |
| 14 | 28 | $\overline{52}$ | $\overline{224}$ | 328 | 328 | $\overline{432}$ | $\overline{168}$ | 164 | 28 | | | | | | | 14 |
| 13 | $\overline{56}$ | 28 | 672 | $\overline{308}$ | $\overline{1512}$ | 448 | 1260 | $\overline{112}$ | $\overline{364}$ | $\overline{56}$ | | | | | | 13 |
| 12 | 70 | 84 | $\overline{1120}$ | $\overline{546}$ | 3152 | 1080 | $\overline{2840}$ | $\overline{1250}$ | 824 | 476 | 70 | | | | | 12 |
| 11 | $\overline{56}$ | $\overline{196}$ | 1120 | 1988 | $\overline{3624}$ | $\overline{4280}$ | 3280 | 3792 | $\overline{356}$ | $\overline{1248}$ | $\overline{364}$ | $\overline{56}$ | | | | 11 |
| 10 | 28 | 196 | $\overline{672}$ | $\overline{2772}$ | 2080 | 7044 | $\overline{1608}$ | $\overline{6060}$ | $\overline{928}$ | 1700 | 824 | 140 | 28 | | | 10 |
| 9 | $\overline{8}$ | $\overline{108}$ | 224 | 2282 | 72 | $\overline{7076}$ | $\overline{1330}$ | 6990 | 2204 | $\overline{1968}$ | $\overline{1158}$ | $\overline{112}$ | $\overline{4}$ | $\overline{8}$ | | 9 |
| 8 | 1 | 32 | $\overline{32}$ | $\overline{1232}$ | $\overline{1104}$ | 4672 | 3456 | $\overline{5850}$ | $\overline{3536}$ | 2264 | 1512 | | $\overline{168}$ | $\overline{16}$ | 1 | 8 |
| 7 | | $\overline{4}$ | | 456 | 952 | $\overline{1992}$ | $\overline{3456}$ | 3084 | 4124 | $\overline{1640}$ | $\overline{1944}$ | | 320 | 96 | 4 | 7 |
| 6 | | | | $\overline{112}$ | $\overline{456}$ | 448 | 2032 | $\overline{680}$ | $\overline{3016}$ | 320 | 1752 | 168 | $\overline{312}$ | $\overline{128}$ | $\overline{16}$ | 6 |
| 5 | | | | 14 | 128 | 16 | $\overline{700}$ | $\overline{190}$ | 1268 | 332 | $\overline{910}$ | $\overline{252}$ | 200 | 80 | 14 | 5 |
| 4 | | | | | $\overline{16}$ | $\overline{32}$ | 120 | 140 | $\overline{272}$ | $\overline{216}$ | 248 | 140 | $\overline{80}$ | $\overline{32}$ | | 4 |
| 3 | | | | | | 4 | $\overline{8}$ | $\overline{20}$ | 28 | 36 | $\overline{36}$ | $\overline{28}$ | 20 | 8 | $\overline{4}$ | 3 |
| 2 | | | | | | | 1 | | $\overline{4}$ | | 6 | | $\overline{4}$ | | 1 | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

**3**

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | | $\overline{2}$ | 8 | $\overline{10}$ | 2 | 4 | $\overline{2}$ | | | | | | | | | 16 |
| 15 | | 16 | $\overline{48}$ | 16 | 64 | $\overline{48}$ | $\overline{16}$ | 16 | | | | | | | | 15 |
| 14 | | $\overline{56}$ | 104 | 148 | $\overline{336}$ | $\overline{24}$ | 224 | $\overline{4}$ | $\overline{56}$ | | | | | | | 14 |
| 13 | | 112 | $\overline{56}$ | $\overline{644}$ | 476 | 840 | $\overline{560}$ | $\overline{420}$ | 140 | 112 | | | | | | 13 |
| 12 | | $\overline{140}$ | $\overline{168}$ | 1110 | 362 | $\overline{2288}$ | $\overline{164}$ | 1538 | 254 | $\overline{364}$ | $\overline{140}$ | | | | | 12 |
| 11 | | 112 | 392 | $\overline{920}$ | $\overline{1916}$ | 2456 | 2700 | $\overline{2056}$ | $\overline{1652}$ | 296 | 476 | 112 | | | | 11 |
| 10 | | $\overline{56}$ | $\overline{392}$ | 214 | 2524 | $\overline{456}$ | $\overline{4706}$ | 274 | 3144 | 516 | $\overline{642}$ | $\overline{364}$ | $\overline{56}$ | | | 10 |
| 9 | | 16 | 216 | 252 | $\overline{1684}$ | $\overline{1496}$ | 3722 | 2360 | $\overline{2922}$ | $\overline{1624}$ | 454 | 476 | 164 | 16 | | 9 |
| 8 | | $\overline{2}$ | $\overline{64}$ | $\overline{236}$ | 504 | 1584 | $\overline{1288}$ | $\overline{2958}$ | 1182 | 2032 | $\overline{132}$ | $\overline{420}$ | $\overline{160}$ | $\overline{40}$ | $\overline{2}$ | 8 |
| 7 | | | 8 | 80 | $\overline{32}$ | $\overline{728}$ | $\overline{28}$ | 1720 | 120 | $\overline{1480}$ | $\overline{144}$ | 392 | 72 | 16 | 4 | 7 |
| 6 | | | | $\overline{10}$ | $\overline{16}$ | 184 | 168 | $\overline{590}$ | $\overline{340}$ | 692 | 250 | $\overline{308}$ | $\overline{64}$ | 32 | 2 | 6 |
| 5 | | | | | 2 | $\overline{32}$ | $\overline{64}$ | 140 | 178 | $\overline{216}$ | $\overline{182}$ | 140 | 76 | $\overline{32}$ | $\overline{10}$ | 5 |
| 4 | | | | | | 4 | 16 | $\overline{20}$ | $\overline{56}$ | 36 | 72 | $\overline{28}$ | $\overline{40}$ | 8 | 8 | 4 |
| 3 | | | | | | | $\overline{2}$ | | 8 | | $\overline{12}$ | | 8 | | $\overline{2}$ | 3 |
| 2 | | | | | | | | | | | | | | | | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

**2**

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | | | 1 | $\overline{4}$ | 6 | $\overline{4}$ | 1 | | | | | | | | | 16 |
| 15 | | | $\overline{8}$ | 24 | $\overline{16}$ | $\overline{16}$ | 24 | $\overline{8}$ | | | | | | | | 15 |
| 14 | | | 28 | $\overline{52}$ | $\overline{44}$ | 136 | $\overline{44}$ | $\overline{52}$ | 28 | | | | | | | 14 |
| 13 | | | $\overline{56}$ | 28 | 252 | $\overline{224}$ | $\overline{224}$ | 252 | 28 | $\overline{56}$ | | | | | | 13 |
| 12 | | | 70 | 84 | $\overline{442}$ | $\overline{108}$ | 792 | $\overline{108}$ | $\overline{442}$ | 84 | 70 | | | | | 12 |
| 11 | | | $\overline{56}$ | $\overline{196}$ | 328 | 752 | $\overline{828}$ | $\overline{828}$ | 752 | 328 | $\overline{196}$ | $\overline{56}$ | | | | 11 |
| 10 | | | 28 | 196 | 6 | $\overline{960}$ | $\overline{50}$ | 1560 | $\overline{50}$ | $\overline{960}$ | 6 | 196 | 28 | | | 10 |
| 9 | | | $\overline{8}$ | $\overline{108}$ | $\overline{196}$ | 524 | 804 | $\overline{1016}$ | $\overline{1016}$ | 804 | 524 | $\overline{196}$ | $\overline{108}$ | $\overline{8}$ | | 9 |
| 8 | | | 1 | 32 | 148 | $\overline{56}$ | $\overline{680}$ | 20 | 1070 | 20 | $\overline{680}$ | $\overline{56}$ | 148 | 32 | 1 | 8 |
| 7 | | | | $\overline{4}$ | $\overline{48}$ | $\overline{72}$ | 224 | 300 | $\overline{400}$ | $\overline{400}$ | 300 | 224 | $\overline{72}$ | $\overline{48}$ | $\overline{4}$ | 7 |
| 6 | | | | | 6 | 32 | $\overline{12}$ | $\overline{140}$ | 6 | 216 | 6 | $\overline{140}$ | $\overline{12}$ | 32 | 6 | 6 |
| 5 | | | | | | $\overline{4}$ | $\overline{8}$ | 20 | 28 | $\overline{36}$ | $\overline{36}$ | 28 | 20 | $\overline{8}$ | $\overline{4}$ | 5 |
| 4 | | | | | | | 1 | | $\overline{4}$ | | 6 | | $\overline{4}$ | | 1 | 4 |
| 3 | | | | | | | | | | | | | | | | 3 |
| 2 | | | | | | | | | | | | | | | | 2 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

8. In the preceding expansion it is easily verified that the only terms $^{\lambda_1}_{1} z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4}$ having zero coefficients, and therefore the only reducible acobians $\left((a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}(a_4a_5)^{\lambda_4},\ a_6\right)$ of weight 31, are those for hich—

(1) The sum of $\kappa$ indices is less than $2^{\kappa+1}-2$, $(\kappa = 1, 2, 3)$;

(2) (i.) $\begin{cases} \lambda_2 = 2, \text{ and } \lambda_1 \text{ is odd;} \\ \lambda_3 = 2, \text{ and } \lambda_4 \text{ is odd;} \end{cases}$

(ii.) $\begin{cases} \lambda_1+\lambda_3 = 6, \text{ and } \lambda_2 \text{ is odd;} \\ \lambda_2+\lambda_3 = 6, \text{ and } \lambda_1 \text{ is odd;} \\ \lambda_2+\lambda_4 = 6, \text{ and } \lambda_1 \text{ is odd;} \end{cases}$

3) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are given by the scheme:—

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|
| 16 | 6 | 4 | 4 |
| 6 | 16 | 4 | 4 |
| 6 | 4 | 16 | 4 |
| 6 | 4 | 4 | 16 |
| $\lambda_4$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ |

(4) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are given by the scheme:—

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|
| 15 | 7 | 4 | 4 |
| 15 | 5 | 4 | 6 |
| 14 | 4 | 4 | 8 |
| 14 | 3 | 8 | 5 |
| 13 | 4 | 8 | 5 |
| 12 | 4 | 4 | 10 |
| 6 | 7 | 13 | 4 |
| 5 | 8 | 13 | 4 |
| $\lambda_4$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ |

9. Of these Jacobians of weight 31, those in the set (1) are included in the first general class of § 2; so also are those in the set (3), for we have seen (§ 4) that each of the covariants

$$(a_1a_2)^6(a_2a_3)^4(a_3a_4)^4(a_4a_5), \qquad (a_1a_2)^4(a_2a_3)^4(a_3a_4)^6(a_4a_5),$$

of degree 5 is reducible; the covariants of the set (2) are included in the second general class of § 2.

Hence the only new reducible Jacobians

$$\left((a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}(a_4a_5)^{\lambda_4},\ a_6\right)$$

of weight 31 are those 16 whose indices have the values given by the scheme (4) in § 8.

It is easily seen that the only reducible Jacobians of this nature and of weight greater than 31 are included in the first general class of § 2.

### **Note added to the preceding paper March 24th,* 1905.

10. For the sake of brevity let us put

$$z_s \equiv \frac{\partial}{\partial x_s}; \; (s = 1, 2, \ldots, \delta),$$

and
$$\phi(z_1, z_2, \ldots, z_\delta) \equiv \prod_{(2^\delta-1)} \left(z_{r(1)} - z_{r(2)} + \ldots + (-)^{\epsilon-1} z_{r(\epsilon)}\right),$$
$$\equiv P_\delta,$$

in the notation of the preceding paper.

The reducibility of all covariants of degree 5 and of Jacobians

$$\left((a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4}, \; a_6\right)$$

of degree 6 has already been completely determined, and the present note is devoted to a discussion of the remaining Jacobians of degree 6, namely, those of the two classes

$$\left((a_1 a_2)^{\lambda_1}, \; (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4} (a_5 a_6)^{\lambda_5}\right), \quad \text{or} \quad (C_2, C_4),$$

and
$$\left((a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2}, \; (a_4 a_5)^{\lambda_4} (a_5 a_6)^{\lambda_5}\right), \quad \text{or} \quad (C_3, C_3).$$

### 11. *Jacobians* $(C_2, C_4)$.

The necessary and sufficient condition that the Jacobian

$$\left((a_1 a_2)^{\lambda_1}, \; (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4} (a_5 a_6)^{\lambda_5}\right),$$

is reducible is (§ 1) that the expression $x_1^{\lambda_1} x_2 x_3^{\lambda_3} x_4^{\lambda_4} x_5^{\lambda_5}$ should be annihilated by $\phi(z_1, z_2, z_3, z_4, z_5)$, that is, that the expression $x_1^{\lambda_1} x_3^{\lambda_3} x_4^{\lambda_4} x_5^{\lambda_5}$ should be annihilated by

$$\left[\frac{\phi(z_1, z_2, z_3, z_4, z_5)}{z_2}\right]_{z_2=0} \equiv \phi(z_1, z_3, z_4, z_5)\, \phi(-z_1, z_3, z_4, z_5).$$

Now the expansion $\phi(z_1, z_2, z_3, z_4)$ has been already calculated (§ 3) for the discussion of covariants of degree 5, so that the expansion now required may be obtained from it with less labour than the complete expansion of $\phi(z_1, z_2, z_3, z_4, z_5)$. The actual expansion is not given here, as we are primarily concerned only with the zero coefficients of the expansion.

I find that the only terms $z_1^{\lambda_1} z_3^{\lambda_3} z_4^{\lambda_4} z_5^{\lambda_5}$ in the expansion of

$$\phi(z_1, z_3, z_4, z_5)\, \phi(-z_1, z_3, z_4, z_5)$$

---

* The following results were originally communicated in a paper "On Reducible Jacobians of Degree 6" (read April 13th, 1905).

with zero coefficients are those for which—

I. The sum of $\kappa$ of the indices $\lambda_1, \lambda_3, \lambda_4, \lambda_5$ is less than

$$(2^{\kappa+1}-2), \quad (\kappa = 1, 2, 3);$$

II. (i.) $\lambda_1$ is odd,

(ii.) $\lambda_4 = 2$, and $\lambda_3$ is odd,

(iii.) $\lambda_1 = 2$, $\lambda_4 = 4$, and $\lambda_3$ is odd;

III. $\lambda_1, \lambda_3, \lambda_4, \lambda_5$ are given by the scheme :—

| $\lambda_1$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|---|---|---|---|
| 16 | 6 | 4 | 4 |
| 14 | 5 | 8 | 3 |
| $\lambda_1$ | $\lambda_5$ | $\lambda_4$ | $\lambda_3$ |

Now the Jacobian $\left((a_1a_2)^6(a_2a_3)^4(a_3a_4)^4,\ a_5\right)$ of degree 5 is known to be reducible (§ 4); it follows therefore that the only reducible Jacobians

$$\left((a_1a_2)^{\lambda_1},\ (a_3a_4)^{\lambda_3}(a_4a_5)^{\lambda_4}(a_5a_6)^{\lambda_5}\right)$$

of degree 6 and weight 31, not included in the two general classes of § 2, are those two for which

$$\lambda_1 = 14, \quad \lambda_3 = 5, \quad \lambda_4 = 8, \quad \lambda_5 = 3,$$

$$\lambda_1 = 14, \quad \lambda_3 = 3, \quad \lambda_4 = 8, \quad \lambda_5 = 5.$$

It is easily verified that the only reducible Jacobians of this nature of degree 6 and of weight greater than 31 are included in the first general class of § 2.

## 12. *Jacobians* $(C_3, C_3)$.

The necessary and sufficient condition that the Jacobian

$$\left((a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2},\ (a_4a_5)^{\lambda_4}(a_5a_6)^{\lambda_5}\right)$$

is reducible is (§ 1) that the expression $x_1^{\lambda_1}x_2^{\lambda_2}x_3x_4^{\lambda_4}x_5^{\lambda_5}$ should be annihilated by $\phi(z_1, z_2, z_3, z_4, z_5)$, that is, that the expression $x_1^{\lambda_1}x_2^{\lambda_2}x_4^{\lambda_4}x_5^{\lambda_5}$ should be annihilated by

$$\left[\frac{\phi(z_1, z_2, z_3, z_4, z_5)}{z_3}\right]_{z_3=0} \equiv \phi(z_1, z_2, z_4, z_5)\,\phi(z_1, z_2, -z_4, -z_5).$$

I find that the only terms $z_1^{\lambda_1} z_2^{\lambda_2} z_4^{\lambda_4} z_5^{\lambda_5}$ in this expansion with zero coefficients are those for which—

I. The sum of $\kappa$ of the indices $\lambda_1, \lambda_2, \lambda_4, \lambda_5$ is less than

$$(2^{\kappa+1}-2), \quad (\kappa = 1, 2, 3);$$

II. (i.) $\lambda_1+\lambda_2$ is odd,

(ii.) $\lambda_1 = \lambda_4 = 3$, and $\lambda_2$ is odd.

It follows therefore that all reducible Jacobians

$$\left((a_1 a_3)^{\lambda_1} (a_2 a_3)^{\lambda_2},\ (a_4 a_5)^{\lambda_4} (a_5 a_6)^{\lambda_5}\right)$$

of degree 6 and weight 31 are included in the two general classes of § 2, and it is easily verified that all such Jacobians of weight greater than 31 are included in the first general class of § 2.

13. The two preceding expansions are susceptible of various tests for their accuracy (cf. §§ 3, 7) : *e.g.*, each expansion vanishes if we equate any two of the variables $z_1, z_2, z_3, z_4$; the first expansion (§ 11) is unaltered by the interchange of $z_3$ and $z_5$; the second expansion (§ 12) is unaltered by the interchange of $z_1$ and $z_2$, or of $z_4$ and $z_5$, and is altered only in sign if we interchange $z_1$ and $z_5$, $z_2$ and $z_4$ simultaneously. Moreover many tests may be obtained by comparing the three expansions of §§ 7, 11, 12 with one another.

These tests have all been applied, and I venture to guarantee the accuracy of both expansions.

# ALTERNATIVE EXPRESSIONS FOR PERPETUANT TYPE FORMS

*By* P. W. WOOD.

[Received March 3rd, 1905—Read March 9th, 1905.]

IN dealing with covariants linear in the coefficients of each of the $\delta$ binary quantics $a_{1x}^{n_1}, a_{2x}^{n_2}, \dots, a_{\delta x}^{n_\delta}$, where $n_1, n_2, \dots, n_\delta$ are all infinite, it is known* that the type forms may be taken either as

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$$

or as

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\delta)^{\lambda_{\delta-1}},$$

where $\lambda_1 \geqslant 2^{\delta-2}$, $\lambda_2 \geqslant 2^{\delta-3}$, ..., $\lambda^{\delta-1} \geqslant 1$, and the sequence of the letters is fixed beforehand. The question† arises as to whether there are other expressions equally adapted to the representation of type forms. The present paper deals with this problem and gives a very general class of such expressions.

## 1.

THEOREM.—*Given any* $\lambda$ *distinct partial differential operators* $O_1, O_2, \dots, O_\lambda$ *in the variables* $x_1, x_2, \dots, x_\delta$ *with constant coefficients, then any quantic* $C_x^\omega$ *of order* $\omega$ *in these variables is expressible in the form*

$$C_x^\omega \equiv z_1^{\mu_1} z_2^{\mu_2} \dots z_\kappa^{\mu_\kappa} p_x^{\omega-\lambda} + \Sigma R,$$

*where*

(i.) $\mu_1, \mu_2, \dots, \mu_\kappa$ *are certain positive integers such that their sum is* $\lambda$;

(ii.) $z_1, z_2, \dots, z_\kappa$ *are* $\kappa$ *distinct linear functions of* $x_1, x_2, \dots, x_\delta$, *with the following properties:—If the operators* $O$ *are divided into* $\kappa$ *sets* $S_1, S_2, \dots, S_\kappa$, *then* $(r = 1, 2, \dots, \kappa)$ *the set* $S_r$ *contains* $\mu_r$ *operators* $O$ *such that each of these* $\mu_r$ *operators annihilates each*

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* Grace, *Proc. London Math. Soc.*, Vol. xxxv.

† The existence of alternative expressions which may be taken as type forms has been pointed out by Prof. Elliott in a paper "On an Integration Theorem as to Rational Integral Functions, with the Bearing on the Theory of Forms" (*Quarterly Journal*, 1904, p. 124). The general principles of Prof. Elliott's paper and the present one are similar, but the results of the latter were obtained quite independently of the former.

*of* $z_1, z_2, \ldots, z_{r-1}$, *but no one of these* $\mu_r$ *operators annihilates* $z_r$ (*thus* $z_1$ *may be taken as any linear function annihilated by none of the* $\mu_1$ *operators of the set* $S_1$);

(iii.) $p_x^{\omega-\lambda}$ *is some quantic of order* $(\omega-\lambda)$ *in the variables* $x_1, x_2, \ldots, x_\delta$;

(iv.) *The aggregate of terms* $R$ *is annihilated by the successive application of the* $\lambda$ *operators* $O_1, O_2, \ldots, O_\lambda$.

Consider the partial differential equation

$$(O_1 O_2 \ldots O_\lambda) P_x^\omega = 0, \tag{1}$$

of which the solution is some quantic $P_x^\omega$; it has exactly

$$\binom{\omega+\delta-1}{\delta-1} - \binom{\omega-\lambda+\delta-1}{\delta-1} \equiv M$$

linearly independent quantic solutions $R_1, R_2, \ldots, R_M$ (say).

We proceed to show that the expression

$$z_1^{\mu_1} z_2^{\mu_2} \ldots z_\kappa^{\mu_\kappa} p_x^{\omega-\lambda}$$

is not a solution of the equation (1) unless $p_x^{\omega-\lambda}$ is identically zero.

In the first place, it is easily proved that, if

$$O(PQ) \equiv \left\{ \sum_{r=1}^{r=\delta} a_r \frac{\partial}{\partial x_r} \right\} (PQ) = 0,$$

where $P$ and $Q$ are quantics, then $O(P) = 0$, and $O(Q) = 0$.

Now consider the effect of operating on $z_1^{\mu_1} z_2^{\mu_2} \ldots z_\kappa^{\mu_\kappa} p_x^{\omega-\lambda}$ with one of the operators of the set $S_\kappa$. Since this operator annihilates each of $z_1, z_2, \ldots, z_{\kappa-1}$, but does not annihilate $z_\kappa$, the result is of the form $z_1^{\mu_1} z_2^{\mu_2} \ldots z_\kappa^{\mu_\kappa-1} p_{1_x}^{\omega-\lambda}$, where $p_{1_x}^{\omega-\lambda}$ is not identically zero unless $p_x^{\omega-\lambda}$ is identically zero. Application of another operator of the set $S_\kappa$ gives us $z_1^{\mu_1} z_2^{\mu_2} \ldots z_\kappa^{\mu_\kappa-2} p_{2_x}^{\omega-\lambda}$, where $p_{2_x}^{\omega-\lambda}$ is not identically zero unless $p_{1_x}^{\omega-\lambda}$ is identically zero, that is, unless $p_x^{\omega-\lambda} \equiv 0$: so the application of the $\mu_\kappa$ operators of the set $S_\kappa$ gives us $z_1^{\mu_1} z_2^{\mu_2} \ldots z_{\kappa-1}^{\mu_{\kappa-1}} p_{\mu_{\kappa_x}}^{\omega-\lambda}$, where $p_{\mu_{\kappa_x}}^{\omega-\lambda} \not\equiv 0$, unless $p_x^{\omega-\lambda} \equiv 0$. If we proceed in the same way, by applying successively the operators of the sets $S_{\kappa-1}, S_{\kappa-2}, \ldots, S_2, S_1$, we obtain as our final result some quantic $\bar{p}_x^{\omega-\lambda}$, which is not identically zero unless $p_x^{\omega-\lambda} \equiv 0$.

The terms of the expression $z_1^{\mu_1} z_2^{\mu_2} \ldots z_\kappa^{\mu_\kappa} p_x^{\omega-\lambda}$ are linearly independent among themselves, and we have chosen our expressions $R_1, R_2, \ldots, R_M$ so that they are linearly independent among themselves. Hence there can be no linear relation of the form

$$z_1^{\mu_1} z_2^{\mu_2} \dots z_\kappa^{\mu_\kappa} p_x^{\omega-\lambda} + \sum_{r=1}^{r=M} R_r \equiv 0, \tag{2}$$

unless $p_x^{\omega-\lambda}$ is identically zero, and the terms $\Sigma R$ are all absent; but any quantic of order $\omega$ in $\delta$ variables is expressible in terms of $\binom{\omega+\delta-1}{\delta-1}$ linearly independent forms of the same order, and therefore any quantic $C_x^\omega$ is expressible in the form

$$C_x^\omega \equiv z_1^{\mu_1} z_2^{\mu_2} \dots z_\kappa^{\mu_\kappa} p_x^{\omega-\lambda} + \sum_{r=1}^{r=M} R_r;$$

for the right-hand side contains

$$\binom{\omega+\lambda+\delta-1}{\delta-1} + M = \binom{\omega+\delta-1}{\delta-1}$$

forms, which we have proved to be linearly independent.*

If we denote by $P_{r_x}^\omega$ the most general quantic annihilated by $O_r$ ($r = 1, 2, \dots, \lambda$), we can replace our terms $\sum_{r=1}^{r=M} R_r$ by $\sum_{r=1}^{r=\lambda} P_{r_x}^\omega$; but these last terms are not linearly independent; there will, in fact, be

$$\lambda \binom{\omega+\delta-2}{\delta-2} - \binom{\omega+\delta-1}{\delta-1} + \binom{\omega-\lambda+\delta-1}{\delta-1}$$

linear relations connecting them.

In the application to perpetuants, which we make immediately, it will appear that the determination of the relations connecting solutions of the equation $(O_1 O_2 \dots O_\lambda) P_x^\omega = 0$ is substantially the same problem as the determination of the syzygies connecting products of perpetuants.†

## 2. *Application to Binary Perpetuants.*

Consider any quantic $Q$ in the variables $x_1, x_2, \dots, x_\delta$ which is annihilated by the operator

$$O \equiv \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \frac{\partial}{\partial x_{r^{(3)}}} + \dots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}},$$

where $r^{(1)}, r^{(2)}, \dots, r^{(\epsilon)}$ are any $\epsilon$ of the suffixes $1, 2, 3, \dots, \delta$ such that

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* It can be easily proved that this expression for $C_x^\omega$ is unique, when we have determined the expression $z_1^{\mu_1} z_2^{\mu_2} \dots z_\kappa^{\mu_\kappa}$ and the $M$ linearly independent quantic solutions $R_1, R_2, \dots, R_M$ of the equation $(O_1 O_2 \dots O_\lambda) P_x^\omega = 0$.

† Cf. Young and Wood, "On Perpetuant Syzygies," *Proc. London Math. Soc.*, Ser. 2, Vol. 2.

$r^{(1)} < r^{(2)} < r^{(3)} < \ldots < r^{(\epsilon)}$. In this quantic $Q$ make the substitutions

$$x_s \equiv \frac{(a_s a_{s+1})}{a_{s_z} a_{s+1_z}} \Pi; \quad (s = 1, 2, \ldots, \delta),$$

where $\Pi \equiv a_{1_z} a_{2_z} \ldots a_{\delta+1_z}$ and $a_1, a_2, \ldots, a_{\delta+1}$ are the symbols of $(\delta+1)$ binary quantics of infinite order in the variables $z_1, z_2$: finally, multiply by the proper (infinite) number of factors $a_{1_z}, a_{2_z}, \ldots, a_{\delta_z}, a_{\delta+1_z}$, so that, for the values $1, 2, \ldots, \delta+1$ of $r$, the number of symbols $a_r$ is equal to the (infinite) order of the binary quantic to which $a_r$ refers. Then, by virtue of $Q$ being annihilated by $O$, it is known* that the resulting expression is equal to a linear function of terms each of which is a product of covariants of lower total degrees (cf. § 7).

The operator $O$ is completely determined by the suffixes $r^{(1)}, r^{(2)}, \ldots, r^{(\epsilon)}$ involved: the number of such operators is therefore $(2^\delta - 1)$. Moreover,† if in any quantic which is annihilated by the successive application of these $(2^\delta - 1)$ operators we make the preceding substitutions, the resulting expression is a covariant of total degree $(\delta+1)$, which is a sum of products of covariants of lower total degrees.

## 3.

In the theorem of § 1 let us take as our operators $O_1, O_2, \ldots, O_\lambda$ the $(2^\delta - 1)$ operators defined in § 2. We divide these $(2^\delta - 1)$ operators into $\delta$ sets in such a way that the set $S_r$ contains those $2^{\delta-r} (= \mu_r)$ operators, each involving the variable $x_r$ and some, all, or none of the variables $x_{r+1}, x_{r+2}, \ldots, x_\delta$. It is clear that with this arrangement we can take $x_r = z_r$ for the values $1, 2, \ldots, \delta$ of $r$: for no operator of the set $S_r$ annihilates $x_r$, and each operator of this set annihilates each of $x_1, x_2, \ldots, x_{r-1}$. The theorem of § 1 may therefore be enunciated in the less general form :—

Any quantic $C_x^\omega$ of order $\omega$ in the variables $x_1, x_2, \ldots, x_\delta$ is expressible in the form

$$C_x^\omega \equiv x_1^{2^{\delta-1}} x_2^{2^{\delta-2}} \ldots x_{\delta-1}^2 x_\delta \cdot p_x^{\omega-(2^\delta-1)} + \sum_{r=1}^{r=2^\delta-1} p_{r_x}^\omega,$$

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* Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 481.

† Wood, "On the Reducibility of Covariants," *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 354.

where $p^{\infty}_{r_x}$ is any quantic annihilated by the typical operator

$$\frac{\partial}{\partial x_{r(1)}} - \frac{\partial}{\partial x_{r(2)}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r(\epsilon)}}.$$

If we substitute for the variables $x$, we obtain immediately Grace's Perpetuant Type Theorem : it should be noticed that this method of establishing the result ensures its *exactness* in the sense that no linear function of the type forms is expressible in terms of product forms. This principle holds good for all the theorms of this paper which are deduced from the theorem of § 1.

### 4.

We shall next extend our definition of reducibility and prove, by means of the theorem of § 1, an extension of Grace's Perpetuant Type Theorem.*

Let the symbol $[a_{r_1} a_{r_2} \ldots a_{r_l}]$ denote any covariant involving the symbolical letters $a_{r_1}, a_{r_2}, \ldots, a_{r_l}$ : and let the symbol $[a_{r_1} a_{r_2} \ldots a_{r_l}]'$ denote any covariant containing all the symbolical letters $a_1, a_2, \ldots, a_{s+1}$ except $a_{r_1}, a_{r_2}, \ldots, a_{r_l}$. We shall define as reducible any transvectant

$$([a_{r_1} a_{a_2} \ldots a_{r_l}],\ [a_{r_1} a_{r_2} \ldots a_{r_l}]')^{\lambda},$$

if $\lambda < \mu_{r_1 r_2 \ldots r_l}$, and we may suppose that $a_1$ always appears in

$$[a_{r_1} a_{r_2} \ldots a_{r_l}]'$$

The number of such quantities $\mu$ is $2^s-1$ : we take $r_1 < r_2 < \ldots < r_l$ and denote by $\sigma_r$ the sum of all the $\mu$'s whose first suffix is $r$ (*e.g.*, $\sigma_{s+1} \equiv \mu_{s+1}$, $\sigma_s \equiv \mu_s + \mu_{s,\,s+1}$). The extension to be established is

> Any perpetuant linear in the coefficients of each of the binary quantics $a_{1_x}^{n_1}, a_{2_x}^{n_2}, \ldots, a_{s+1_x}^{n_{s+1}}$ of infinite order is expressible in terms of
>
> (i.) symbolical products
>
> $$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_s a_{s+1})^{\lambda_s},$$
>
> where $$\lambda_1 \geqslant \sigma_2,\ \lambda_2 \geqslant \sigma_3,\ \ldots,\ \lambda_s \geqslant \sigma_{s+1} ;$$
>
> (ii.) covariants defined above as reducible.

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* Young and Wood, "On Perpetuant Syzygies," *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 224.

The theorem of § 1 is still true if we suppose that the operator $O_r$ occurs $\nu_r$ times ($r = 1, 2, \ldots, \lambda$): in this case we may replace our terms $\sum_{r=1}^{r=M} R_r$ by the terms $\sum_{r=1}^{r=\lambda} \kappa_{r_x}^{\nu_r - 1} p_{r_x}^{\omega-(\nu_r-1)}$, where $p_{r_x}^{\omega-(\nu_r-1)}$ is the most general quantic annihilated by $O_r$, and $\kappa_{r_x}^{\nu_r-1}$ is the most general quantic *not* annihilated by $O_r$.

Writing suffixes in the place of symbols, we have defined the transvectant

$$\{[r^{(1)}+1, \ldots, r^{(2)}, r^{(3)}+1, \ldots, r^{(4)}, \ldots], [r^{(1)}+1, \ldots, r^{(2)}, r^{(3)}+1, \ldots, r^{(4)}, \ldots]'\}^{\lambda}$$

as reducible if
$$\lambda < \mu_{r^{(1)}+1, \ldots, r^{(2)}, r^{(3)}+1, \ldots; r^{(4)}, \ldots},$$
$$< \varpi, \text{ say.}$$

It is easily shewn * that all terms such as

$$\kappa_x^{\varpi-1} p_x^{\omega-(\varpi-1)},$$

where $p_x^{\omega-(\varpi-1)}$ is annihilated by $O \equiv \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}}$, yield on substitution only forms which, by virtue of the above definition, are reducible: and the same is true for all such terms similarly derived from the remaining operators $O$.

Now let the operators of the theorem of § 1 be the $(2^\delta - 1)$ operators used already, and let the operator

$$\frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \frac{\partial}{\partial x_{r^{(3)}}} - \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}}$$

occur $\mu_{r^{(1)}+1, \ldots, r^{(2)}, r^{(3)}+1, \ldots, r^{(4)}, \ldots}$ times, while each of the other operators occurs a corresponding number of times. Divide the operators into $\delta$ sets in such a way that, for $r = 1, 2, \ldots, \delta$, the set $S_r$ contains all those operators each of which involves $x_r$, and some, all, or none of $x_{r+1}, x_{r+2}, \ldots, x_\delta$: the number of operators in the set $S_r$ is clearly $\sigma_{r+1}$: as before, with this arrangement, we can take $x_r = z_r$ for $r = 1, 2, \ldots, \delta$. Hence the theorem of § 1 leads to the

THEOREM.—*Any quantic $C_x^\omega$ of order $\omega$ in the variables $x_1, x_2, \ldots, x_\delta$ is expressible in the form*

$$C_x^\omega \equiv x_1^{\sigma_2} x_2^{\sigma_3} \ldots x_\delta^{\sigma_{\delta+1}} . p_x^{\omega - \sum_{r=2}^{r=\delta+1} \sigma_r} + \sum_{r=1}^{r=2^\delta-1} p_{r_x}^\omega,$$

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* Wood, "On the Reducibility of Covariants," *loc. cit.*, p. 356.

*where* $p_z^w$ *is any quantic annihilated by applying the operator*

$$\frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \ldots + (-)^{e-1} \frac{\partial}{\partial x_{r^{(e)}}}$$

*exactly* $\mu_{r^{(1)}+1, \ldots, r^{(2)}, r^{(3)}+1, \ldots, r^{(4)}, \ldots}$ *times.*

On substitution for the variables $x$, we obtain the required extension of Grace's Perpetuant Type Theorem. As before, the method of establishing the result ensures its exactness.

## 5. *Alternative Expressions for Type Forms.*

The problem of determining such expressions may be attacked in two ways.

(i.) We may first find the linear forms $z$ and then seek for the corresponding grouping of the operators into sets.

(ii.) We may group the operators into sets and then seek for the linear forms $z$ defined by certain facts of annihilation or non-annihilation by the operators of the various sets.

The first method is employed in this paper, and we shall shew immediately that the linear forms $z$ may be taken to be the variables $x_1, x_2, \ldots, x_\delta$ in any order. For, let $y_1, y_2, \ldots, y_\delta$ be the variables $x_1, x_2, \ldots, x_\delta$ in any order and, for the values $1, 2, \ldots, \delta$ of $r$, let the set $S_r$ contain the $2^{\delta-r}$ operators, each of which involves the variable $y_r$ and some, all, or none of the variables $y_{r+1}, y_{r+2}, \ldots, y_\delta$: it is clear that we may put $y_r = z_r$ $(r = 1, 2, \ldots, \delta)$ and satisfy all the necessary conditions.

Before seeking for more general classes of expressions adapted to the representation of type forms, it is necessary to consider the relations between our $(2^\delta - 1)$ operators and linear functions of the variables each consisting of the sum of consecutive variables, that is, linear functions such as

$$x_{r+1} + x_{r+2} + \ldots + x_{r+\theta}.$$

## 6.

It is easily proved that, for all values of $r$ and $\theta$, the linear expression

$$x_{r+1} + x_{r+2} + \ldots + x_{r+\theta}$$

is annihilated by each of $(2^{\delta-1} - 1)$ of the operators, and is not annihilated by any one of the remaining $2^{\delta-1}$ operators.

Any operator annihilating this expression must, in fact, contain an even number or none of the variables $x_{r+1}, x_{r+2}, \ldots, x_{r+\theta}$. The number of ways of choosing an even number or none of these $\theta$ variables is $2^{\theta-1}$; the number of ways of choosing some, all, or none of the $(\delta-\theta)$ variables $x_1, x_2, \ldots, x_{r-1}, x_{r+\theta+1}, \ldots, x_\delta$ is $2^{\delta-\theta}$: and therefore, rejecting the case where no variables at all are chosen from either set, we see that the required number of operators is

$$2^{\theta-1} . 2^{\delta-\theta}-1 = 2^{\delta-1}-1.$$

Obviously no other operator of our set can annihilate the expression, and therefore there are *exactly* $(2^{\delta-1}-1)$ of our operators each of which annihilates the expression.

## 7.

LEMMA I.—If we are given $\delta$ linearly independent linear expressions $a_1, a_2, \ldots, a_m, \beta_1, \beta_2, \ldots, \beta_{\delta-m}$, each consisting of a sum of consecutive variables, then, if $m$ have any of the values 0, 1, 2, ..., $\delta-1$, there is a unique operator among our $(2^\delta-1)$ operators, which annihilates each of $a_1, a_2, \ldots, a_m$ and annihilates no one of $\beta_1, \beta_2, \ldots, \beta_{\delta-m}$.

For, consider a typical operator

$$O \equiv \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}}$$

of our set: it is known[*] that any linear expression annihilated by it is a linear function of

$$y_1, y_2, \ldots, y_{\delta-1},$$

where

$$\left\{ \begin{array}{l} y_{r^{(\theta)}} = x_{r^{(\theta)}} + x_{r^{(\theta)}+1} + \ldots + x_{r^{(\theta+1)}}, \quad (\theta = 1, 2, 3, \ldots, \epsilon-1), \\ \left. \begin{array}{l} y_\kappa = x_\kappa, \quad \text{if} \quad \kappa < r^{(\epsilon)}, \\ y_{\kappa-1} = x_\kappa, \quad \text{if} \quad \kappa > r^{(\epsilon)}, \end{array} \right\} \quad (\kappa \neq r^{(\theta)}, \quad \theta = 1, 2, 3, \ldots, \epsilon). \end{array} \right\},$$

Also any linear expression can be expressed in terms of $y_1, y_2, \ldots, y_{\delta-1}$ and $x_{r^{(1)}}$.

Now take any $\delta$ linearly independent linear expressions $\pi_1, \pi_2, \ldots, \pi_\delta$, each consisting of a sum of consecutive variables, and express them in terms of $y_1, y_2, \ldots, y_{\delta-1}$ and $x_{r^{(1)}}$, each coefficient being positive or negative unity or zero. Suppose each of $\pi_1, \pi_2, \ldots, \pi_\mu$ involves $x_{r^{(1)}}$, while each of $\pi_{\mu+1}, \pi_{\mu+2}, \ldots, \pi_\delta$ involves only $y_1, y_2, \ldots, y_{\delta-1}$. Then $O$ annihilates each of $\pi_{\mu+1}, \pi_{\mu+2}, \ldots, \pi_\delta$ and annihilates no one of

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[*] Wood, "On the Irreducibility of Perpetuant Types," *loc. cit.*, p. 481.

$\pi_1, \pi_2, \ldots, \pi_\mu$, and, since $\pi_1, \pi_2, \ldots, \pi_\delta$ are linearly independent, $\mu$ must have one of the values $1, 2, \ldots, \delta$.

Conversely, if we are given $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_{\delta-m}$, we can find linear expressions $y'_1, y'_2, \ldots, y'_{\delta-1}$ of similar nature, and so find the corresponding variables $x_{\bar{\tau}(1)}, x_{\bar{\tau}(2)}, \ldots, x_{\bar{\tau}(\bar{\epsilon})}$, which uniquely determine an operator

$$\bar{O} \equiv \frac{\partial}{\partial x_{\bar{\tau}(1)}} - \frac{\partial}{\partial x_{\bar{\tau}(2)}} + \ldots + (-)^{\bar{\epsilon}-1} \frac{\partial}{\partial x_{\bar{\tau}(\bar{\epsilon})}}$$

of the required nature.

8. Lemma II.—If we are given $(m+n)$ linearly independent linear expressions $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n$, each consisting of a sum of consecutive variables $x_1, x_2, \ldots, x_\delta$, then [for the values $0, 1, 2, \ldots, \delta-1$ of $m$, and $1, 2, \ldots, \delta$ of $n$; $(m+n) \leqslant \delta$] there are exactly $2^{\delta-m-n}$ operators among our set of $2^\delta-1$, such that each of them annihilates each of $\alpha_1, \alpha_2, \ldots, \alpha_m$ and annihilates no one of $\beta_1, \beta_2, \ldots, \beta_n$.

First consider the case of two variables: the only linear expressions possible are 1 or 2 of $x_1, x_2, x_1+x_2$, while the operators are $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_1 - \partial/\partial x_2$. The lemma is easily verified in this case: thus

(i.) $\partial/\partial x_1 - \partial/\partial x_2$ annihilates $x_1+x_2$, but neither $x_1$ nor $x_2$.

(ii.) $\partial/\partial x_1$ annihilates $x_2$, but neither $x_1$ nor $x_1+x_2$.

(iii.) $\partial/\partial x_2$ annihilates $x_1$, but neither $x_2$ nor $x_1+x_2$.

Now, assuming the lemma true for $\delta$ variables, let us consider the case of $(\delta+1)$ variables: suppose the linear forms in $(\delta+1)$ variables are $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n$; we have to find the number of operators annihilating every $A$ and no $B$. If $(m+n) = \delta+1$, the lemma is the same as the preceding one; if $(m+n) \leqslant \delta$, we suppose that

$$A_r = A'_r + x_{\delta+1}, \quad (r = 1, 2, \ldots, \theta),$$

$$B_r = B'_r + x_{\delta+1}, \quad (r = 1, 2, \ldots, \phi),$$

and that no one of $A_{\theta+1}, A_{\theta+2}, \ldots, A_m, B_{\phi+1}, B_{\phi+2}, \ldots, B_n$ involves $x_{\delta+1}$; $\theta$ and $\phi$ may have any values from zero to $m$ or $n$ respectively.

Now $A'_1, A'_2, \ldots, A'_\theta, A_{\theta+1}, A_{\theta+2}, \ldots, A_m, B'_1, B'_2, \ldots, B'_\phi, B_{\phi+1}, B_{\phi+2}, \ldots, B_n$ are $(m+n)$ linearly independent sums of consecutive variables $x_1, x_2, \ldots, x_\delta$, and so, by hypothesis, there are $2^{\delta-(m+n)}$ operators involving only $x_1, x_2, \ldots, x_\delta$, each annihilating each of $A'_1, A'_2, \ldots, A_\theta, A_{\theta+1}, A_{\theta+2}, \ldots, A_m$, and no one of $B'_1, B'_2, \ldots, B'_\phi, B_{\phi+1}, B_{\phi+2}, \ldots, B_n$, and therefore there are

$2^{\delta-(m+n)}$ operators involving only $x_1, x_2, \ldots, x_\delta$, each annihilating each of $A_1, A_2, \ldots, A_m$, and no one of $B_1, B_2, \ldots, B_n$.

Again there are $2^{\delta-(m+n)}$ operators $\Theta$ involving only $x_1, x_2, \ldots, x_\delta$, such that every $\Theta$ annihilates each of $A_{\theta+1}, A_{\theta+2}, \ldots, A_m, B'_1, B'_2, \ldots, B'_\phi$ and annihilates no one of $A'_1, A'_2, \ldots, A'_\theta, B_{\phi+1}, B_{\phi+2}, \ldots, B_n$; if to each of these operators $\Theta$ we add a term $\pm\,\partial/\partial x_{\delta+1}$ so as to give another operator of our set of $(2^{\delta+1}-1)$ operators, then the resulting operator $(\Theta \pm \partial/\partial x_{\delta+1})$ enjoys the property that it annihilates each of $A_1, A_2, \ldots, A_m$ and no one of $B_1, B_2, \ldots, B_n$. In this way we obtain $2^{\delta-(m+n)}$ operators involving $x_{\delta+1}$, and these are necessarily distinct from those already obtained. Hence altogether we find $2^{\delta+1-(m+n)}$ operators of the required nature.

The lemma, being true for two variables, is therefore true universally.

9. In the theorems of § 1 let us take $\kappa = \delta$, and $z_1, z_2, \ldots, z_\delta$ as *any* $\delta$ linearly independent sums of consecutive variables. From the preceding lemmas it follows that:—

> (1) There is a unique operator annihilating each of $z_1, z_2, \ldots, z_{\delta-1}$ but not $z_\delta$.
>
> (2) There are two operators annihilating each of $z_1, z_2, \ldots, z_{\delta-2}$ but not $z_{\delta-1}$, and these operators must be different from the operator of (1).
>
> (3) There are four operators annihilating each of $z_1, z_2, \ldots, z_{\delta-3}$ but not $z_{\delta-2}$, and these operators must be different from the operators of (1) and (2);

and so on.

Finally there are $2^{\delta-2}$ operators annihilating $z_1$ but not $z_2$ (these are certainly not included among the operators of previous sets all of which annihilate $z_2$), and there remain $2^{\delta-1}$ operators not annihilating $z_1$; for we have seen that there are exactly $(2^{\delta-1}-1)$ operators annihilating $z_1$ (§ 6), and these are all included in the previous sets.

Hence, with these choices for $z_1, z_2, \ldots, z_\delta$, it is possible to group the operators so as to satisfy all the necessary conditions of the theorem of § 1.

10. On substituting for the variables $x$, we obtain the following result, which includes Grace's Perpetuant Type Theorem as a special case:—

> Any perpetuant linear in the coefficients of each of the binary quantics $a_{1x}^{n_1}, a_{2x}^{n_2}, \ldots, a_{\delta x}^{n_\delta}$ of infinite orders is linearly expressible in

terms of

(i.) Symbolical products of the form

$$(b_1 b_2)^{\lambda_1} (b_3 b_4)^{\lambda_2} \dots (b_{2\delta-3} b_{2\delta-2})^{\lambda_{\delta-1}},$$

where $\lambda_1 \geqslant 2^{\delta-2}$, $\lambda_2 \geqslant 2^{\delta-3}$, ..., $\lambda_{\delta-1} \geqslant 1$, and $b_1, b_2, \dots, b_{2\delta-2}$ are any of the letters $a_1, a_2, \dots, a_\delta$ chosen so that, for all values of $\kappa$, it is impossible to find $\kappa$ symbolical determinants involving $\kappa$ or fewer of the symbols $a_1, a_2, \dots, a_\delta$.

(ii.) Products of perpetuants of lower total degrees.

The extension of the theorem of § 4 may be effected in a similar manner, provided we modify suitably our definitions of reducibility as determined by certain transvectants.

# THEOREMS ON THE LOGARITHMIC POTENTIAL

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[Read March 9th, 1905.—Received March 16th, 1905.]

THE following investigation contains, in § 2, conditions, necessary and sufficient, for the existence of the second derivates of the logarithmic potential produced by a surface distribution at points within the area which carries the surface density. The surface density is assumed to be generally continuous, the discontinuities occurring along regular curves, but no assumption is made as to the existence of first derivates of the density. As a preliminary, an investigation is given, in § 1, of the first derivates of the potential; this contains no essential novelty, although it does not appear in most of the ordinary text-books.

The corresponding problem for the first derivates of the logarithmic potential of a line distribution, at a point on the curve carrying the density, is discussed in § 4, and these are compared, in § 5, with the imiting values of the derivates at a point which is made to approach he curve along a path (not touching the curve). For the work in these paragraphs we need a lemma of the integral calculus, given in § 3.

The mode of investigation used in §§ 2, 4, 5 is suggested by the papers of Petrini[*] on the corresponding problems of the Newtonian potential, but many changes in details have been made. Although, as may be anticipated, there is considerable similarity between Petrini's results and hose given below, yet there are some unexpected points of difference.

In § 6 the results are compared with those of previous writers, and in § 7 an indication is given as to certain simplifications possible when the density is continuous.[†]

## 1. *The First Derivates of the Logarithmic Potential of a Surface Distribution.*

If $V$ is the logarithmic potential at $(x, y)$ of a surface distribution $\rho$ over a plane area $S$, we have[‡]

$$V = -2\gamma \iint \rho \log r \, d\xi \, d\eta,$$

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[*] *Öfversigt af K. Vet. Akad. Förh. Stockholm*, Bd. LVII., 1900, pp. 225 and 867.

[†] §§ 6 and 7 have been added to the paper as originally written.—*April 30th*, 1905.

[‡] The constants are chosen so that $\rho$ may represent the volume density of the equivalent ylindrical distribution, attracting according to the Newtonian law, $\gamma$ being the gravitation onstant.

where $r^2 = (x-\xi)^2+(y-\eta)^2$, and the integral is to be extended over the whole of the area $S$. Thus, if $V'$ is the potential at $(x+h, y)$,

$$V' = -2\gamma \iint \rho \log r' \, d\xi \, d\eta,$$

where $r'^2 = (x+h-\xi)^2+(y-\eta)^2$.

Then, if $(x, y)$ is external to the area $S$, there is no difficulty in seeing that the equation

$$\lim_{h=0} (V'-V)/h = 2\gamma \iint [\rho(\xi-x)/r^2] \, d\xi \, d\eta$$

is correct; in fact, $(\log r' - \log r)/h$ converges to the value $(x-\xi)/r^2$ as $h$ tends to zero, and the convergence is easily seen to be uniform for all points $(\xi, \eta)$ of the area $S$.

But the truth of the equation is not quite so readily established in case $(x, y)$ falls within the area $S$; for then the convergence of the above expression to its limit is no longer uniform throughout the area.

To deal with this case we proceed on the lines of the proof given by Bouquet for the corresponding three-dimensional case.[*] Plainly, if the theorem can be established for a small circular area, with its centre at $(x, y)$, it will be true in general; for the preceding result can be applied to the portion of the area which falls outside the circle.

Now
$$\frac{V'-V}{h} = 2\gamma \iint \rho \left[\frac{1}{h} \log \frac{r}{r'}\right] d\xi \, d\eta,$$

and denote by $X$ the integral

$$2\gamma \iint \rho \frac{\xi - x}{r^2} \, d\xi \, d\eta,$$

both being extended over the circular area. It is easily seen that the integral $X$ is convergent (and, indeed, absolutely convergent) by expressing the integral in polar coordinates $r$, $\theta$. Now, since $|r'-r| \leqslant |h|$, it follows that[†]

$$\left|\log \frac{r}{r'}\right| < \tfrac{1}{2} \left|\frac{r}{r'} - \frac{r'}{r}\right| < \tfrac{1}{2} |h| \left(\frac{1}{r} + \frac{1}{r'}\right).$$

Hence
$$\left|\frac{V'-V}{h}\right| < \gamma \iint |\rho| \left(\frac{1}{r} + \frac{1}{r'}\right) d\xi \, d\eta.$$

Now $\iint (|\rho|/r) \, d\xi \, d\eta \leqslant 2\pi a M$, if $a$ is the radius of the circle and $M$ is the upper limit of $|\rho|$ (supposed *finite*) within the circle. Also $\iint (|\rho|/r') \, d\xi \, d\eta$ is less than $\iint (M/r') \, d\xi \, d\eta$ extended over a circle of radius

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[*] See, for instance, Picard's *Traité d'Analyse* (t. I., p. 161); the earliest place in which the proof occurs seems to be the first edition of Briot's *Théorie Mécanique de la Chaleur* (p. 194).

[†] For, if $p = \log q$, then $q - 1/q = 2 \sinh p$, or $|q - 1/q| > 2|p|$, the numbers being supposed *real*.

$a+|h|$ with centre at $(x+h, y)$; and this gives

$$\iint (|\rho|/r')\, d\xi\, d\eta < 2\pi(a+|h|)M < 4\pi a M, \quad \text{if} \quad |h| < a;$$

so that

$$|(V'-V)h^{-1}| < 6\pi\gamma a M.$$

Next consider the integral $X$; we have

$$|X| < 2\gamma \iint (|\rho|/r)\, d\xi\, d\eta$$

or

$$|X| < 4\pi\gamma a M.$$

By combining these two results we find

$$|h^{-1}(V'-V)-X| < 10\pi\gamma a M.$$

If, therefore, $\epsilon$ is any assigned small positive number, let $a$ be chosen so that $a \leqslant \frac{1}{10}\epsilon/\pi\gamma M$; and this can be done, because $M$, or at least a number greater than $M$, can be fixed beforehand.

Then $|h^{-1}(V'-V)-X| < \epsilon$, if $|h| < a$, and consequently

$$\lim_{h=0} (V'-V)/h = X,$$

which is the desired result.*

It may be noticed incidentally that the condition, used above, that $|\rho|$ is to have a finite upper limit in the part of the area $S$ which is in the neighbourhood of the point $(x, y)$ is by no means necessary, and it is easy to see that the final conclusion is correct provided that the two integrals

$$\iint (\rho/r)\, d\xi\, d\eta, \qquad \iint (\rho/r')\, d\xi\, d\eta$$

are convergent.

## 2. *The Second Derivates of the Logarithmic Potential of a Surface Distribution.*

If the density $\rho$ is always less in absolute value than a fixed number, we have seen (§ 1) that the first derivates of the potential are given by the integrals

$$\frac{\partial V}{\partial x} = 2\gamma \iint \rho \frac{\xi - x}{r^2}\, d\xi\, d\eta, \qquad \frac{\partial V}{\partial y} = 2\gamma \iint \rho \frac{\eta - y}{r^2}\, d\xi\, d\eta,$$

and that these results are correct, whether $(x, y)$ is internal or external to the distribution.

Thus, to find $\partial^2 V/\partial x^2$, we have to consider the limit (should it exist)

$$\lim_{h=0} \frac{2\gamma}{h} \iint \rho \left[\frac{\xi - x - h}{r'^2} - \frac{\xi - x}{r^2}\right] d\xi\, d\eta,$$

where $r'^2 = (\xi - x - h)^2 + (\eta - y)^2$.

---

* Harnack (*Theorie des logarithmischen Potentiales*, Kap. 1, § 5) has obtained this result, using the method of integrating $X$, instead of differentiating $V$.

If $(x, y)$ is not an internal point, there is no difficulty in proving (by the aid of uniform convergence or otherwise) that the second derivate is obtained by differentiating under the integral sign, so that

$$\frac{\partial^2 V}{\partial x^2} = 2\gamma \iint \rho \frac{(\xi - x)^2 - (\eta - y)^2}{r^4} d\xi\, d\eta,$$

with similar values for the other second derivates.

But the discussion is entirely different if $(x, y)$ is an internal point, and it will be necessary to postulate some further restriction on $\rho$, in addition to the condition that $|\rho|$ is to have a finite upper limit. In order to discuss the second derivates at the edge of the distribution, we shall not restrict $\rho$ to be continuous in the neighbourhood of $(x, y)$, but *it will be assumed that $\rho$ is continuous on each radius through* $(x, y)$. Thus, if we write

$$\xi - x = r \cos\theta, \qquad \eta - y = r \sin\theta,$$

we shall assume that the limit $\lim\limits_{r=0} \rho$ gives a definite function of $\theta$; and if $\rho$ happens to be completely continuous, the function of $\theta$ will reduce to a constant, in which case the method of § 7 can be used.

We may restrict the distribution to lie within a circle, with centre $(x, y)$, of an arbitrarily small radius $a$; for the second derivates of the potential due to the distribution outside the circle can be calculated by differentiating under the integral sign. Now the integral to be considered is absolutely convergent, and may therefore be written in the form

$$\frac{2\gamma}{h} \int_0^{2\pi} d\theta \int_0^a \rho \left[ \frac{r^2 \cos\theta - hr}{r^2 - 2hr\cos\theta + h^2} - \cos\theta \right] dr.$$

For the sake of definiteness, suppose $h$ to be *positive* and less than $a$; the necessary modifications when $h$ is negative will be recognized without difficulty. Then write $r = ht$, and the integral becomes

$$2\gamma \int_0^{2\pi} d\theta \int_0^{a/h} \rho \left( \frac{t^2 \cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta \right) dt,$$

where it is to be remembered that $\rho$ is a function of $ht$ and $\theta$. But this is the same as $A + B + C$, where

$$A = 2\gamma \int_0^{2\pi} d\theta \int_0^1 \rho \left( \frac{t^2 \cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta \right) dt,$$

$$B = 2\gamma \int_0^{2\pi} d\theta \int_1^{a/h} \rho \left( \frac{t^2 \cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta - \frac{\cos 2\theta}{t} \right) dt,$$

$$C = 2\gamma \int_0^{2\pi} \cos 2\theta\, d\theta \int_1^{a/h} (\rho/t)\, dt = 2\gamma \int_0^{2\pi} \cos 2\theta\, d\theta \int_h^a (\rho/r)\, dr.$$

Write now $f(\theta) = \lim_{r=0} \rho$, and then there is no difficulty in seeing that

$$\lim_{h=0} A = 2\gamma \int_0^{2\pi} f(\theta)\, d\theta \int_0^1 \left( \frac{t^2 \cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta \right) dt,$$

because* $$\left| \frac{t^2 \cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta \right| < (t^2 - 2t\cos\theta + 1)^{-\frac{1}{2}} + 1$$

and $$\int_0^1 [(t^2 - 2t\cos\theta + 1)^{-\frac{1}{2}} + 1]\, dt = 1 + \log(1 + \operatorname{cosec} \tfrac{1}{2}\theta).$$

Now, since the integral $\int_0^{2\pi} [1 + \log(1 + \operatorname{cosec} \frac{1}{2}\theta)]\, d\theta$ is convergent, it is clear that the value given for $\lim_{h=0} A$ will be certainly correct if the function $\rho$ approaches its limit $f(\theta)$ uniformly for all values of $\theta$ from 0 to $2\pi$.

To deal with $B$, let the integration with respect to $t$ be taken in two stages, from 1 to 2 and from 2 to $a/h$ (assuming, as we may, that $h < \frac{1}{2}a$); then, if for the present we write

$$F(t, \theta) = \frac{t^2 \cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta - \frac{\cos 2\theta}{t},$$

we have $$\frac{B}{2\gamma} = \int_0^{2\pi} d\theta \int_1^2 \rho F(t, \theta)\, dt + \int_0^{2\pi} d\theta \int_2^{a/h} \rho F(t, \theta)\, dt.$$

By an argument exactly similar to that which was used for finding $\lim_{h=0} A$, we see that†

$$\lim_{h=0} \int_0^{2\pi} d\theta \int_1^2 \rho F(t, \theta)\, dt = \int_0^{2\pi} f(\theta)\, d\theta \int_1^2 F(t, \theta)\, dt.$$

Next, we have the identity

$$\int_0^{2\pi} d\theta \int_2^{a/h} \rho F(t, \theta)\, dt - \int_0^{2\pi} f(\theta)\, d\theta \int_2^{\infty} F(t, \theta)\, dt$$

$$= \int_0^{2\pi} d\theta \int_2^{N} [\rho - f(\theta)] F(t, \theta)\, dt + \int_0^{2\pi} d\theta \int_N^{a/h} \rho F(t, \theta)\, dt - \int_0^{2\pi} f(\theta)\, d\theta \int_N^{\infty} F(t, \theta)\, dt,$$

where $N$ is any number subject to the restriction $N < a/h$. Thus

$$\left| \int_0^{2\pi} d\theta \int_2^{a/h} \rho F(t, \theta)\, dt - \int_0^{2\pi} f(\theta)\, d\theta \int_2^{\infty} F(t, \theta)\, dt \right|$$

$$< \int_0^{2\pi} d\theta \int_2^{N} |\rho - f(\theta)|\, |F(t, \theta)|\, dt + M \int_0^{2\pi} d\theta \left[ \int_N^{a/h} |F(t, \theta)|\, dt + \int_N^{\infty} |F(t, \theta)|\, dt \right],$$

where $M$ is the upper limit to $|\rho|$ (and therefore to $|f(\theta)|$ also) within the circle $r = a$.

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* For $|t^2 \cos\theta - t| \leqslant |1 - t\cos\theta| \leqslant (1 - 2t\cos\theta + t^2)^{\frac{1}{2}}$.

† In fact, here

$|F(t, \theta)| < 2[1 + (t^2 - 2t\cos\theta + 1)^{-\frac{1}{2}}]$ and $\int_1^2 [1 + (t^2 - 2t\cos\theta + 1)^{-\frac{1}{2}}]\, dt < 1 + \log(3 \operatorname{cosec} \frac{1}{2}\theta)$.

Now $$F(t, \theta) = \frac{t\cos 3\theta - \cos 2\theta}{t(t^2 - 2t\cos\theta + 1)};$$

so that $$|F(t, \theta)| < \frac{t+1}{t(t-1)^2},$$

and consequently* $$\int_2^N |F(t, \theta)|\,dt < \int_2^\infty \frac{t+1}{t(t-1)^2} < 2,$$

while $$\int_N^{a/h} |F(t, \theta)|\,dt < \int_N^\infty |F(t, \theta)|\,dt < \frac{2}{N-1}.$$

Let $\epsilon$ be any arbitrarily assigned small positive number; then choose $N$ so that $$8\pi M/(N-1) < \tfrac{1}{2}\epsilon, \quad \text{or} \quad N-1 > 16\pi M/\epsilon.$$
Next choose $H$ so that
$$H < a/N, \quad \text{and} \quad |\rho - f(\theta)| < \epsilon/8\pi, \quad \text{if} \quad h < H,$$
for all values of $\theta$ from 0 to $2\pi$, and for all values of $t$ from 2 to $N$. Such a choice of $H$ is possible, because the limit of $\rho$ is $f(\theta)$ when the product $ht$ tends to zero, and the convergence of $\rho$ to its limit $f(\theta)$ has been supposed uniform with respect to $\theta$; and $N$ has been fixed above.

It is now clear that
$$\left| \int_0^{2\pi} d\theta \int_2^{a/h} \rho F(t, \theta)\,dt - \int_0^{2\pi} f(\theta)\,d\theta \int_2^\infty F(t, \theta)\,dt \right|$$
$$< \int_0^{2\pi} d\theta \int_2^N (\epsilon/8\pi)\,|F(t, \theta)|\,dt + 2M\int_0^{2\pi} d\theta \int_N^\infty |F(t, \theta)|\,dt$$
$$< 4\pi(\epsilon/8\pi) + 2M[4\pi/(N-1)] < \epsilon,$$
provided that $h < H$.

Hence $$\lim_{h=0} \int_0^{2\pi} d\theta \int_2^{a/h} \rho F(t, \theta)\,dt = \int_0^{2\pi} f(\theta)\,d\theta \int_2^\infty F(t, \theta)\,dt.$$

If we combine the value found for $\lim\limits_{h=0} A$ with the values obtained from the two parts of $B$, it will be clear that
$$\lim_{h=0} (A+B) = 2\gamma \int_0^{2\pi} f(\theta)\,d\theta \left[ \int_0^1 \left( \frac{t^2\cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta \right) dt \right.$$
$$\left. + \int_1^\infty \left( \frac{t^2\cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta - \frac{\cos 2\theta}{t} \right) dt \right].$$

The integrals with respect to $t$ can be found without difficulty, and they give†

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* The indefinite integral is given by $\int \frac{t+1}{t(t-1)^2}\,dt = -\frac{2}{t-1} + \log\frac{t}{t-1}$.

† If the upper limit of the second integral is $N$ (instead of $\infty$), the result is
$$\tfrac{1}{2}\cos 2\theta \log\left(1 - \frac{2}{N}\cos\theta + \frac{1}{N^2}\right) - \sin 2\theta \arctan\frac{N\sin\theta}{1 - N\cos\theta},$$
where the angle is between 0 and $\pi$ if $\sin\theta$ is positive, between 0 and $-\pi$ if $\sin\theta$ is negative.

$$\int_0^1 \left(\frac{t^2\cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta\right) dt + \int_1^\infty \left(\frac{t^2\cos\theta - t}{t^2 - 2t\cos\theta + 1} - \cos\theta - \frac{\cos 2\theta}{t}\right) dt$$
$$= (\theta - \pi)\sin 2\theta,$$

where $\theta$ is to be taken between 0 and $2\pi$. Thus

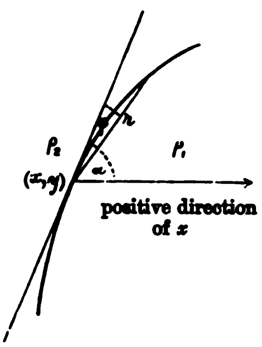
$$\lim_{h=0}(A+B) = 2\gamma \int_0^{2\pi} (\theta - \pi)\sin 2\theta f(\theta)\, d\theta.$$

It is now clear that *the existence of the second derivate depends on the existence of the limit* $\lim_{h=0} C$, *and, if this limit exists, we have*

$$\frac{\partial^2 V}{\partial x^2} = 2\gamma \int_0^{2\pi} (\theta - \pi)\sin 2\theta f(\theta)\, d\theta + \lim_{h=0} 2\gamma \int_0^{2\pi} \cos 2\theta\, d\theta \int_h^a (\rho/r)\, dr.$$

It is to be observed that this second derivate is at present only *one-sided*; that is to say, $h$ has been restricted to be positive. But, as we shall see on p. 352, it is not usually possible to obtain a second derivate which is valid for both signs of $h$, unless $\rho$ is completely continuous, so that $f(\theta)$ is a constant.

*The simplest kind of discontinuous density* occurs when $\rho$ is constant, say $\rho_1$, on one side of a regular curve passing through $(x, y)$; and a different constant, say $\rho_2$, on the other side of the curve.



Suppose that the positive direction of $x$ falls in the $\rho_1$ region: we have then only to consider the value of that part* of the integral $C$ which arises from the space included between the curve and its tangent at $(x, y)$. If we take as the other boundaries two circles of radii $r_1$, $r_2$, the corresponding part of $C$ is numerically less than

$$2\gamma\,|\rho_1 - \rho_2| \int_{r_1}^{r_2} (\phi/r)\, dr \qquad (r_2 > r_1),$$

where $\phi$ is the angle between the radius $r$ and the tangent to the curve at $(x, y)$. But $\lim_{r=0}(2\phi/r)$ is the curvature at $(x, y)$, which we assume to be finite; and consequently the last integral tends to zero with $r_2$, whatever $r_1/r_2$ may be; thus $C$ has a limit when $h$ tends to zero. Further this limit is less than

$$2\gamma\,|\rho_1 - \rho_2| \int_0^a (\phi/r)\, dr,$$

---

* For the rest of the integral is equal to

$$2\gamma(\log a/h)\left[\rho_1 \int_0^\alpha \cos 2\theta\, d\theta + \rho_2 \int_\alpha^{\pi+\alpha} \cos 2\theta\, d\theta + \rho_1 \int_{\pi+\alpha}^{2\pi} \cos 2\theta\, d\theta\right] = 0.$$

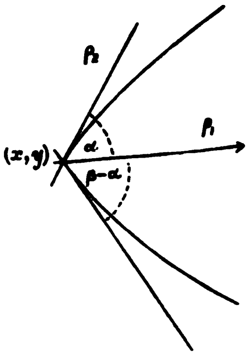
$\phi$ being regarded as always positive; and consequently the limit of $C$ may be made as small as we please by proper choice of $a$. Thus the derivate $\partial^2 V/\partial x^2$ exists and is given effectively by the integral

$$\begin{aligned}2\gamma \int_0^{2\pi} (\theta-\pi) \sin 2\theta f(\theta)\, d\theta &= 2\gamma\left[\rho_1 \int_0^{\alpha} (\theta-\pi) \sin 2\theta\, d\theta + \rho_2 \int_{\alpha}^{\pi+\alpha} (\theta-\pi) \sin 2\theta\, d\theta \right.\\ &\qquad\left. + \rho_1 \int_{\pi+\alpha}^{2\pi} (\theta-\pi) \sin 2\theta\, d\theta\right]\\ &= -2\pi\gamma\rho_1 + \pi\gamma(\rho_1-\rho_2)\cos 2\alpha.\end{aligned}$$

From this result it is plain that, as stated on p. 351, *the derivate is, in general, one-sided only;* for, if the sign of $h$ is changed, we produce the same effect on the derivate as by interchanging $\rho_1$ and $\rho_2$. The new derivate is

$$-2\pi\gamma\rho_2 + \pi\gamma(\rho_2-\rho_1)\cos 2\alpha,$$

which is only equal to the former one if $\cos 2\alpha = 1$, or if the axis of $x$ is parallel to the tangent to the curve bounding the regions $\rho_1$ and $\rho_2$.



However, *if the point $(x, y)$ is at an angle of the boundary formed by two regular curves, even the one-sided derivates do not exist; except for two special directions.* We note first that the curves may be replaced by their tangents (provided that the curvature of each curve is finite); the argument on this point is the same as above (p. 351), and need not be repeated. Then the integral $C$ becomes

$$\begin{aligned}2\gamma \log(a/h)\left[\rho_1 \int_0^{\alpha} \cos 2\theta\, d\theta + \rho_2 \int_{\alpha}^{2\pi+\alpha-\beta} \cos 2\theta\, d\theta + \rho_1 \int_{2\pi+\alpha-\beta}^{2\pi} \cos 2\theta\, d\theta\right]\\ = \gamma \log(a/h)(\rho_1-\rho_2)[\sin 2\alpha + \sin(2\beta-2\alpha)]\\ = 2\gamma \log(a/h)(\rho_1-\rho_2)\sin\beta\cos(\beta-2\alpha),\end{aligned}$$

where $\beta$ is the angle between the two tangents. Hence, excluding the case $\sin\beta = 0$ (which is not essentially different from the case of a single regular curve), we see that $\lim\limits_{h=0} C$ can only exist if $\alpha = \frac{1}{2}\beta \pm \frac{1}{4}\pi$; this gives two special directions, each making an angle of $\frac{1}{4}\pi$ with the bisector of the angle between the two tangents.[*]

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[*] In the above calculation it was tacitly assumed that $\beta$ was equal to $\frac{1}{2}\pi$ at least, but it is easily seen, by a fresh integration, that the result is true even if $\beta$ is less than $\frac{1}{2}\pi$.

This conclusion can be verified for the special case of a right angle by reference to Maxwell's results* on the "geometrical mean distance" of a point and a rectangle; his formula shews that, at a corner of the rectangle, the second derivates exist only for the two directions which coincide with the sides of the rectangle. Care must be taken, however, to notice that the second derivates of the function obtained from Maxwell's formula are *not* precisely the same as those given by the results on the last page; for our results refer only to the potential of the area contained within a small circle of radius $a$, whose centre is at the corner; while Maxwell's calculation relates to the whole rectangle. Still, the two formulæ should give the same *difference* between the derivates taken in opposite directions along the same side of the rectangle; and this appears to be verified, the difference being $2\pi\gamma(\rho_1-\rho_2)$.

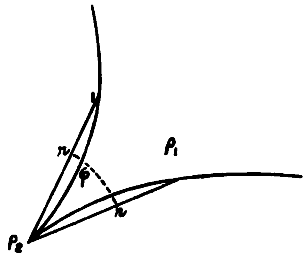
The results obtained for the logarithmic potential naturally lead to the conjecture that some analogous properties hold for the Newtonian potential. It appears, by using Petrini's test (*l.c.*), that, at a conical point on the surface of a solid (of uniform density), the potential has second derivates only for directions which lie on the surface of a quadric cone. For instance, at the corner of a cube, the quadric cone passes through the three edges of the cube which meet there.† Thus the various conjectures which were made by some of the older investigators as to the value of $\Delta V$ at the corner of a cube are now seen to be all meaningless, unless we attach some special kind of definition to the symbol.

*If the point* $(x, y)$ *is an ordinary cusp on the boundary, the second derivates certainly exist.* For it is easy to see that $|C|$ is less than

$$2\gamma\,|\rho_1-\rho_2|\int_h^a (\phi/r)\,dr,$$

as on p. 351. But here $\lim\limits_{r=0}(\phi/r^{\frac{1}{2}})$ is finite, so that we may write $\phi \leqslant Ar^{\frac{1}{2}}$ throughout the integral.

Hence $|C| < 4\gamma A\,|\rho_1-\rho_2|\,[a^{\frac{1}{2}}-h^{\frac{1}{2}}]$, and so $\lim\limits_{h=0} C$ exists, and may be made arbitrarily small by taking $a$ small enough.



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* *Trans. Roy. Soc. Edin.*, Vol. XXVI., 1872, p. 729; *Electricity and Magnetism*, Vol. II., §§ 691–693. The particular result required is (4) in § 692.

† With these edges as axes of coordinates, the equation to the cone is $yz+zx+xy=0$. This equation is obtained by integrating out

$$\iint \left[3\left\{(x\cos\phi+y\sin\phi)\sin\theta+z\cos\theta\right\}^2-(x^2+y^2+z^2)\right]\sin\theta\, d\theta\, d\phi$$

over the appropriate solid angle, and equating the result to zero.

A simple example of a density which is continuous, but which gives a potential without second derivates at the origin, is

$$\rho = (x^2-y^2)\sin\lambda/r^2\lambda,$$

where $r^2 = x^2+y^2$ and $\lambda = [\log(1/r)]^{\frac{1}{2}}$; and, to make the density continuous, we must have $\rho = 0$ at the origin. For then the integral $C$ is equal to

$$2\gamma\int_0^{2\pi}\cos^2 2\theta\, d\theta\int_h^a \frac{\sin\lambda}{r\lambda}dr = 4\pi\gamma(\cos\lambda_1-\cos\lambda_2),$$

where $\lambda_1 = [\log(1/h)]^{\frac{1}{2}}$, $\lambda_2 = [\log(1/a)]^{\frac{1}{2}}$. Hence the limit $\lim\limits_{h=0} C$ is quite indeterminate; and thus the second derivates $\partial^2 V/\partial x^2$, $\partial^2 V/\partial y^2$ cannot exist.

### 3. *A Lemma on the Limit of a Definite Integral.*

In the following sections (§§ 4, 5) we shall have to find the limiting values of certain definite integrals of the type

$$\lim_{r=0}\int_b^{a/r} f(t,r)\,dt,$$

where $r$ approaches zero through *positive* values, and $a$, $b$ are fixed, $a$ being positive.

Suppose that $$\lim_{r=0} f(t,r) = g(t);$$

then the truth of the equation

$$\lim_{r=0}\int_b^{a/r} f(t,r)\,dt = \int_b^\infty g(t)\,dt$$

depends (i.) on the convergence of the second integral, and (ii.) on some further condition. Without attempting to obtain the most general form of the second condition, the following will be found sufficient for the applications in view:—

*It will be supposed that a number $N$ can be found, corresponding to any prescribed positive number $\epsilon$, such that*

$$\left|\int_T^{a/r} f(t,r)\,dt\right| < \epsilon,$$

*if $N < T < a/r$; this inequality holding for all values of $r$ less than $a/N$.*

That some such condition is necessary is shown by the example

$$f(t,r) = \frac{1}{1+t^2} - \frac{r}{1+r^2t^2},$$

which gives $g(t) = 1/(1+t^2)$. But the integral

$$\int_0^{1/r} f(t, r)\,dt = \text{arc tan}\,(1/r) - \tfrac{1}{4}\pi\,;$$

so that

$$\lim_{r=0} \int_0^{1/r} f(t, r)\,dt = \tfrac{1}{4}\pi,$$

although

$$\int_0^{\infty} g(t)\,dt = \tfrac{1}{2}\pi.$$

To see that the conditions above are sufficient, we observe that when they are satisfied $N$, $N'$ can be found so that

$$\left| \int_T^{a/r} f(t, r)\,dt \right| < \tfrac{1}{3}\epsilon,$$

if $N \leqslant T < a/r$, and

$$\left| \int_T^{\infty} g(t)\,dt \right| < \tfrac{1}{3}\epsilon,$$

if $N' \leqslant T$.

Let $P$ denote the greater of $N, N'$; then it is easily seen that

$$\left| \int_P^{a/r} f(t, r)\,dt - \int_P^{\infty} g(t)\,dt \right| < \tfrac{2}{3}\epsilon,$$

provided that $r$ is less than $a/P$.

Since $P$ is now fixed, and since

$$\lim_{r=0} f(t, r) = g(t),$$

we can find $\delta$ so that

$$|f(t, r) - g(t)| < \tfrac{1}{3}\epsilon/(P-b), \text{ if } r < \delta,$$

where $t$ is free to vary from $b$ to $P$. Then

$$\left| \int_b^{a/r} f(t, r)\,dt - \int_b^{\infty} g(t)\,dt \right|$$

$$\leqslant \left| \int_b^{P} f(t, r)\,dt - \int_b^{P} g(t)\,dt \right| + \left| \int_P^{a/r} f(t, r)\,dt - \int_P^{\infty} g(t)\,dt \right| < \epsilon,$$

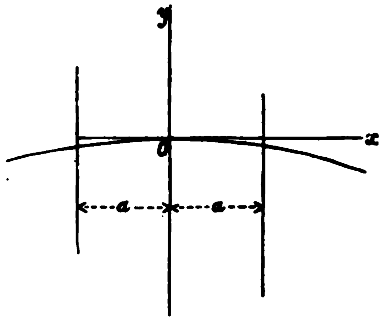
provided that $r$ is less than both $a/P$ and $\delta$. Consequently

$$\lim_{r=0} \int_b^{a/r} f(t, r)\,dt = \int_b^{\infty} g(t)\,dt.$$

It is plain that exactly similar tests will justify the equation

$$\lim_{r=0} \int_{-a/r}^{b} f(t, r)\,dt = \int_{-\infty}^{b} g(t)\,dt.$$

### 4. *The First Derivates of the Logarithmic Potential of a Curve (at a point on the curve).*



We shall assume that the curve has a tangent at all points with which we are concerned; further, that the curvature is *finite* and that the density ($\sigma$) is *continuous* on the two sides of $O$, with a possible discontinuity at $O$.

Take the origin at the point $O$, at which the derivates are to be calculated, and let the axis of $x$ be the tangent at $O$, the axis of $y$ being the normal. Draw the two lines $x = \pm a$, which will cut out a small portion of the curve round $O$; then the potential of the remaining part of the curve is continuous and has continuous derivates of all orders,* for points which are sufficiently close to $O$. Write, for the coordinates of points on the curve, $\eta = \xi \tan\psi$, so that $\psi$ is a function of $\xi$ with the property that $\lim_{\xi=0} (\psi/\xi)$ is finite.

Then the potential at $(x, y)$ is

$$-2\gamma \int_{-a}^{+a} k\sigma \log R\, d\xi,$$

where $R^2 = (x-\xi)^2+(y-\eta)^2$, and $k$ is the secant of the angle between the tangent and the axis of $x$; in the same way the potential at $O$ is

$$-\gamma \int_{-a}^{+a} k\sigma \log(\xi^2+\eta^2)\, d\xi.$$

Write $x = rl$, $y = rm$, where $l^2+m^2 = 1$; then, to find the derivate at $O$, in the direction of $l$, $m$, we have to examine

$$-\frac{\gamma}{r}\int_{-a}^{+a} k\sigma \log\left(\frac{R^2}{\xi^2+\eta^2}\right) d\xi$$

and see whether this has a limit when $r$ tends to zero. If we write $\xi = rt$, the integral to be considered becomes

$$-\gamma \int_{-a/r}^{+a/r} k\sigma \log\left(\frac{\lambda^2}{t^2 \sec^2\psi}\right) dt,$$

---

* The proof of this statement offers no difficulty: the method of § 1 needs but little change to apply here.

in which it must be remembered that $k$, $\sigma$, and $\psi$ are functions of $rt$, and

$$\lambda^2 = t^2 \sec^2 \psi - 2t\,(l + m \tan \psi) + 1.$$

The integral may be written in the form $A + B$, where

$$\begin{aligned} A = &-\gamma \int_1^{a/r} k\sigma \left[ \log\left(\frac{\lambda^2}{t^2 \sec^2 \psi}\right) + \frac{2}{t}\,(l \cos \psi + m \sin \psi) \cos \psi \right] dt \\ &-\gamma \int_{-a/r}^{-1} k\sigma \left[ \log\left(\frac{\lambda^2}{t^2 \sec^2 \psi}\right) + \frac{2}{t}\,(l \cos \psi + m \sin \psi) \cos \psi \right] dt \\ &-\gamma \int_{-1}^{+1} k\sigma \log\left(\frac{\lambda^2}{t^2 \sec^2 \psi}\right) dt \end{aligned}$$

and

$$\begin{aligned} B = &\ 2\gamma \int_r^a k\sigma\,(l \cos \psi + m \sin \psi) \cos \psi \, d\xi/\xi \\ &+ 2\gamma \int_{-a}^{-r} k\sigma\,(l \cos \psi + m \sin \psi) \cos \psi \, d\xi/\xi. \end{aligned}$$

To the integral $A$ we can apply the lemma of § 3, and we find

$$\begin{aligned} \lim_{r=0} A = &-\gamma\sigma_1 \left[ \int_0^1 \log\left(\frac{\lambda_0^2}{t^2}\right) dt + \int_1^\infty \left\{ \log\left(\frac{\lambda_0^2}{t^2}\right) + \frac{2l}{t} \right\} dt \right] \\ &-\gamma\sigma_2 \left[ \int_{-\infty}^{-1} \left\{ \log\left(\frac{\lambda_0^2}{t^2}\right) + \frac{2l}{t} \right\} dt + \int_{-1}^{0} \log\left(\frac{\lambda_0^2}{t^2}\right) dt \right], \end{aligned}$$

where $\sigma_1$, $\sigma_2$ are the limiting values of $\sigma$, on the two sides of $O$, and

$$\lambda_0^2 = t^2 - 2lt + 1.$$

To verify that the condition of the lemma is actually satisfied here, we observe that, if $h$ is between 0 and 1,

$$\log(1 - 2h\cos\theta + h^2) + 2h\cos\theta = -2\left(\tfrac{1}{2}h^2 \cos 2\theta + \tfrac{1}{3}h^3 \cos 3\theta + \ldots\right);$$

so that

$$|\log(1 - 2h\cos\theta + h^2) + 2h\cos\theta| < h^2/(1-h)^2.$$

Hence

$$\left| \log\left(\frac{\lambda^2}{t^2 \sec^2 \psi}\right) + \frac{2}{t}\,(l \cos \psi + m \sin \psi) \cos \psi \right| < 1/(t \sec\psi - 1)^2 < 1/(t-1)^2,$$

by taking $h = 1/(t \sec \psi)$; and consequently, if $N < T < a/r$,

$$\left| \gamma \int_T^{a/r} k\sigma \left[ \log\left(\frac{\lambda^2}{t^2 \sec^2 \psi}\right) + \frac{2}{t}\,(l \cos \psi + m \sin \psi) \cos \psi \right] dt \right| < \int_T^{a/r} \frac{L\,dt}{(t-1)^2} < \frac{L}{N-1},$$

where $L$ is the upper limit to $\gamma k\sigma$ in the small arc of the curve. It is now clear that the lemma may be applied to the first term in $A$; and a similar argument can be used for the second term in $A$; the third term requires no special discussion, since the range of the integral is fixed and includes no singularity of the integrand other than $t=0$ (which offers no difficulty).

After carrying out the integrations, it will be found that

$$\lim_{r=0} A = -2\gamma\sigma_1 (1-l^2)^{\frac{1}{2}}\left[\frac{\pi}{2} + \operatorname{arc\,tan}\frac{l}{(1-l^2)^{\frac{1}{2}}}\right]$$

$$-2\gamma\sigma_2 (1-l^2)^{\frac{1}{2}}\left[\frac{\pi}{2} - \operatorname{arc\,tan}\frac{l}{(1-l^2)^{\frac{1}{2}}}\right] - 4\gamma l(\sigma_1-\sigma_2).$$

As regards $B$, we note that

$$\int_r^a k\sigma(l\cos\psi + m\sin\psi)\cos\psi\, d\xi/\xi$$

$$= \int_r^a k\sigma(m\cos\psi - l\sin\psi)\sin\psi\, d\xi/\xi + \int_r^a l(k-1)\sigma\, d\xi/\xi + \int_r^a l\sigma\, d\xi/\xi.$$

In the first of these three integrals we have

$$|\, m\cos\psi - l\sin\psi\,| < 1,$$

and $\lim\limits_{\xi=0}(\sin\psi/\xi)$ is finite; so that the limit

$$\lim_{r=0}\int_r^a k\sigma(m\cos\psi - l\sin\psi)\sin\psi\, d\xi/\xi$$

exists, and may be made arbitrarily small by choice of $a$. In the second integral $\int_r^a l(k-1)\sigma\, d\xi/\xi$, $|l| < 1$ and $\lim\limits_{\xi=0}(k-1)/\xi = 0$, in consequence of the finite curvature. Thus this integral has also a limit (as $r$ tends to zero) which may be made small by proper choice of $a$.

Thus the existence of the limit $\lim\limits_{r=0} B$ depends on that of

$$\lim_{r=0}\left[\int_r^a l\sigma\, d\xi/\xi + \int_{-a}^{-r} l\sigma\, d\xi/\xi\right] = \lim_{r=0}\left[\int_r^a l\,\{\sigma(\xi) - \sigma(-\xi)\}\, d\xi/\xi\right].$$

Now, if, as we have assumed above, the two limits

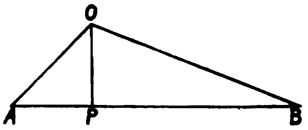
$$\lim_{\xi=0}\sigma(+\xi) = \sigma_1, \qquad \lim_{\xi=0}\sigma(-\xi) = \sigma_2$$

are different, it is plain that the last integral has no limit unless $l$ is zero. Thus we obtain the result:

*If $O$ is a point of discontinuity in $\sigma$, the general derivate does not exist, but the normal derivate does exist, and is given by*

$-\pi\gamma(\sigma_1+\sigma_2)+$ *terms which do not depend on the distribution near $O$.*

*It may be noted that this derivate is calculated for the directions away from $O$; so that the discontinuity at $O$ in $\partial V/\partial y$ is $-2\pi\gamma(\sigma_1+\sigma_2)$.*



As a verification we may refer to Maxwell's result for a uniform rod [*l.c.*, § 692 (1)]; the derivate at $A$, in the direction $AO$, is given by[*]

$$-2\gamma\sigma \lim_{OA=0} \frac{[AP\log OA+PB\log OB+OP(\angle AOB)]-AB\log AB}{OA},$$

where $OP$ is perpendicular to $AB$.

In general this limit does not exist, but, if $OA$ is perpendicular to $AB$, so that $P$ is coincident with $A$, it gives the limit

$$-2\gamma\sigma \lim_{OA=0}(\angle AOB) = -\pi\gamma\sigma,$$

which agrees with our last two results, by taking $\sigma_1=\sigma$, $\sigma_2=0$.

For the existence of the general derivate it is, therefore, necessary that $\sigma$ should be continuous at $O$; but this condition is not sufficient, as may be seen from the example

$$\sigma = 1+1/\log(1/\xi) \quad (\xi>0),$$
$$\sigma = 1 \quad (\xi \leqslant 0).$$

Here $\int_r^a [\sigma(+\xi)-\sigma(-\xi)]\,d\xi/\xi = \int_r^a [\log(1/\xi)]^{-1}d\xi/\xi = \log\left[\frac{\log(1/r)}{\log(1/a)}\right]$,

which manifestly has no limit as $r$ tends to zero.

Thus we may conclude that:

*The derivate (in any direction other than the normal) can only exist if the limit*

$$\lim_{r=0}\int_r^a [\sigma(\xi)-\sigma(-\xi)]\,d\xi/\xi$$

*exists, a condition which implies the continuity of $\sigma$. When this condition is satisfied, the derivate is given by*

$-2\pi\gamma\sigma_0(1-l^2)^{\frac{1}{2}}+$ *terms which do not depend on the distribution near $O$,*

*where $\sigma_0$ is the value of $\sigma$ at $O$.*

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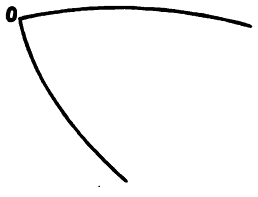
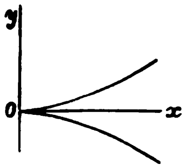
[*] In Maxwell's notation $AB\log R$ represents the value of the integral $\int \log r\,d\xi$, integrated along the rod; this is to be multiplied by $-2\gamma\sigma$ to agree with the form used here for the potential.



It is easy to extend the results obtained here to cover the case when the curve consists of two distinct branches (each regular) meeting at $O$ in different directions. A short calculation shews that there is generally only *one* direction in which the derivate can exist,* and even for this one a further condition (on $\sigma$) has to be satisfied.



For the existence of derivates in all directions, it is necessary that $\sigma$ should tend to zero at $O$ (on both branches), and that the integral $\int_0 \sigma(\xi)\,d\xi/\xi$ should be convergent (for both branches separately). The case when $O$ is an ordinary cusp is also easy to discuss; † the derivate in a direction perpendicular to the tangent always exists. For other directions the existence of the limit

$$\lim_{r=0}\int_r^a [\sigma_1(\xi)+\sigma_2(\xi)]\,d\xi/\xi$$

is necessary.

### 5. *Components of Force (at a Point on the Curve) due to the Logarithmic Potential of a Curve.*

We investigated, in the last paragraph, the values of $\partial V/\partial x$, $\partial V/\partial y$ at $O$; but it does not follow, without special investigation, that these results give the components of force‡ at $O$. We shall now examine, first, the limiting values of $\partial V/\partial x$, $\partial V/\partial y$ as $(x, y)$ moves up to $O$ along a regular arc which does not touch the curve at $O$; and, secondly, the values of the components of force at $O$.

We take the same axes and notation generally as in § 4; then the force at $(x, y)$ in the direction $l, m$ is given by the integral

$$2\gamma\int_{-a}^{+a}\frac{k\sigma}{R^2}[l(\xi-x)+m(\eta-y)]\,d\xi,$$

where $R^2=(\xi-x)^2+(\eta-y)^2$. Here, of course, this only represents the force due to the small arc of the curve round $O$; the force due to the remaining part of the curve is continuous within a sufficiently small area round the point $O$.

---

* In the case when $\sigma$ has the same limit at $O$ for both branches, this direction bisects the angle between the two tangents at $O$.

† Of course, certain obvious changes must be made in the previous work, owing to the discontinuity in the curvature at $O$. Compare the work at the foot of p. 353.

‡ For the general proof that $\partial V/\partial x$, $\partial V/\partial y$ represent the components of force requires that $(x, y)$ shall not be a point of the curve.

Write now $x = rf$, $y = rg$, where $f^2+g^2 = 1$; and, without loss of generality, $f$, $g$ may be supposed to remain constant as $r$ tends to zero; of course, it must be remembered that $g$ is not to be zero, though $f$ may be zero. Then, as before, put $\xi = rt$, $\eta = \xi \tan\psi$, and the limit to be considered becomes

$$\lim_{r=0} 2\gamma \int_{-a/r}^{+a/r} k\sigma \frac{t(l+m\tan\psi)-(lf+mg)}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}\,dt,$$

where now, of course, $k$, $\sigma$, $\psi$ are functions of the product $rt$.

Now this integral may be put in the form $A+B$, where

$$\begin{aligned} A = {} & 2\gamma\int_{-a/r}^{-1} k\sigma\left[\frac{t(l+m\tan\psi)}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}-\frac{1}{t}(l\cos\psi+m\sin\psi)\cos\psi\right]dt \\ & +2\gamma\int_{-1}^{+1} k\sigma\frac{t(l+m\tan\psi)}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}\,dt \\ & +2\gamma\int_{+1}^{+a/r} k\sigma\left[\frac{t(l+m\tan\psi)}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}-\frac{1}{t}(l\cos\psi+m\sin\psi)\cos\psi\right]dt \\ & -2\gamma\int_{-a/r}^{+a/r} k\sigma\frac{(lf+mg)\,dt}{t^2\sec^2\psi-2t(f+g\tan\psi)+1} \end{aligned}$$

and

$$\begin{aligned} B = {} & 2\gamma\int_{-a}^{-r} k\sigma(l\cos\psi+m\sin\psi)\cos\psi\,d\xi/\xi \\ & +2\gamma\int_{r}^{a} k\sigma(l\cos\psi+m\sin\psi)\cos\psi\,d\xi/\xi. \end{aligned}$$

It will be seen that this $B$ is the same as the one used in the last paragraph; so that no fresh investigation of the limit of $B$ will be required. Of course, $A$ is not now the same as before. To evaluate the limit of $A$, we apply the lemma of § 3, and we consider first

$$\left|\int_T^{a/r}\frac{(lf+mg)\,dt}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}\right|.$$

Here $|lf+mg| \leqslant 1$, $|f+g\tan\psi| \leqslant \sec\psi$; so that this expression is less than

$$\int_T^{a/r}\frac{dt}{(t\sec\psi-1)^2} < \int_T^{a/r}\frac{dt}{(t-1)^2} < \frac{1}{T-1};$$

thus, if $N = 1+1/\epsilon$, and $a/r > T > N$, the test of the lemma is satisfied, provided that $|k\sigma|$ remains less than a fixed number on the small arc of the curve.

A similar argument applies to the case

$$\left|\int_{-a/r}^{-T} \frac{(lf+mg)\,dt}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}\right|;$$

and consequently

$$\lim_{r=0} 2\gamma\int_{-a/r}^{+a/r} k\sigma\frac{(lf+mg)\,dt}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}$$

$$= 2\gamma\sigma_1\int_0^{+\infty}\frac{(lf+mg)\,dt}{t^2-2ft+1}+2\gamma\sigma_2\int_{-\infty}^{0}\frac{(lf+mg)\,dt}{t^2-2ft+1}$$

$$= 2\gamma\sigma_1\frac{lf+mg}{(1-f^2)^{\frac{1}{2}}}\left(\frac{\pi}{2}+\alpha\right)+2\gamma\sigma_2\frac{lf+mg}{(1-f^2)^{\frac{1}{2}}}\left(\frac{\pi}{2}-\alpha\right),$$

where

$$\alpha = \text{arc tan}\,[f/(1-f^2)^{\frac{1}{2}}].$$

Take next

$$\left|\int_T^{a/r}\left[\frac{(l+m\tan\psi)\,t}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}-\frac{1}{t}(l\cos\psi+m\sin\psi)\cos\psi\right]dt\right|$$

$$= \left|\int_T^{a/r}\frac{(l\cos\psi+m\sin\psi)\,[2t(f\cos\psi+g\sin\psi)-\cos\psi]}{t\,[t^2\sec^2\psi-2t(f+g\tan\psi)+1]}\,dt\right|;$$

here $|l\cos\psi+m\sin\psi| \leqslant 1$, $|f\cos\psi+g\sin\psi| \leqslant 1$; so that the expression is less than

$$\int_T^{a/r}\frac{2t+1}{t\,(t\sec\psi-1)^2}<\int_T^{a/r}\frac{3}{(t-1)^2}<\frac{3}{T-1},$$

if $T>1$. Thus, just as above, the test of the lemma is satisfied; so that

$$\lim_{r=0} 2\gamma\int_1^{a/r} k\sigma\left[\frac{(l+m\tan\psi)\,t}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}-\frac{1}{t}(l\cos\psi+m\sin\psi)\cos\psi\right]dt$$

$$= 2\gamma\sigma_1\int_1^{\infty}\left(\frac{lt}{t^2-2ft+1}-\frac{l}{t}\right)dt.$$

Similarly,

$$\lim_{r=0} 2\gamma\int_{-a/r}^{-1} k\sigma\left[\frac{(l+m\tan\psi)\,t}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}-\frac{1}{t}(l\cos\psi+m\sin\psi)\cos\psi\right]dt$$

$$= 2\gamma\sigma_2\int_{-\infty}^{-1}\left(\frac{lt}{t^2-2ft+1}-\frac{l}{t}\right)dt,$$

and, without any special investigation,

$$\lim_{r=0} 2\gamma\int_{-1}^{+1} k\sigma\frac{(l+m\tan\psi)\,t}{t^2\sec^2\psi-2t(f+g\tan\psi)+1}\,dt$$

$$= 2\gamma\sigma_1\int_0^1\frac{lt}{t^2-2ft+1}\,dt+2\gamma\sigma_2\int_{-1}^{0}\frac{lt}{t^2-2ft+1}\,dt.$$

The sum of these four integrals is found to be

$$2\gamma\sigma_1 \frac{lf}{(1-f^2)^{\frac{1}{2}}}\left(\frac{\pi}{2}+\alpha\right)+2\gamma\sigma_2 \frac{lf}{(1-f^2)^{\frac{1}{2}}}\left(\frac{\pi}{2}-\alpha\right).$$

Hence $\lim\limits_{r=0} A = -2\gamma\sigma_1 \frac{mg}{(1-f^2)^{\frac{1}{2}}}\left(\frac{\pi}{2}+\alpha\right)-2\gamma\sigma_2 \frac{mg}{(1-f^2)^{\frac{1}{2}}}\left(\frac{\pi}{2}-\alpha\right)$,

where it should be observed that the value of $g/(1-f^2)^{\frac{1}{2}}$ is either $+1$ or $-1$, according to the sign of $g$.

We saw above that when $\sigma_1$ and $\sigma_2$ are unequal the limit of $B$ could only exist if $l=0$; if we write $l=0$ and $m=1$, we see that—

*In case $\sigma$ is discontinuous at $O$, the limit of the normal force as $(x, y)$ approaches $O$ is only equal to the normal derivate of the potential at $O$, if $(x, y)$ approaches $O$ along the normal.* In general, the value of the limit depends on the direction of the path followed by $(x, y)$.*

*The limit of the tangential force cannot exist.*

As a verification, consider the case of a uniform straight rod; the result of Maxwell's, already used (p. 359), shows that

$$\frac{\partial V}{\partial x} = -2\gamma\sigma \log(OA/OB), \qquad \frac{\partial V}{\partial y} = -2\gamma\sigma(\angle AOB),$$

where now $O$ is the point $(x, y)$. The origin is taken at the end $A$ of the rod, with the axis of $x$ along the rod. It is clear that $\partial V/\partial x$ has no limit when $O$ approaches $A$, while $\partial V/\partial y$ approaches the limit

$$-2\gamma\sigma\left[\frac{\pi}{2}+\text{arc}\tan(f/g)\right],$$

assuming that $y$ (and therefore $g$) is positive. This agrees with the limit calculated above, if we write $\sigma_1=\sigma$, $\sigma_2=0$.

Turning now to the case when $\sigma$ is continuous at $O$, so that

$$\sigma_1=\sigma_2=\sigma_0, \text{ say,}$$

it is clear that $\lim\limits_{r=0} A = \mp 2\pi\gamma\sigma_0 m$, where the ambiguous sign is opposite to the sign of $g$. Comparing this with the result at the foot of p. 359, we can therefore state:—

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* Because $f$ is then zero; so that $\lim\limits_{r=0} A = \mp\pi\gamma(\sigma_1+\sigma_2)$, the sign $\mp$ being opposite to the sign of $g$.

In general, $f=\sin\alpha$ and the value is found to be $\mp[\pi\gamma(\sigma_1+\sigma_2)+2\alpha\gamma(\sigma_1-\sigma_2)]$.

*The force (in a direction not parallel to the normal at O) has a limiting value only if the limit*

$$\lim_{r=0}\int_r^a [\sigma(+\xi)-\sigma(-\xi)]\,d\xi/\xi$$

*exists, a condition which implies the continuity of $\sigma$. When this condition is satisfied, the limit of the force is*

$$\mp 2\pi\gamma\sigma_0 m + \textit{terms which do not depend on the distribution near } O,$$

*which is the same as the derivate at O, in the same direction.*

The discussion of modifications of these results, consequent on geometrical singularities of the curve at $O$, may be carried out exactly as in the last paragraph.

Finally, *to calculate the components of force at O*, we must consider the limiting value of

$$\int_r^a k\sigma\frac{l\xi+m\eta}{\xi^2+\eta^2}\,d\xi+\int_{-a}^{-s} k\sigma\frac{l\xi+m\eta}{\xi^2+\eta^2}\,d\xi,$$

when $r$ and $s$ tend to zero independently.

If, as before, we write $\eta=\xi\tan\psi$, we have to evaluate

$$\lim_{r,\,s=0}\left[\int_r^a k\sigma(l\cos\psi+m\sin\psi)\cos\psi\,d\xi/\xi + \int_{-a}^{-s} k\sigma(l\cos\psi+m\sin\psi)\cos\psi\,d\xi/\xi\right].$$

Proceeding exactly as before (p. 358), we see that this is essentially the same as

$$\lim_{r,\,s=0}\left[\int_r^a l\sigma\,d\xi/\xi+\int_{-a}^{-s} l\sigma\,d\xi/\xi\right].$$

Now, unless $\sigma$ tends to zero at $O$, this limit is not determinate when $l$ is different from zero. Thus we may state the result:

*The normal force at O is determinate, and is independent of the distribution near O. But the tangential force at O is generally indeterminate even when $\sigma$ is continuous, unless $\sigma_0=0$.*

We may call the value obtained by writing $s=r$ the *principal value* of the tangential force. It is then clear that *the conditions for the existence of this principal value are exactly the same as for the existence of the general derivate of the potential at O.*

### 6. *A Comparison of the foregoing Results with those of Previous Writers.*

The only writer who has *explicitly* discussed these problems for the logarithmic potential is Harnack in his *Theorie des logarithmischen Potentiales* (Leipzig, 1887). However I shall refer to some authors who have obtained results for the corresponding problems in the Newtonian potential theory, because their conclusions generally apply to the logarithmic potential, if suitable modifications are made.

It should be observed, once for all, that all these writers make the preliminary assumption that *the density is completely continuous;* and consequently their conclusions cannot be compared with such of the foregoing results as relate to discontinuities in the density.

It will be convenient for the purposes of comparison to examine the surface distribution and line distribution separately; and to modify the conditions found previously, by introducing the hypothesis of continuity at the beginning (compare § 7 below).

(I.) *Surface Distributions.*—The existence of the second derivate $\partial^2 V/\partial x^2$ was proved (§ 2 above) to depend on the limit

$$\lim_{h=0} C = \lim_{h=0} \int_0^{2\pi} \cos 2\theta \, d\theta \int_h^a (\rho/r)\, dr.$$

Now, if $\rho$ is continuous and $\lim_{r=0} \rho = \rho_0$, we have

$$\int_0^{2\pi} \cos 2\theta \, d\theta \int_h^a (\rho_0/r)\, dr = \rho_0 \log (a/h) \int_0^{2\pi} \cos 2\theta \, d\theta = 0\,;$$

so that the limit above is the same as

$$\lim_{h=0} \int_0^{2\pi} \cos 2\theta \, d\theta \int_h^a [(\rho-\rho_0)/r]\, dr = \lim_{h=0} C'.$$

(I. *a*) *Hölder's Condition* (*Inaugural Dissertation,** Tübingen, 1882).—
The condition is that $$|\rho-\rho_0| < Gr^g,$$
where $G$, $g$ are positive constants, and $r$ is less than some fixed value, which we may take to be greater than $a$. If this holds,

$$\left| \int_0^{2\pi} \cos 2\theta \, d\theta \int_{r_1}^{r_2} [(\rho-\rho_0)/r]\, dr \right| < 2\pi \int_{r_1}^{r_2} |\rho-\rho_0|\, r^{-1} dr < 2\pi G \int_{r_1}^{r_2} r^{g-1} dr$$
$$< 2\pi (G/g)\,(r_2^g - r_1^g).$$

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* This dissertation has unfortunately been inaccessible, but all subsequent writers agree in quoting the same form for the condition given by Hölder.

Thus the limit of $C'$ exists and can be made as small as we please by diminishing $a$; it is easily seen, further, that the condition is also satisfied when $h$ is *negative*, so that the derivate is not merely one-sided.

Harnack (*l.c.*, p. 24, footnote) has remarked that Hölder's method for the Newtonian potential can be modified to apply to the logarithmic potential; and C. Neumann* has given an alternative discussion of Hölder's results which can also be modified, although the method of § 7 would seem to be preferable.

(I.*b*) *Harnack's Condition* (*Theorie &c.*, Kap. 1, § 6).—It is here supposed that $\rho$ has finite and integrable first derivates—a condition analogous to the one used by Gauss in discussing the Newtonian potential. Then, since

$$\frac{\cos 2\theta}{r^2} = -\frac{\partial}{\partial\xi}\left(\frac{\xi-x}{r^2}\right),$$

the limit to be examined may be replaced by

$$\lim_{h=0}\iint(\rho-\rho_0)\frac{\partial}{\partial\xi}\left(\frac{\xi-x}{r^2}\right)d\xi\,d\eta,$$

where the integral is extended over the ring between two circles of radii $h$, $a$ whose common centre is $(x, y)$.

This integral may be integrated by parts, and it becomes then

$$\left[\int_0^{2\pi}(\rho-\rho_0)\cos^2\theta\,d\theta\right]_{r=h}^{r=a}-\iint\frac{\cos\theta}{r}\,\frac{\partial\rho}{\partial\xi}\,d\xi\,d\eta.$$

The integrals round the circles are together less, in absolute value, than $4\pi\omega$, if $\omega$ is the greatest value of $|\rho-\rho_0|$ on either circle; and, since $\rho$ is continuous, $\omega$ can be made as small as may be required by diminishing $a$. The surface integral is less,† in absolute value, than $2\pi aM$, where $M$ is the greatest value of $|\partial\rho/\partial\xi|$ in the ring. Thus

$$\iint(\rho-\rho_0)\frac{\partial}{\partial\xi}\left(\frac{\xi-x}{r^2}\right)d\xi\,d\eta$$

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* *Leipzig Berichte*, Bd. XLII., 1890, p. 327.

† This is almost obvious; reference may be made, if necessary, to the methods of § 1.

may be made arbitrarily small by diminishing $a$; therefore the limit

$$\lim_{h=0} \iint (\rho-\rho_0)\frac{\partial}{\partial \xi}\left(\frac{\xi-x}{r^2}\right) d\xi\, d\eta$$

must exist, and consequently $\partial^2 V/\partial x^2$ also exists.

(I.$c$) *Morera's Condition.**—It is here assumed that the limit

$$\lim_{h=0} \int_h^a \frac{(\rho-\rho_0)}{r}\, dr$$

exists for all values of $\theta$; if this condition is satisfied, it is plain that the limit

$$\lim_{h=0} \int_0^{2\pi} \cos 2\theta\, d\theta \int_h^a [(\rho-\rho_0)/r]\, dr$$

exists, and the condition of § 2 is therefore satisfied.

(II.) *Line Distributions.*—In this connection, the previous results appear to relate entirely to the problem of § 5, the calculation of the derivates at a point of the curve (as in § 4) not having been undertaken. In many cases, too, the point $(x, y)$ is supposed to approach the curve along the *normal*; as was proved above (§ 5), this limitation of the mode of approach does affect the force, if there is a discontinuity in the density. But in the older work the density is invariably assumed to be continuous, and consequently the limitation does not really make the results less general.

(II. $a$) *Hölder's Condition* (*l.c.*).—This is $|\sigma-\sigma_0| < G|\xi|^g$, where $G$ and $g$ are *positive* constants, and $|\xi|$ is less than some fixed value. When this holds the condition of § 5 is easily seen to be satisfied. For then

$$\left| \int_r^{r^2} [\sigma(+\xi)-\sigma(-\xi)]\, d\xi/\xi \right| < 2G \int_{r_1}^{r_2} \xi^{g-1}\, d\xi < 2Gr_2^g/g;$$

so that

$$\lim_{r=0} \int_r^a [\sigma(+\xi)-\sigma(-\xi)]\, d\xi/\xi$$

exists and may be made arbitrarily small by choice of $a$.

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* G. Morera, *Rend. del R. Ist. Lomb.*, Ser. 2, T. xx., 1887, p. 302, and *Il Nuovo Cimento*, Ser. 3, T. xxii., 1887, p. 240. Reference may also be made to G. A. Maggi, *Rend. del R. Ist. Lomb.*, Ser. 2, T. xxii., 1889, p. 647; *Il Nuovo Cimento*, Ser. 3, T. xxvii., 1890, p. 21; T. xxxiii., 1893, p. 113.

(II.*b*) *Harnack's Condition* (*Theorie*, *&c.*, Kap. 1, § 9).—It is assumed that $\sigma$ has a finite and integrable first derivate, say $\sigma'(\xi)$; and then

$$\lim_{\xi=0}\left[\sigma(+\xi)-\sigma(-\xi)\right]/\xi = 2\sigma'(0);$$

so that the limit $\displaystyle\lim_{r=0}\int_r^a\left[\sigma(+\xi)-\sigma(-\xi)\right]d\xi/\xi$

certainly exists.

(II.*c*) *Morera's Condition.*[*]—It is assumed that the limits

$$\lim_{r=0}\int_r^a\left|(\sigma-\sigma_0)/\xi\right|d\xi \quad\text{and}\quad \lim_{r=0}\int_{-a}^{-r}\left|(\sigma-\sigma_0)/\xi\right|d\xi$$

both exist. If this is so, it follows at once that the limit

$$\lim_{r=0}\int_r^a\left|\sigma(+\xi)-\sigma(-\xi)\right|d\xi/\xi$$

exists; and consequently that the condition of § 5 is satisfied.

It is manifest that of the conditions enumerated Morera's are the nearest to those obtained in §§ 2, 5 above. However, they do not include all cases in which derivates exist; thus, with a surface distribution such that $\rho_0 = 0$ and $\rho = 1/\lambda$, where $\lambda = [\log(1/r)]^{\frac{1}{2}}$, we have

$$\int_0^{2\pi}\cos 2\theta\, d\theta\int_h^a(\rho/r)\,dr = 0;$$

so that the condition of § 2 is satisfied, and the potential has second derivates. But

$$\int_h^a\frac{\lfloor\rho-\rho_0\rfloor}{r}dr = 2[\log(1/h)]^{\frac{1}{2}}-2[\log(1/a)]^{\frac{1}{2}};$$

so that Morera's condition (I. *c*) would exclude this case.

Similarly, a line distribution such that

$$\sigma_0 = 0, \qquad \sigma = [\log(1/|\xi|)]^{-1}$$

would satisfy the condition of § 5, but not Morera's condition (II. *c*).

## 7. *Simplified Investigations in case the Density is Continuous.*

If $\rho$ is completely continuous, the work of § 2 can be shortened considerably by the following method. Let $U$ denote the logarithmic potential of the circular area when it carries a *uniform* density $\rho_0$ (the value of $\rho$ at the centre)*; it is then obvious that

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[*] G. Morera, *Rend. del R. Ist. Lomb.*, Ser. 2, T. xx., 1887, p. 543; *Il Nuovo Cimento*, Ser. 3, T. xxiii., 1888, p. 1. Compare G. A. Maggi, *Rend. del R. Ist. Lomb.*, Ser. 2, T. xxiv., 1891, pp. 87, 220, 960; *Il Nuovo Cimento*, Ser. 3, T. xxxiii., 1893, p. 249, T. xxxiv., 1894, pp. 22, 81.

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 U}{\partial x^2} + \lim_{h=0} \iint \frac{2\gamma}{h} (\rho - \rho_0) \left( \frac{\xi - x - h}{r'^2} - \frac{\xi - x}{r^2} \right) d\xi\, d\eta,$$

and a short calculation shows that†

$$U = -2\pi\gamma\rho_0 \left[a^2 \log a + \tfrac{1}{2}(r^2 - a^2)\right].$$

Thus
$$\frac{\partial^2 V}{\partial x^2} = -2\pi\gamma\rho_0 + \lim_{h=0} I,$$

if $I$ denotes the integral written above.

For brevity let $c$ denote $|h|$, and let $\omega$ denote the maximum of $|\rho - \rho_0|$ within the circle of radius $a$; let a circle of radius $2c$ be drawn, with centre $(x, y)$, and let $I_1$, $I_2$ represent the parts of $I$ which arise from the area within the smaller circle, and from the ring between the circles. Then, by a method exactly like the methods used in § 1, we can prove that‡

$$|I_1| < \frac{2\gamma\omega}{c} \iint \left( \frac{1}{r} + \frac{1}{r'} \right) d\xi\, d\eta < 12\pi\gamma\omega.$$

To deal with $I_2$, we observe that

$$\frac{\xi - x - h}{r'^2} - \frac{\xi - x}{r^2} = \frac{r\cos\theta - h}{r^2 - 2rh\cos\theta + h^2} - \frac{\cos\theta}{r}$$

$$= \frac{h\cos 2\theta}{r^2} + \frac{h^2 (r\cos 3\theta - h\cos 2\theta)}{r^2 (r^2 - 2rh\cos\theta + h^2)}.$$

Thus
$$I_2 - 2\gamma \int_0^{2\pi} \cos 2\theta\, d\theta \int_{2c}^{a} [(\rho - \rho_0)/r]\, dr = K,$$

where
$$K = 2\gamma h \int_0^{2\pi} d\theta \int_{2c}^{a} \frac{(\rho - \rho_0)(r\cos 3\theta - h\cos 2\theta)}{r(r^2 - 2rh\cos\theta + h^2)}\, dr.$$

Hence
$$|K| < 4\pi\gamma c\omega \int_{2c}^{\infty} \frac{r + c}{(r - c)^3}\, dr,$$

or
$$|K| < 4\pi\gamma\omega \int_1^{\infty} \frac{t + 2}{t^3}\, dt \quad \text{(putting } r = c + ct\text{)},$$

that is
$$|K| < 8\pi\gamma\omega.$$

It is now clear that $\partial^2 V/\partial x^2 + 2\pi\gamma\rho_0$ differs from

$$\lim_{h=0} 2\gamma \int_0^{2\pi} \cos 2\theta\, d\theta \int_{2c}^{a} [(\rho - \rho_0)/r]\, dr$$

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* In fact, here the function $f(\theta)$ used in § 2 is the constant $\rho_0$.

† This does not appear to be given by Maxwell (*l.c.*), but is easily deduced from his results.

‡ It may be remarked that $I_1$ corresponds to $A + B$ of § 2, and $I_2$ to $C$.

by less than $|I_1|+|K|$; and therefore the difference is less than $20\pi\gamma\omega$. But $\omega$ can be made arbitrarily small, by diminishing $a$, so that the existence of $\partial^2 V/\partial x^2$ depends only on the existence of the limit

$$\lim_{c=0} \int_0^{2\pi} \cos 2\theta \, d\theta \int_c^a [(\rho-\rho_0)/r] \, dr,$$

which is the result of § 2 as modified for continuous density on p. 365.

If $\sigma$ is continuous, it is similarly possible to simplify the work of §§ 4, 5 by comparing the potential of the actual curve with that of a short portion of the tangent at $O$, carrying a *uniform* density $\sigma_0$; but the simplification is not so great as to make it worth while to put down the details here.

In conclusion, it may be added that Petrini's investigations on the Newtonian potential of a volume density can be simplified in a manner similar to that described in this paragraph, if the density is known to be continuous at all points within a small sphere enclosing the point at which the second derivates are to be calculated.

# ORDINARY INNER LIMITING SETS IN THE PLANE OR HIGHER SPACE

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[Received April 10th, 1905.—Read April 13th, 1905.]

## PART I.

### 1.

A knowledge of the properties of *a region* (using the word in its most general sense) in the plane or higher space enables us to extend to $n$ dimensions the theorems about inner limiting sets which I gave for the straight line in various earlier papers.

In giving these generalisations I have availed myself of the opportunity to simplify the proof of the fundamental theorem originally given in my Leipzig paper.*

I quote the following theorems with respect to regions and sets of regions, confining myself, for simplicity of expression, throughout this paper to the plane: the results and reasoning are, however, perfectly general.

**THEOREM** A.—*If $R_1$, $R_2$, ... be a countably infinite series of closed regions, each lying entirely within the preceding, and if the span*† *of these regions decreases without limit, there is one, and only one, internal point common to all these regions.*

*If, instead of lying entirely within one another, only the internal points of each region are known to be internal to the preceding region, there is, if the span decrease indefinitely, again one, and only one, point determined by the regions, but it may be a common boundary point.*

*If the span of the regions do not decrease indefinitely, the common points form a perfect connected set.*†

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* "Zur Lehre d. nicht abgeschlossenen Punktmengen," *Leipziger Bericht*, August, 1903.

† W. H. Young, "On Regions and Sets of Regions," *Quart. Jour. of Math.*, 1905. The span of a region in any direction is the width of a strip which is bounded by lines perpendicular to that direction, contains every internal point of the region, and has on each of its bounding lines at least one boundary point of the region. The upper limit of the span in every direction is called *the span*.

THEOREM B.—*The potency of a perfect set is c.*[*]

THEOREM C. (Cantor's Theorem).—*Any set of non-overlapping regions is countable.*[†]

THEOREM D.—*Any set of regions can be replaced by a set of non-overlapping regions, having the same internal points.*[†]

THEOREM E.—*Any set which has no component dense in itself is countable.*[‡]

## 2.

*Definition.*—Given a finite or countably infinite series of sets of regions, overlapping in any way, the set of all the points each of which is internal to at least one region of each set is called *the inner limiting set of the series of sets of regions*, or *an ordinary limiting set*

THEOREM 1. *Of the Normal Regions.—Any ordinary inner limiting set may be defined by means of normal regions, that is, by means of a series of sets of regions, such that* (*a*) *the regions of each set are non-overlapping and therefore countable, and* (*b*) *each region is internal to a region of the preceding set, with, however, possibly common boundary points.*

The property (*a*) of the normal regions follows from Theorems C and D, already quoted; (*b*) is evident, since we only have to cut off from the regions of the second set all parts external to the regions of the first set, and so with each successive set; it is clear that, in so doing, a region of the second set which is reduced will have boundary points common with a region of the first set, and so for the other sets.

In the case when the span of the normal regions decreases without limit, every series of normal regions, one from each successive set, lying each inside its predecessor, defines one and only one point, which is a point of the inner limiting set, unless, from and after some stage, all the regions have one common boundary point.

LEMMA.—*If an inner limiting set is such that the span of its normal intervals decreases without limit, and also that we can assign a series of constantly increasing integers* $r_1, r_2, \ldots,$ *and corresponding to these a*

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[*] Theorem enunciated by Cantor and proved by Bendixon, *Bih. Svensk. Vet. Handl.*, Vol. IX., No. 6 (1884). Cp. also Schoenflies, *Bericht über die Mengenlehre*, p. 86, and W. H. Young, *Quart. Jour. of Math.*, No. 143, 1905.

[†] See note † on previous page.

[‡] Cantor, *Acta Math.*, Vol. VII. See also W. H. Young, *Quart. Jour. of Math.*, No. 138 (1903), where the theorem occurs as Cor. to the Theorem of the Nucleus; the reasoning for linear space only requires small verbal alterations to be valid for the plane or higher space.

*region of the $r_1$-th normal set which contains two regions of the $r_2$-th normal set entirely internal to it, each of which contains two regions of the $r_3$-th normal set entirely internal to them, and so on, then the potency of the inner limiting set is c.*

To prove this, let us denote the normal region of the $r_1$-th set by $d_{01}$, and the two regions of the $r_2$-th set, which it contains, by $d_{001}$ and $d_{011}$, and, continuing this system of notation, let us denote by $d_{N01}$ and $d_{N11}$ the two normal regions, referred to in the enunciation, which lie inside the region already denoted by $d_{N1}$, where $N$ denotes any combination of zeros and ones with $n$ figures.

Since these regions have no common boundary points to complicate the issue, every series of them lying one inside the other defines a point of the inner limiting set. To each such series of regions, however, by our system of notation, corresponds one non-terminating binary fraction, which, if $d_{N1}$ be any region of the series, begins with the figures denoted by $N$; conversely, to each non-terminating fraction there corresponds such a series of the regions. Thus there is a (1, 1)-correspondence between certain points of our inner limiting set and the non-terminating binary fractions; so that the inner limiting set is of potency $c$.

THEOREM 2.—*An ordinary inner limiting set which has a component dense in itself has the potency c; otherwise it is countable.*

By the Theorem of the Nucleus, due to Cantor, any set which has no component dense in itself is countable; therefore we only have to prove the first part of the theorem.

Suppose the given ordinary inner limiting set to have a component $U$ which is dense in itself. Let $P$ be any point of $U$; then there is a region $d_{01}$ of the first normal set containing $P$ as internal point, and, since $P$ is a limiting point of $U$, there will be another point $Q$ of $U$ inside the region $d_{01}$.

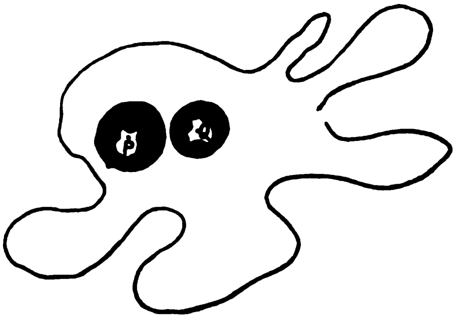

FIG. 1.

Round these two points $P$ and $Q$ as centres we can then, since they are both internal points of $d_{01}$, describe circles external to one another, and entirely internal to $d_{01}$ (Fig. 1). If the span of the normal regions decreases without limit, we can assign a stage at which the span of all the normal regions is less than the radius of the smaller of these two circles; at this stage the normal regions containing $P$ and $Q$ will be distinct, and both entirely internal to $d_{01}$. Denoting these two regions by $d_{001}$ and $d_{011}$, there will be in each a component of $U$, which is therefore dense in itself, and the argument can be repeated. Thus the condition of the lemma is satisfied, and the inner limiting set is therefore of potency $c$.

If the span of the normal regions does not decrease without limit, we take, instead of the actual regions, those parts of the regions of the second set which contain $P$ and $Q$ respectively, and are internal to the two circles, and call these $d_{001}$ and $d_{011}$. Then, as before, in each of these the argument can be repeated, and we get a new series of normal regions, obeying the conditions of the lemma, whose inner limiting set has therefore the potency $c$. But this new inner limiting set is clearly a component of the old one, since the defining regions at each stage are internal to the original regions. Thus the given ordinary inner limiting set has in this case also the potency $c$.

Thus the theorem is proved.

### 3.

The converse theorems given by me in my paper on "Sequences of Sets of Intervals containing a given Set of Points," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, Part 4, now follow immediately for ordinary inner limiting sets in the plane or in $n$-dimensional space. The proofs there given only require verbal modification, "region" being used instead of "interval," and "span" for length."

*If any set of points $E$ be taken, and regions described round the points of $E$, so as to form a series of sets of regions whose inner limiting set contains $E$ as a component,*

> (1) *The inner limiting set consists of $E$, together with certain points of the first derived set $E'$; the latter points may sometimes be absent.*
>
> (2) *The regions may be so constructed that the inner limiting set contains every point of $E'$.*
>
> (3) *If the content of the regions*[*] *is ever less than that of $E'$, there is a more than countable set of points of $E'$ not contained in the inner limiting set.*
>
> (4) *The potency of the inner limiting set is necessarily the same as the potency of $E'$, unless $E$ contains no component dense in itself, while $E'$ is more than countable.*
>
> (5) *If $E$ contains no component dense in itself, we can so arrange the regions that the inner limiting set consists of $E$ alone; if, more-*

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[*] That is, the content of the regions without their boundary points.

*over, $E'$ be more than countable, the regions can also be so arranged that the inner limiting set is either countable or has the potency c.*

(6) *In general, we can so arrange the regions that those points of the inner limiting set which are not points of $E$ are limiting points only of $U$, the greatest component of $E$ which is dense in itself.*

Theorem V. of this article, in conjunction with Theorem II. of the preceding article, gives *the necessary and sufficient condition that a countable set should be an inner limiting set, viz., the countable set must contain no component dense in itself.*

## PART II.

### 4.

It is one of the merits of the theory of sets of points that it brings into close parallelism the theory of functions of several variables with that of a single variable. In this way theorems for one variable are often easily generalised; we need only cite the theory of multiple integration as an example. The results already arrived at in this paper will, in the same way, not only have an interest in themselves, but also an important application to the theory of functions of several variables. I propose now to show that the points of continuity of such a function form, in fact, an ordinary inner limiting set of the kind here discussed, while, conversely, given any such set, a function may be constructed having the points of that set, and no others, as its points of continuity.

It follows then, from the results of this paper, that *the set of points of continuity of a function of any number of variables cannot be at the same time countable and contain a component dense in itself.* A particular case of this result is that there is no function of two variables whose points of continuity are the rational points of the plane; this is a generalisation of the well known theorem for a single variable.

### 5.

THEOREM.—*The points of continuity of a function of any number of variables form an ordinary inner limiting set.*

Take any small positive quantity $e$, and describe round every point of continuity a region in which the oscillation of the function is less than $e$, which is, by the definition of continuity, always possible.*

---

* If the point lies on the boundary of the fundamental region, the part of the small region within the fundamental region alone concerns us, and the oscillation *in this part* must, for continuity at the boundary point, be less than $e$.

In this way we get a set of regions containing all the points of continuity as internal points. If we now let $e$ assume every value of a sequence

$$e_1, e_2, \ldots \qquad (1)$$

having zero as limit, we get a series of sets of regions whose inner limiting set certainly includes every point of continuity of the given function. That it contains no other points is immediately clear, since any point $P$ of this inner limiting set is such that, given any small quantity $e$, we can determine not only a value $e_r$ of the sequence (1) less than $e$ but also corresponding to $e_r$, a region containing $P$ as internal point, in which* the oscillation of the function is less than $e$; thus, by the definition of continuity, $P$ is a point of continuity of the given function. Thus the theorem is proved.†

THEOREM.—*Given any ordinary inner limiting set in the straight line, or plane, or higher space, a function of one, or two, or more variables can be constructed having precisely the points of that set, and no others, as points of continuity.*‡

Construct first a series of sets of normal regions having the given set $G$ as inner limiting set, (§ 2, Theorem I.).

At every point $P$ of $G$ let the function be zero;

$$f(P) = 0. \qquad (1)$$

If there be any limiting point $Q$ of $G$ which does not belong to $G$, there is a definite smallest positive integer $i_Q$ such that $Q$ is not internal to any normal region of the $i$-th set, for all values of $i \geqslant i_Q$. In each such point $Q$ let the function have the value $e^{i_Q}$, where $e$ is any small positive quantity less than 1;

$$f(Q) = e^{i_Q}. \qquad (2)$$

This function is then certainly discontinuous at every such point $Q$.

Any point $R$ which does not belong to the set $\Gamma$ got by closing $Q$ has a certain minimum distance $r$ from the points of that closed set $\Gamma$, by a well known theorem easy of proof. Thus any such point can be surrounded by a small region, of span less than $r$, containing only points of the set complementary to $\Gamma$. It follows that, if we divide the points of this complementary set into those which are rational and those which are irrational, there will, in every sufficiently small region containing $R$, be both rational and irrational points of that complementary set. Thus, if at every rational point $R$ of that complementary set the function has the corresponding value $r$, and at every irrational point $R$ the value zero, the function will be discontinuous at every point of that complementary set;

---

* See note on p. 375.

† Another way of proof is by considering the closed set of all the points at which the oscillation of the function $\geqslant k$; the complementary set, which includes all the points of continuity, consists of all the internal points of one or more regions. Thus, giving $k$ all values of a sequence having zero as limit, we get the points of continuity as an ordinary inner limiting set. This argument may also be applied to determine the analogous theorem for the points of continuity on the boundary.

‡ This theorem was proved by me for the straight line in the *Wiener Bericht*, October, 1903, in a manner closely analogous to the present.

$$f(R) = r, \tag{3}$$

$$f(R') = 0. \tag{4}$$

All the points of continuity of this function $f$ are then certainly among the points of $G$, and it is easy to prove, as follows, that every point $P$ of $G$ is, in fact, a point of continuity of $f$.

Take any large integer $m$ and any small positive quantity $h$. Draw the normal region of the $m$-th set containing $P$, and, inside this, draw a region $d$ of span less than $h$, containing $P$ as internal point.

Then, for every internal point of the region $d$, $r$ is less than $h$, since the point is within a distance $h$ of $P$ itself; also, for every point $Q$ of $\Gamma$, but not of $G$, inside the region $d$, $i_Q \geqslant m$. Therefore, if $X$ be any internal point of the region $d$,

$$|f(P) - f(X)| < h + e^m,$$

which, by choosing $m$ sufficiently large and $h$ sufficiently small, can be made smaller than any assigned quantity; so that the condition that $P$ should be a point of continuity of the function $f$ is satisfied.

Thus the function $f$ has the given ordinary inner limiting set for the set of its points of continuity.

COR.—If $F$ be any continuous function, the function $F+f$ will then also have the points of the given ordinary inner limiting set, and no others, for its points of continuity.

## 6.

It is natural to base a classification of functions on the theorems just obtained. We get in this way three chief classes:—

(1) No points of continuity (*totally discontinuous functions*).

(2) A countable set of points of continuity; this set contains then no component dense in itself.

(3) The points of continuity are of potency $c$.

The latter class includes three important special cases :—

> (3*a*) The points of continuity are dense nowhere in a certain region (the function is *pointwise continuous* in that region).
>
> (3*b*) The points of continuity are dense everywhere inside a certain region, but do not fill it up (the function is *pointwise discontinuous* in that region).
>
> (3*c*) The set of the points of continuity contains every internal point of a certain region (the function is *continuous* in that region).

Nothing is here asserted as to the continuity or discontinuity of the functions at points outside or on the boundary of the region considered.

There are various special cases of interest in the classes (3*a*) and (3*b*); the points of continuity may, for instance in the case (3*a*), include all the points of a curve, or of a set of curves ; they may, however, be entirely disconnected and contain no curve, and yet be such that for every value of $x$ there is a point of continuity, *i.e.*, they may project into a straight line, as in the case of the set of points discussed by me in the *Mathematischen Annalen* of this year. There are, of course, various other possibilities. Remarks of a similar nature apply to the discontinuities of a function of the type (3*b*).

### 7. *Ordinary Inner Limiting Sets in Curved Space.*

In order to discuss satisfactorily the distribution of the points of continuity of a function on the boundary of a region in which, for example, it is continuous, we need an easy extension of the idea of an ordinary inner limiting set, which is also interesting in itself ; this is the idea of *an ordinary inner limiting set with respect to a curve, or surface, or other curved space.*

To take, for instance, a plane curve as fundamental region ; a curve, in the most general sense, has been defined as a special case of an ordinary inner limiting set[*] :—

*A plane set of points, dense nowhere, and such that, given any small quantity e, and describing round each of the points as centre a circle of radius less than or equal to e, these circles always generate a single*

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[*] "On Regions," &c., *loc. cit.*

*region whose span does not decrease indefinitely as e decreases, is called a curved arc, or, shortly, a curve.*[*]

On such a curve we may have a set of curved arcs, like a set of intervals on a straight line, and a countable series of sets of arcs, and the set of points each of which is internal to at least one arc of every set will be called *an ordinary inner limiting set with respect to the curve.* A similar definition will apply to any curved space.

It follows from the fact that a curved arc is an inner limiting set, and that an inner limiting set of inner limiting sets is an inner limiting set,[†] that *an ordinary inner limiting set with respect to a plane curve is itself an ordinary plane limiting set*, and has therefore the properties of such a set. It is now easily proved that *the section by a plane curve of any plane ordinary inner limiting set is an ordinary inner limiting set with respect to the curve.*

For, let $G$ be the given plane set, and $E$ the section, $P$ any point of $E$, and $d$ any one of the regions defining $G$ as inner limiting set and containing $P$. Then the curve cuts the region $d$ in one or more curved arcs, one of which contains $P$ as internal point. Thus the series of sets of regions defining $G$ give a series of sets of arcs on the curve, each of which certainly contains $E$, so that every point of $E$ belongs to their inner limiting set. On the other hand, if $Q$ be any point of this latter inner limiting set, every arc of these sets which contains $Q$ is a part of one of the given regions; so that $Q$ certainly belongs to the inner limiting set $G$, and therefore to $E$. Thus $E$ is an ordinary inner limiting set with respect to the curve. Q. E. D.

Conversely, *if $F$ be a set on a curve which is not an inner limiting set with respect to the curve, it is not an ordinary plane inner limiting set.*

## 8.

We have now certain information with respect to the distribution of the points of continuity of a function on the outer rim[‡] of a region in which it is continuous. These must form an ordinary inner limiting set with respect to the outer rim, which is itself a curve.

---

[*] This includes a segment of a straight line if the span in one direction only decreases without limit.

[†] "Open Sets and the Theory of Content," *Proc. London Math. Soc.*, Ser. 2, Vol. 2, Part 1, p. 36.

[‡] I use "outer rim" here instead of "boundary" because the former is always a curve, the latter not. The outer rim of an ordinary simply connected region is the same as its boundary, *e.g.*, the outer rim of a circle is the circumference; the boundary in general is obtained by cutting out holes, (whose rims are parts of the boundary, as are also limiting points of such

Moreover, *a function can be constructed, continuous in a given region, and at the points of any ordinary inner limiting set with respect to the outer rim.* For instance, the points of continuity on the outer rim may be countable, but, if so, they must form a set without any component dense in itself. If not countable, they are of potency $c$.

---

rims), and dotting in other boundary points, ("On Regions," &c.). A precisely similar theorem, however, holds for the boundary, since it is a closed set of points.

We may, indeed, define *an ordinary inner limiting set with respect to a closed set of points $E$* as the set $G$ of all the common points of a series of closed components of $E$, or *an ordinary inner limiting set with respect to any inner limiting set $E$* as the set $G$ of all the common points of a series of ordinary inner limiting sets contained in $E$. In both these cases the arguments used in the case of a curve apply; so that $G$ is itself an ordinary inner limiting set.

# A METHOD FOR DETERMINING THE BEHAVIOUR OF CERTAIN CLASSES OF POWER SERIES NEAR A SINGULAR POINT ON THE CIRCLE OF CONVERGENCE

*By* G. H. HARDY.

[Read February 9th, 1905.—Received April 7th, 1905.—Revised May 18th, 1905.]

1. Suppose that the coefficients $a_\nu$ of a power series

$$f(x) = a_0+a_1x+a_2x^2+\dots, \tag{1}$$

whose radius of convergence is unity, are analytic functions of certain parameters $\alpha$, $\beta$, $\gamma$, and are capable, when the real parts of $\alpha+\beta-1$ and $\gamma$ are positive, of being expressed in the form

$$a_\nu = \int_0^1 \left\{ \log\left(\frac{1}{u}\right)\right\}^{\alpha-1} (1-u)^{\beta-1} u^{\gamma-1+\nu}\phi(u)\,du, \tag{2}$$

where $\phi(u)$ is an arbitrary function of $u$ which we shall assume to be regular at and in the immediate neighbourhood of all points of the straight line (0, 1).[*]

Consider the contour integral

$$\int_{(C)} (\log u)^{\alpha-1}(u-1)^{\beta-1}u^{\gamma-1+\nu}\phi(u)\,du, \tag{3}$$

$C$ denoting a loop starting from and ending at the origin and enclosing the line (0, 1). If this contour is so chosen that $\phi(u)$ is regular on and within it, and those values of the many valued functions are taken which are defined by the equations

$$(\log u)^{\alpha-1} = e^{(\alpha-1)\log\log u}, \qquad (u-1)^{\beta-1} = e^{(\beta-1)\log(u-1)}, \qquad u^{\gamma-1} = e^{(\gamma-1)\log u},$$

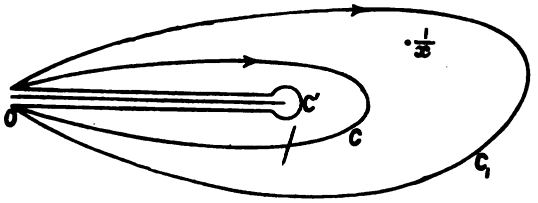
wherein $\log u$, $\log\log u$, $\log(u-1)$ are all real at the point where $C$ crosses the line $(1, \infty)$, it is evident that (3) is an analytic function of $\alpha$, $\beta$, and $\gamma$

---

[*] That is to say that $\phi(u)$ is regular within a region which includes the line, while the lower limit of the distances between points on the line and points on the boundary of the region is positive.

regular for all values of $\alpha$ and $\beta$ and all values of $\gamma$ whose real part is positive. Moreover, if the real part of $\alpha+\beta-1$ is also positive, the contour $C$ may be deformed into a contour $C'$ consisting of the two sides of the line (0, 1), and the contour integral is equal to

$$2i\sin\{(\alpha+\beta)\pi\}\int_0^1\left\{\log\left(\frac{1}{u}\right)\right\}^{\alpha-1}(1-u)^{\beta-1}u^{\gamma-1+\nu}\,du = 2i\sin\{(\alpha+\beta)\pi\}\,a_\gamma,$$



since $\log u/(u-1)$ is regular near $u=1$. Hence (unless $\alpha+\beta$ is an integer)

$$(4)\qquad a_\nu = \frac{1}{2i\sin\{(\alpha+\beta)\pi\}}\int_C(\log u)^{\alpha-1}(u-1)^{\beta-1}u^{\gamma-1+\nu}\phi(u)\,du,$$

provided the real part of $\gamma$ is positive.

Now suppose that $x$ has any value other than a real value greater than unity. Then the contour $C$ can be so chosen that the point $u=1/x$ lies outside it. But in the particular case in which $|x|<1$

$$(5)\ f(x)=\sum_0^\infty a_\nu x^\nu = \frac{1}{2i\sin\{(\alpha+\beta)\pi\}}\int_C(\log u)^{\alpha-1}(u-1)^{\beta-1}\phi(u)\frac{u^{\gamma-1}\,du}{1-xu}.$$

This equation therefore holds for all values of $x$ save real values greater than unity.

We now draw another contour $C_1$ starting from and ending at the origin and enclosing the contour $C$ *and the point* $u=1/x$, but no other singularities of $\phi(u)$. Then it is obvious from Cauchy's theorem that

$$(6)\quad f(x)=\frac{1}{2i\sin\{(\alpha+\beta)\pi\}}\left[\int_{C_1}(\log u)^{\alpha-1}(u-1)^{\beta-1}\frac{u^{\gamma-1}\phi(u)\,du}{1-xu}\right.$$
$$\left.-2\pi i\left(\log\frac{1}{x}\right)^{\alpha-1}\left(\frac{1}{x}-1\right)^{\beta-1}x^{-\gamma}\phi\left(\frac{1}{x}\right)\right]$$
$$=-\frac{\pi}{\sin\{(\alpha+\beta)\pi\}}\left(\log\frac{1}{x}\right)^{\alpha-1}(1-x)^{\beta-1}x^{1-\beta-\gamma}\phi\left(\frac{1}{x}\right)$$
$$+\frac{1}{2i\sin\{(\alpha+\beta)\pi\}}\int_{C_1}(\log u)^{\alpha-1}(u-1)^{\beta-1}$$ [illegible]

Now when $x$ tends to unity the second term on the right-hand side of (6) tends to the finite limit

$$A = \frac{i}{2\sin\{(\alpha+\beta)\pi\}}\int_{(C_1)} (\log u)^{\alpha-1}(u-1)^{\beta-2}u^{\gamma-1}\phi(u)\,du.$$

Hence

$$(7)\quad f(x) = -\frac{\pi}{\sin\{(\alpha+\beta)\pi\}}\left(\log\frac{1}{x}\right)^{\alpha-1}(1-x)^{\beta-1}x^{1-\beta-\gamma}\phi\left(\frac{1}{x}\right)+A+\epsilon_x$$

where $\epsilon_x$ is a quantity whose limit is zero when $x$ tends to 1 along any path which has no point in common with the straight line $(1, \infty)$, the many valued functions being fixed by the equations

$$\left(\log\frac{1}{x}\right)^{\alpha-1} = e^{(\alpha-1)\log\log(1/x)}, \qquad (1-x)^{\beta-1} = e^{(\beta-1)\log(1-x)},$$

$$x^{1-\beta-\gamma} = e^{(1-\beta-\gamma)\log x}$$

where $\log\log(1/x)$, $\log(1-x)$, and $\log x$ are real when $0 < x < 1$, the real part of $\gamma$ is positive, and $\alpha+\beta$ is not an integer.

2. I shall now illustrate the application of the general formula (7) by considering some interesting special cases.

(i.) Suppose $\alpha = 1$, $\phi(u) \equiv 1$. Then equation (2) becomes

$$a_\nu = \int_0^1 (1-u)^{\beta-1}u^{\gamma-1+\nu}\,du$$

$$= \frac{\Gamma(\beta)\Gamma(\gamma+\nu)}{\Gamma(\beta+\gamma+\nu)} = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\,\frac{\gamma(\gamma+1)\dots(\gamma+\nu-1)}{(\beta+\gamma)(\beta+\gamma+1)\dots(\beta+\gamma+\nu-1)},$$

and $$f(x) = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\left\{1+\frac{\gamma}{\beta+\gamma}x+\frac{\gamma(\gamma+1)}{(\beta+\gamma)(\beta+\gamma+1)}x^2+\dots\right\}.$$

Hence, if

$$F(x) = 1+\frac{\gamma}{\beta+\gamma}x+\frac{\gamma(\gamma+1)}{(\beta+\gamma)(\beta+\gamma+1)}x^2+\dots,$$

$$F(x) - \frac{\Gamma(\beta+\gamma)\Gamma(1-\beta)}{\Gamma(\gamma)}(1-x)^{\beta-1}x^{1-\beta-\gamma}$$

tends to a finite limit when $x$ approaches unity along any point lying within or along the circle of convergence. This may be verified by means of the relations which hold between the particular solutions of the hypergeometric equation.

Moreover the equation (5) furnishes the analytic continuation of the function $f(x)$ for a region exterior to the circle of convergence; a region whose extent depends on the extent of the region in which $\phi(u)$ is regular. In the present case $\phi(u)$ is a constant, and, if $f(x)$ is made uniform by a cut along $(1, \infty)$, the branch of $f(x)$ thus defined is regular all over the plane except at $x = 1$, and its singularity there is completely specified by the expression

$$\{\Gamma(\beta+\gamma)\Gamma(1-\beta)/\Gamma(\gamma)\}\ (1-x)^{\beta-1}\, x^{1-\beta-\gamma}.$$

The restriction that the real part of $\gamma$ must be positive is easily removed by the help of the recurrence formula

$$f_{\beta,\gamma}(x) - x f_{\beta,\gamma+1}(x) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\,\Gamma(\gamma)},$$

except in the trivial case in which $\gamma$ is a negative integer or zero. We also supposed that $\beta$ was not an integer. If $\beta$ (but not $\beta+\gamma$) is zero or a negative integer, $F(x)$ reduces to a sum of binomial expansions. If $\beta$ is a positive integer greater than unity, $F(x)$ remains convergent for $x = 1$; if $\beta = 1$, the infinite term is easily found to be

$$\gamma \log\{1/(1-x)\}.$$

(ii.) Suppose $\beta = 1$, $\phi(u) \equiv 1$. Then

$$a_\nu = \int_0^1 \left\{\log\left(\frac{1}{u}\right)\right\}^{\alpha-1} u^{\gamma-1+\nu}\, du = \frac{\Gamma(\alpha)}{(\gamma+\nu)^\alpha}$$

and
$$f(x) = \Gamma(\alpha)\left\{\frac{1}{\gamma^\alpha} + \frac{x}{(\gamma+1)^\alpha} + \frac{x^2}{(\gamma+2)^\alpha} + \dots\right\},$$

$(\gamma+\nu)^\alpha$ being equal to $\exp\{\alpha\log(\gamma+\nu)\}$ where the logarithm has its principal value. Hence

$$(8)\qquad \frac{1}{\gamma^\alpha} + \frac{x}{(\gamma+1)^\alpha} + \frac{x^2}{(\gamma+2)^\alpha} + \dots - \Gamma(1-\alpha)\left(\log\frac{1}{x}\right)^{\alpha-1} x^{-\gamma}$$

tends for $x = 1$ to the finite limit

$$\frac{\Gamma(1-\alpha)}{2\pi i}\int_{C_1} (\log u)^{\alpha-1}\,\frac{u^{\gamma-1}\,du}{1-u}.$$

In this case, too, the restriction as to the real part of $\gamma$ is easily removed. The restriction that $\alpha$ is not to be an integer is more fundamental.

To consider this case we must go back to the equation

$$(9)\qquad f(x)=\frac{\pi}{\sin a\pi}\left\{\log\left(\frac{1}{x}\right)\right\}^{a-1}x^{-\gamma}+\frac{i}{2\sin a\pi}\int_{C_1}(\log u)^{a-1}\frac{u^{\gamma-1}du}{1-xu}.$$

Suppose that $a=k+\epsilon$, where $k$ is a positive integer and $\epsilon$ is small. It is easy to see that we are at liberty to expand in powers of $\epsilon$ and equate coefficients. Thus we obtain

$$\frac{i}{2\pi}\int_{C_1}(\log u)^{k-1}\frac{u^{\gamma-1}du}{1-xu}=\left\{\log\left(\frac{1}{x}\right)\right\}^{k-1}x^{-\gamma}$$

and

$$(10)\quad \Gamma(k)\left\{\frac{1}{\gamma^k}+\frac{x}{(\gamma+1)^k}+\dots\right\}=(-)^k x^{-\gamma}\left(\log\frac{1}{x}\right)^{k-1}\log\log\left(\frac{1}{x}\right)$$

$$+\frac{(-)^k i}{2\pi}\int_{C_1}(\log u)^{k-1}\log\log\left(\frac{1}{u}\right)\frac{u^{\gamma-1}du}{1-xu}.$$

The equations (9) and (10) may be used to study the behaviour of $f(x)$ all over the plane. I do not propose to go into this, as the results have, in this case, been already obtained by other methods. But there is one point to which it is worth while to call attention.

The integral $$\frac{i}{2\sin a\pi}\int_{C_1}(\log u)^{a-1}\frac{u^{\gamma-1}du}{1-xu}$$

is plainly expansible, in the neighbourhood of $x=1$, in the power series

$$(11)\qquad \frac{i}{2\sin a\pi}\sum_0^{\infty}(x-1)^{\nu}\int_{C_1}(\log u)^{a-1}\frac{u^{\gamma-1+\nu}}{(1-u)^{\nu+1}}du.$$

In particular the value of the integral (11) can be assigned without calculation when $\nu=0$. For it is plainly an analytic function of $a$ regular for all non-integral values of $a$ at any rate. But, if the real part of $a$ is greater than unity, the value of the integral is clearly

$$\frac{1}{\gamma^a}+\frac{1}{(\gamma+1)^a}+\dots=\zeta(a,\gamma),$$

the well known generalised Riemann zeta function.

It follows that

$$(12)\qquad \lim_{x=1}\left\{\sum_0^{\infty}\frac{x^{\nu}}{(\gamma+\nu)^a}-\Gamma(1-a)\left(\log\frac{1}{x}\right)^{a-1}x^{-\gamma}\right\}=\zeta(a,\gamma).^*$$

It is interesting to note that the function $\zeta(a,\gamma)$, which may, for all values

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* It was first proved by Appell (*Comptes Rendus*, t. LXXXVII.) that, if

$$f_s(x)=\sum_1^{\infty}\nu^{-s}x^{\nu},$$

of $a$, be defined as the *finite term** in the asymptotic expansion of

$$\sum_1^n (\gamma+\nu)^{-a}$$

in powers of $1/n$, appears also as the finite term in the expansion of $\Sigma\, x^\nu/(\gamma+\nu)^a$ in powers of $(1-x)$.

(iii.) Suppose that $a = 1$ and $\phi(u) = (1-tu)^{-\delta}$, $t$ being any quantity whose modulus is less than unity. Then

$$a_\nu = \int_0^1 (1-u)^{\beta-1} u^{\gamma-1+\nu} (1-tu)^{-\delta}\, du = \frac{\Gamma(\gamma+\nu)\,\Gamma(\beta)}{\Gamma(\beta+\gamma+\nu)} F(\delta,\, \gamma+\nu,\, \beta+\gamma+\nu,\, t).$$

Hence the term which is not regular at $x = 1$ in the case of the series

$$(13)\quad 1+\frac{\gamma}{\beta+\gamma} F(\delta,\, \gamma+1,\, \beta+\gamma+1,\, t)\,x + \frac{\gamma(\gamma+1)}{(\beta+\gamma)(\beta+\gamma+1)} F(\delta,\, \gamma+2,\, \beta+\gamma+2,\, t)\,x^2+\dots$$

is
$$\frac{\Gamma(\beta+\gamma)\,\Gamma(1-\beta)}{\Gamma(\gamma)} (1-x)^{\beta-1} x^{1-\beta-\gamma+\delta} (x-t)^{-\delta}.$$

(iv.) Suppose that $\beta = 1$, and that $\phi(u)$ is an integral function of $\log(1/u)$, defined by

$$\phi(u) = \sum_0^\infty c_p \left\{\log\left(\frac{1}{u}\right)\right\}^p,$$

such that $\phi(u)$ remains continuous even for $u = 0$, provided $\log(1/u)$ approaches infinity by a path which does not recede indefinitely from the real axis in the plane of $\log(1/u)$, *i.e.*, provided $u$ approaches 0 by a path

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then
$$\lim_{x=1} \left[ f_s(x) \Big/ \frac{\Gamma(1-s)}{(1-x)^{1-s}} \right] = 1 \quad [R(s) < 1],$$
when $x$ tends to 1 along the line (0, 1). Various generalisations of this result have been given by Appell, Césaro, Pringsheim, Le Roy, and Lindelöf (Césaro, *Rend. della R. Accademia di Napoli*, December, 1893; Pringsheim, *Acta Math.*, Vol. XXVIII.; Le Roy, *Bulletin des Sc. Math.*, 1900–1, and *Annales de la Faculté des Sciences de Toulouse*, 1900; Lindelöf, *Acta Soc. Fennicæ*, t. XXXI., No. 3).

The method used in this paper was devised with a view to proving the particular formula

$$\lim_{x=1} \left[ \sum_1^\infty \frac{x^n}{n^s} - \Gamma(1-s) \left\{ \log\left(\frac{1}{x}\right) \right\}^{s-1} \right] = \zeta(s),$$

which at the time I supposed to be new. As a matter of fact it was first proved by Lindelöf in the memoir cited above (see his *Calcul des Résidus*, Gauthier-Villars, 1905); M. Lindelöf's method is entirely different and rests on an application of a general summation formula due in principle to Abel. The function $\Sigma\, x^\nu/(\gamma+\nu)^s$ is an infinitely many valued function whose singular points are 0, 1, $\infty$; but the *principal branch* is regular at the first of these points, as is obvious.

* See *e.g.*, Barnes, "Theory of the Gamma Function," *Mess. of Math.*, Vol. XXIX., p. 92.

which does not wind an infinite number of times round the origin. Suppose further that

$$a_\nu = \sum_0^\infty \frac{\Gamma(a+p)}{(\gamma+\nu)^{a+p}} c_p = \int_0^1 \left\{\log\left(\frac{1}{u}\right)\right\}^{a-1} u^{\gamma-1+\nu} \phi(u)\,du,$$

the series being convergent for all values of $\nu$. I shall not delay to discuss what hypotheses as to the nature of the coefficients $c_p$ are necessary to ensure the truth of this last equation; it is enough for my present purpose to say that such functions $\phi(u)$ certainly exist. For instance, M. le Roy* has proved that

$$\sum_0^\infty \frac{x^n}{1+(n+1)^2} = \int_0^1 \sin\left(\log\frac{1}{u}\right)\frac{du}{1-xu},$$

$$\sum_0^\infty \frac{e^{-a/(n+1)}}{n+1} x^n = \int_0^1 J_0\left\{2\sqrt{a}\log\left(\frac{1}{u}\right)\right\}\frac{du}{1-xu},$$

of which the first corresponds to the case in which

$$a = \gamma = 1, \qquad \phi(u) = \sin\{\log(1/u)\}.$$

Then, for example,

$$\sum_0^\infty \frac{x^n}{1+(n+1)^2} - \frac{1}{x}\sin\left(\log\frac{1}{x}\right)\log\log\left(\frac{1}{x}\right) \tag{14}$$

is regular in the neighbourhood of $x = 1$; a result which may be also deduced from (10).

It would be easy to multiply examples: but I prefer to indicate shortly how the same method may be generalised so as to apply to functions of several variables.

8. Suppose that we have a double power series

$$f(x, y)\, \Sigma\Sigma a_{\mu,\nu} x^\mu y^\nu \tag{15}$$

convergent for

$$|x| < 1, \qquad |y| < 1;†$$

and suppose that

$$a_{\mu,\nu} = \int_0^1 \left\{\log\left(\frac{1}{u}\right)\right\}^{a-1} (1-u)^{\beta-1} u^{\gamma-1+\omega\mu+\omega'\nu} \phi(u)\,du$$

(at any rate for some values of the parameters $a$, $\beta$, $\gamma$), $\omega$, $\omega'$ being two quantities whose real parts we shall, for convenience, suppose positive.

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* *Annales de la Faculté des Sciences de Toulouse*, 1900, p. 342.

† That is to say, that (1, 1) form one pair of *rayons associés : v.* Borel, *Leçons sur les Séries à Termes positifs*, p. 86.

Proceeding as in § 1, we find

$$(16)\quad f(x, y) = \frac{1}{2i\sin\{(\alpha+\beta)\pi\}}\int_C (\log u)^{\alpha-1}(u-1)^{\beta-1}\frac{u^{\gamma-1}\phi(u)\,du}{(1-xu^{\omega})(1-yu^{\omega'})}.$$

The argument is now exactly similar to our previous argument, except that, instead of having to consider only the one pole $u = 1/x$, we have to consider *two* poles, viz. those points

$$u = x^{-1/\omega}, \qquad u = y^{-1/\omega},$$

whose amplitudes are very small when $x$ and $y$ are both nearly equal to unity.

We deduce that

$$(17)\ f(x, y)+\frac{\pi}{\sin\{(\alpha+\beta)\pi\}}\left\{\frac{\omega^{-\alpha}\left\{\log\left(\frac{1}{x}\right)\right\}^{\alpha-1}(1-x^{1/\omega})^{\beta-1}x^{(1-\beta-\gamma)/\omega}\phi(x^{-1/\omega})}{1-yx^{-\omega'/\omega}}\right.$$
$$\left.+\frac{\omega'^{-\alpha}\left\{\log\left(\frac{1}{y}\right)\right\}^{\alpha-1}(1-y^{1/\omega'})^{\beta-1}y^{(1-\beta-\gamma)/\omega'}\phi(y^{-1/\omega'})}{1-xy^{-\omega/\omega'}}\right\}$$

tends to a finite limit when $x$ and $y$ tend together in any manner to unity, provided that the process to the limit is made in such a way that the denominators never vanish.

Let us suppose, for example, that $\phi(u) \equiv 1$ and $\beta = 1$. Then

$$a_{\mu,\nu} = \int_0^1\left\{\log\left(\frac{1}{u}\right)\right\}^{\alpha-1}u^{\gamma-1+\omega\mu+\omega'\nu}\,du = \frac{\Gamma(\alpha)}{(\gamma+\omega\mu+\omega'\nu)^{\alpha}},$$

$$f(x, y) = \Gamma(\alpha)\,\Sigma\Sigma\frac{x^{\mu}y^{\nu}}{(\gamma+\omega\mu+\omega'\nu)^{\alpha}}.$$

Then the part of $\Sigma\Sigma\dfrac{x^{\mu}y^{\nu}}{(\gamma+\omega\mu+\omega'\nu)^{\alpha}}$ which is not regular near (1, 1) is

$$(18)\qquad \Gamma(1-\alpha)\left\{\frac{\omega^{-\alpha}\left\{\log\left(\frac{1}{x}\right)\right\}^{\alpha-1}x^{-\gamma/\omega}}{1-yx^{-\omega'/\omega}}+\frac{\omega'^{-\alpha}\left\{\log\left(\frac{1}{y}\right)\right\}^{-1}y^{-\gamma/\omega'}}{1-xy^{-\omega/\omega'}}\right\}.$$

Let us specialise further by supposing $\omega = \omega' = 1$. We obtain

$$(19)\qquad \Gamma(1-\alpha)\left\{\frac{x^{1-\gamma}\left\{\log\left(\frac{1}{x}\right)\right\}^{\alpha-1}-y^{1-\gamma}\log\left(\frac{1}{y}\right)\right\}^{\alpha-1}}{x-y}\right\}.$$

This result may be verified. For

$$f(x, y) = \sum_{=0}^{\infty}\sum_{(\mu+\nu=k)}a_{\mu,\nu}x^{\mu}y^{\nu} = \frac{\Gamma(\alpha)}{x-y}\left\{\sum_{k=0}^{\infty}\frac{x^{k+1}-y^{k+1}}{(\gamma+k)^{\alpha}}\right\},$$

and it follows from § 2 that the irregular part of $\Sigma x^{k+1}/(\gamma+k)^a$ is

$$\Gamma(1-a)\, x^{1-\gamma} \left\{ \log\left(\frac{1}{x}\right) \right\}^{a-1}.$$

It may further be shown that

$$\lim_{x,\, y=1} \left\{ \Sigma\Sigma \frac{x^\mu y^\nu}{(\gamma+\omega\mu+\omega'\nu)^a} - \Phi(x, y) \right\} = \zeta_2(a, a, \omega, \omega'), \tag{20}$$

$\Phi(x, y)$ denoting the quantity (18) and $\zeta_2(a, a, \omega, \omega')$ Mr. Barnes' double zeta function, and the procedure to the limit being conditioned as specified above.

It is evident that the same method may be applied to the multiple series

$$\Sigma \frac{x_1^{\nu_1} x_\nu^{\nu_2} \ldots x_k^{\nu_k}}{(\gamma+\nu_1\omega_1+\ldots+\nu_k\omega_k)^a}$$

n the neighbourhood of the point (1, 1, 1, ..., 1). The same is true of such series as

$$\Sigma \frac{\Gamma(\gamma+\nu_1\omega_1+\ldots+\nu_k\omega_k)}{\Gamma(\beta+\gamma+\nu_1\omega_1+\ldots+\nu_k\omega_k)}\, x_1^{\nu_1} x_2^{\nu_2} \ldots x_k^{\nu_k},$$

the natural generalisation of the series considered in § 2 (i.). These multiple series are exceedingly interesting, and a large variety of results can be proved concerning them; my present object is merely to show as shortly as possible that by the method employed in this paper valuable nformation may be obtained conerning them, and I hope to return to the subject on some future occasion.

# THE INTERSECTION OF TWO CONIC SECTIONS

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[Received April 30th, 1905.—Read May 11th, 1905.—Received in revised form May 29th, 1905.]

IT is stated in Salmon's *Conic Sections*, sixth edition, p. 337, that these cases of four real and four imaginary intersections of two conics have not been distinguished by any simple criterion.

In the notes (p. 391) of the same volume it is further stated that this discrimination has been made by Kemmer (Giessen, 1878), and his results are quoted.

The problem has been discussed subsequently by Storey (*American Journal*, Vol. VI.) and by Gundelfinger (*Vorlesungen*, Teubner, Leipzig, 1895), whose results differ from each other and are distinct from those of Kemmer.

It is the object of this paper, whose treatment is independent of theirs in method, to establish simple criteria for the cases of intersection of two conics, with corresponding results for real and imaginary tangents.

The forms in which both Kemmer and Storey expressed their results will be derived at once from the treatment of this paper; of Kemmer's four conditions one will be shown to be unnecessary, and the results of Storey will likewise call for modification.

The following notation will be adopted:—The two conics $(abcfgh)(xyz)^2$ and $(a'b'c'f'g'h')(xyz)^2$ will be called $S$ and $S'$.

The minors of the determinants $\Delta$ and $\Delta'$ will be styled, as usual, $A, B, C, F, G, H$ and $A', B', C', \ldots$.

$$\begin{array}{rccl}
(bc'+b'c-2ff') & \text{will} & \text{be called} & K, \\
(ac'+ca'-2gg') & ,, & ,, & L, \\
(ab'+a'b-2hh') & ,, & ,, & M, \\
(gh'+g'h-af'-a'f) & ,, & ,, & K', \\
(fh'+f'h-bg'-b'g) & ,, & ,, & L', \\
(fg'+f'g-ch'-c'h) & ,, & ,, & M'.
\end{array}$$

The tangential equations of $S$ and $S'$, viz., $(ABCFGH)(uvw)^2$ and $(A'B'C'F'G'H')(uvw)^2$, will be called $\Sigma$ and $\Sigma'$, and the contravariant $(KLMK'L'M')(uvw)^2$ called, as usual, $\phi$.

The invariants of the conics will be $\Delta$, $\theta$, $\theta'$, and $\Delta'$, the cubic determining the line pairs $\lambda S+S'$, $\Delta\lambda^3+\theta\lambda^2+\theta'\lambda+\Delta' = 0$, and its discriminant $\theta^2\theta'^2+18\Delta\Delta'\theta\theta'-27\Delta^2\Delta'^2-4\Delta\theta'^3-4\Delta'\theta^3 = D$.

The question of the intersection of the two conics $S$ and $S'$ may be placed upon the simple basis of the nature of the several line pairs $\lambda S+S'$.

If $\lambda S+S'$ break up into straight lines $(lx+my+nz)$ and $(l'x+m'y+n'z)$, then $l$, $m$, $n$ and $l'$, $m'$, $n'$ can be so determined that

$$\begin{aligned}
a\lambda+a' &= ll', \\
b\lambda+b' &= mm', \\
c\lambda+c' &= nn', \\
2(f\lambda+f') &= mn'+m'n, \\
2(g\lambda+g') &= nl'+n'l, \\
2(h\lambda+h') &= lm'+l'm.
\end{aligned}$$

It readily follows that

$$\left.\begin{array}{lrl}
(1) & -(A\lambda^2+K\lambda+A') = & \{(f\lambda+f')^2-(b\lambda+b')(c\lambda+c')\} = \tfrac{1}{4}(mn'-m'n)^2 \\
(2) & -(B\lambda^2+L\lambda+B') = & \{(g\lambda+g')^2-(c\lambda+c')(a\lambda+a')\} = \tfrac{1}{4}(nl'-n'l)^2 \\
(3) & -(C\lambda^2+M\lambda+C') = & \{h\lambda+h')^2-(a\lambda+a')(b\lambda+b')\} = \tfrac{1}{4}(lm'-l'm)^2 \\
(4) & -(F\lambda^2+K'\lambda+F') = & \{(a\lambda+a')(f\lambda+f')-(g\lambda+g')(h\lambda+h')\} \\
 & & \qquad = \tfrac{1}{4}(lm'-l'm)(nl'-n'l) \\
(5) & -(G\lambda^2+L'\lambda+G') = & \{(b\lambda+b')(g\lambda+g')-(f\lambda+f')(h\lambda+h')\} \\
 & & \qquad = \tfrac{1}{4}(mn'-m'n)(lm'-l'm) \\
(6) & -(H\lambda^2+M'\lambda+H') = & \{(c\lambda+c')(h\lambda+h')-(f\lambda+f')(g\lambda+g')\} \\
 & & \qquad = \tfrac{1}{4}(nl'-n'l)(mn'-m'n)
\end{array}\right\}. \tag{I.}$$

From (1), (2), and (3) of (I.) it is readily seen that

$$u\sqrt{\{-(A\lambda^2+K\lambda+A')\}}+v\sqrt{\{-(B\lambda^2+L\lambda+B')\}} \\ +w\sqrt{\{-(C\lambda^2+M\lambda+C')\}} = \tfrac{1}{2}\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}. \tag{II.}$$

Now, if both sides of (1), (2), (3), (4), (5), and (6) be multiplied by $u^2$, $v^2$, $w^2$, $2vw$, $2uw$, and $2uv$ respectively, and the results be added,

$$-(\Sigma\lambda^2+\phi\lambda+\Sigma') = \tfrac{1}{4}\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}^2 \text{ by (II.)}$$

$$= [u\surd\{-(A\lambda^2+\ldots)\}+v\surd\{-(B\lambda^2+\ldots)\}\;.$$

$$+w\surd\{-(C\lambda^2+\ldots)\}]^2. \quad \text{(III)}.$$

If the line pair $\lambda S+S'$ be real, equations (III.), where $u$, $v$, $w$ are any real variable line coordinates, show that, since the squared expressions are positive, $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is essentially negative in sign.

If the line pair be coincident, then, since $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}$ vanishes, it follows that $(\Sigma\lambda^2+\phi\lambda+\Sigma') = 0$.

Again, if the line pair be imaginary, and the values of $\lambda$ be real, then, since $a\lambda+a' = ll'$, $b\lambda+b' = mm'$, ..., $l$ and $l'$, $m$ and $m'$, ... are all pairs of conjugate complex numbers, and therefore $(mn'-m'n)$, $(lm'-l'm)$, ... are all entirely imaginary and of the form $ti$, where $t$ is real, and therefore $\tfrac{1}{2}\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = t'i$, where $t'$ is real. Its square is therefore negative, and consequently, by (III.), the value of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is essentially positive.

In equation (III.) $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$ is plainly the equation to a vertex of the common self-polar triangle, and the three values of $\surd\{-(A\lambda^2+\ldots)\}$, $\surd\{-(B\lambda^2+\ldots)\}$, $\surd\{-(C\lambda^2+\ldots)\}$ are proportional to the coordinates of its three vertices, and so, if we contemplate the case of $\lambda S+S'$ a parallel pair, and these vertices at infinity, the vanishing of $\begin{vmatrix} \sin A & \sin B & \sin C \\ l & m & n \\ l' & m' & n' \end{vmatrix}$ indicates also the vanishing of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ for these special coordinates, and includes this case in the above.

The nature of the line pair $\lambda S+S'$, therefore, depends solely upon the sign of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$. The cases of four real and four imaginary intersections of two conics, it is well known, have thus much in common, that

, the discriminant of the cubic $\Delta\lambda^3+\theta\lambda^2+\theta'\lambda+\Delta' = 0$, is positive, and at $\lambda$ has three real values; if there be two real and two imaginary tersections, $D$ is negative and $\lambda$ has only one real value.

To distinguish the first two cases, we notice that

($\alpha$) For four real intersections, identified geometrically by the existence of three real line pairs $\lambda S+S'$, $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ must by the foregoing have *three real negative values*. (IV.)

($\beta$) For four imaginary intersections, given by

(1) $s+it,\quad s'+it'$;

(2) $s-it,\quad s'-it'$;

(3) $\sigma+i\tau,\quad \sigma'+i\tau'$;

(4) $\sigma-i\tau,\quad \sigma'-i\tau'$,

there is clearly still one real pair of common chords, *i.e.*, the lines joining (1) to (2) and (3) to (4), but the other two pairs are imaginary. The values of $\lambda$ are all real, and, as $\lambda S+S'$ imaginary implies that $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is positive, it follows that of the three values of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ *two are positive and one is negative.* (V.)

The three values of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ may now be regarded as the oots of a cubic equation in $z$, where

$$z = \Sigma\lambda^2+\phi\lambda+\Sigma',$$

ubject to

$$\Delta\lambda^3+\theta\lambda^2+\theta'\lambda+\Delta' = 0.$$

The elimination of $\lambda$ yields

$$\begin{vmatrix} \Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta' & \Delta' \\ \Sigma & \phi & \Sigma'-z & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma'-z & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma'-z \end{vmatrix} = 0, \qquad \text{(VI.)}$$

n which we note that

$$z_1z_2z_3 = \Pi(\Sigma\lambda^2+\phi\lambda+\Sigma') = 1/\Delta^2 \begin{vmatrix} \Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta' & \Delta' \\ \Sigma & \phi & \Sigma' & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma' & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma' \end{vmatrix} = \delta/\Delta^2,$$

and we propose to show that, in all cases of intersection, the determinant $\delta$ is essentially negative in sign.

By (IV.) the case of four real intersections was distinguished by the fact that $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ had three real negative values. The product of its values is therefore negative.

By (V.), four imaginary intersections required that $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ should have one negative and two positive values. The product of the three values is therefore again negative.

In the case of two real and two imaginary intersections the product is also negative; for the cubic in $\lambda$ has now two imaginary roots, but one real pair of common chords remains, *i.e.*, the line through the two real intersections and the line through the two conjugate imaginary intersections. $(\Sigma\lambda_1^2+\phi\lambda_1+\Sigma')$ is therefore negative for the real pair $\lambda_1 S+S'$. The product of $(\Sigma\lambda_2^2+\phi\lambda_2+\Sigma')$ and $(\Sigma\lambda_3^2+\phi\lambda_3+\Sigma')$, being the sum of two squares, is positive. This is clearly the case, since $\lambda_2$ and $\lambda_3$ are conjugate complex numbers. It follows that the product $\Pi(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is again negative.

The foregoing having shown that $\Pi-(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is proportional to $\Pi(ux+vy+wz)^2$, *i.e.*, the square of the tangential equation to the vertices $(x_1, y_1, z_1, \ldots)$ of the common self-polar triangle, it is now possible to identify this product $-\delta$ with $\Gamma^2$, where $\Gamma$ is the well known contravariant of the two conics.[*]

The cubic equation (VI.), when written in full, is

$$z^3+\left\{\frac{\Sigma(-\theta^2+2\theta'\Delta)-3\Delta^2\Sigma'+\theta\Delta\phi}{\Delta^2}\right\}z^2$$

$$+\left\{\frac{\begin{array}{l}\Sigma^2(\theta'^2-2\theta\Delta')+\Sigma'^2 3\Delta^2+\Sigma\Sigma'(2\theta^2-4\theta'\Delta)\\ \qquad\qquad+\Sigma\phi(3\Delta\Delta'-\theta\theta')-\Sigma'\phi\, 2\theta\Delta+\phi^2\theta'\Delta\end{array}}{\Delta^2}\right\}z$$

$$-\left\{\frac{\begin{array}{l}-\Delta\Delta'\phi^3+(\theta\Sigma\Delta'+\theta'\Sigma'\Delta)\phi^2-\{\Sigma^2\Delta'\theta'+\Sigma'^2\Delta\theta+\Sigma\Sigma'(\theta\theta'-3\Delta\Delta')\}\phi\\ \qquad\qquad+\Sigma^3\Delta'^2+\Sigma'^3\Delta^2-\Sigma^2\Sigma'(2\Delta'\theta-\theta'^2)-\Sigma'^2\Sigma(2\Delta\theta'-\theta^2)\end{array}}{\Delta^2}\right\}$$

$$= 0, \quad \text{(VI.)}$$

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[*] The distinctions of the above may be exemplified by reference to the value $T$ of the area of the common self-conjugate triangle, which can be shown to be given by

$$T^2 = -\frac{\mu^4}{4}\,\frac{D}{\delta} = -\frac{\mu^4}{4}\,\frac{D}{\Delta^2\Pi(\Sigma\lambda^2+\phi\lambda+\Sigma')},$$

where $\mu = 4T'^2\sin^2 A\sin^2 B\sin^2 C$, $T' =$ the triangle of reference, and $\Sigma$, $\phi$, and $\Sigma'$ contain line infinity coordinates.

which we shall call $$z^3+pz^2+qz+r = 0, \qquad \text{(VI.)}''$$

where $r = -\delta/\Delta^2$ has been shown to be *positive* in all cases of intersection.

The distinction between the cases of four real and four imaginary intersections is now apparent.

In the former case the cubic has three real negative roots; in the latter one real negative and two real positive roots.

Therefore, for four real points, $p$ and $q$ must both be positive.
For four imaginary points, $p$ or $q$ at least must be negative. (VII.)

This twofold condition may be embodied formally in one, if we note that, at the minimum point of the $z$ cubic, the value is negative in the former case and positive in the latter, inasmuch as $r$ is positive.

This value of $z$ is the common root of the equations

$$(1) \qquad z^3+pz^2+qz+(r-\zeta) = 0,$$

$$(2) \qquad 3z^2+2pz+q = 0,$$

for the minimum case. The elimination of $z$ between (1) and (2) gives a quadratic for $(r-\zeta)$ whose greater root must be chosen, and $(p^2-3q)$ is positive by the conditions. The common value of $z$ in (1) and (2) is easily found to be

$$z = \frac{pq-9(r-\zeta)}{-2(p^2-3q)}, \qquad (3)$$

and therefore $27(r-\zeta)^2+(4p^3-18pq)(r-\zeta)+q^2(4q-p^2) = 0$;

so $$9(r-\zeta) = -\frac{(2p^3-9pq)+2(p^2-3q)^{\frac{3}{2}}}{3},$$

and therefore, by (3), $z = \frac{1}{3}[-p+\sqrt{(p^2-3q)}]$.

For four real points $D$ is positive, $+\sqrt{(p^2-3q)}-p$ is negative. For four imaginary points $D$ is positive, $+\sqrt{(p^2-3q)}-p$ is positive, which embodies the previous twofold conditions.

It can be easily shown that the four points of intersection are given tangentially by

$$(\lambda_2-\lambda_3)\sqrt{(\Sigma\lambda_1^2+\phi\lambda+\Sigma')} \pm (\lambda_3-\lambda_1)\sqrt{(\Sigma\lambda_2^2+\phi\lambda+\Sigma')}$$
$$\pm (\lambda_1-\lambda_2)\sqrt{(\Sigma\lambda_3^2+\phi\lambda_3+\Sigma')} = 0,$$

and this suggests the derivation of another cubic in $z'$ whose coefficients will display collateral symmetry in $\Sigma$, $\phi$, $\Sigma'$ and the invariants.

If, therefore, the roots of the cubic in $z$ be multiplied respectively by

$\Delta^2(\lambda_2-\lambda_3)^2$, $\Delta^2(\lambda_3-\lambda_1)^2$, and $\Delta^2(\lambda_1-\lambda_2)^2$, we shall arrive at a cubic in $z'$ whose coefficients are the expressions used in Kemmer's conditions.

In this case, to form $z'^3+p'z'^2+q'z'+r'=0$,

we have

$$\Sigma\left\{\Delta^2(\lambda_2-\lambda_3)^2(\Sigma\lambda_1^2+\phi\lambda_1+\Sigma')\right\}$$
$$=2\Sigma(\theta'^2-3\theta\Delta')+\phi(\theta\theta'-9\Delta\Delta')+2\Sigma'(\theta^2-3\theta'\Delta);$$

and therefore
$$p'=\begin{vmatrix}\Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta'\end{vmatrix},$$

and similarly

$$q'=\Sigma\left\{\Delta^4(\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)^2(\Sigma\lambda_1^2+\phi\lambda_1+\Sigma')(\Sigma\lambda_2^2+\phi\lambda_2+\Sigma')\right\}$$
$$=\tfrac{1}{4}\left\{p'^2-(\phi^2-4\Sigma\Sigma')D\right\},$$

and
$$r'=rD\Delta^2=-D\delta=-D\begin{vmatrix}\Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta' & \Delta' \\ \Sigma & \phi & \Sigma' & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma' & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma'\end{vmatrix},$$

and the $z'$ cubic appears as

$$z'^3+\begin{vmatrix}\Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta'\end{vmatrix}z'^2+\tfrac{1}{4}\left\{\begin{vmatrix}\Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta'\end{vmatrix}^2-(\phi^2-4\Sigma\Sigma')D\right\}z'$$
$$+rD\Delta^2=0 \qquad \text{(VIII.)}$$

or
$$z'^3+p'z'^2+q'z'+r'=0,$$

and all the former conclusions still hold from the positive nature of the multipliers of the former roots in $z$.

### *Criticism of Kemmer's Results.*

Kemmer's results are that for four real intersections the following four conditions must be satisfied, viz., $D$, $p'$, $q'$, and $r'$ must *all* be positive. Now $r'=rD\Delta^2$, and, if $D$ be positive, this asserts that $r$ must also be positive.

But we have shown clearly that in all possible cases of intersection $r$

is positive and equal to $\Gamma^2/\Delta^2$; and therefore this fourth condition of Kemmer is superfluous.

The case of four real intersections may now be distinguished with advantage by (1) $D$ positive, (2) $+\sqrt{(p'^2-3q')}-p'$ must be negative.

### *The Results of Storey.*

From the original identities (I.) we can deduce very simply the results which Storey obtained by a different method.

$$\left.\begin{array}{ll}
(1) & -(A\lambda^2+K\lambda+A') = \tfrac{1}{4}(mn'-m'n)^2 \\
(2) & -(B\lambda^2+L\lambda+B') = \tfrac{1}{4}(nl'-n'l)^2 \\
(3) & -(C\lambda^2+M\lambda+C') = \tfrac{1}{4}(lm'-l'm)^2 \\
(4) & -(F\lambda^2+K'\lambda+F') = \tfrac{1}{4}(lm'-l'm)(nl'-n'l) \\
(5) & -(G\lambda^2+L'\lambda+G') = \tfrac{1}{4}(mn'-m'n)(lm'-l'm) \\
(6) & -(H\lambda^2+M'\lambda+H') = \tfrac{1}{4}(nl'-n'l)(mn'-m'n)
\end{array}\right\}. \qquad \text{(I.)}'$$

If we choose for $u, v, w$ the quantities $(ax+hy+gz)$, $(hx+by+fz)$, and $(gx+fy+cz)$ as multipliers, and multiply (1) to (6) respectively by $(ax+hy+gz)^2, \ldots, 2(hx+by+fz)(gx+fy+cz), \ldots$, and add once more, we shall find the sums to give, as may be readily verified for the two simplest conics,

$$-\{\Delta S\lambda^2+(\theta S-\Delta S')\lambda+(\theta' S-F)\} = \tfrac{1}{4}\{\Sigma(mn'-m'n)(ax+hy+gz)\}^2, \qquad \text{(IX.)}$$

where $F$ is the covariant conic of $S$ and $S'$, and the right-hand side now represents the square of the equation of a side of the common self-polar triangle when equated to zero.

The apparent anomaly, common to the right-hand sides of (IX.) and (III.), that, despite the fact of *real* vertices and sides of the self-polar triangle in the case of four imaginary intersections, the square of $\Sigma(mn'-m'n)\,u$ should be negative is explained by the statement that $(mn'-m'n), \ldots$ are only *proportional* to the values of the coordinates of the vertices, *e.g.*, $(mn'-m'n) = kx_1 i$, where $x_1$ is real, &c.

The function $-\{\Delta S\lambda^2+(\theta S-\Delta S')\lambda+(\theta' S-F)\}$ now simply replaces the former $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$, and all the conclusions with respect to the sign of the latter apply equally to that of the former.

The most symmetrical form of these new criteria will now be obtained by the simple summations, &c., used for the $z$ cubic, and, just as the sum of three values of $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$ appeared as $p$, so does the sum of the three values of $-\{\Delta S\lambda^2+(\theta S-\Delta S')\lambda+(\theta' S-F)\}$ appear as Storey's

$$S_1 = (-\theta' S-\theta S'+3F).$$

Similarly the equivalent of $q$ is Storey's

$$S_2 = (\theta\Delta' S^2+\theta'\Delta S'^2+3F^2+(\theta\theta'-3\Delta\Delta')\,SS'-2\theta' SF-2\theta S'F),$$

and, lastly, the equivalent of $r$ appears as Storey's

$$S_3 = F^3-F^2(\theta S'+\theta' S)+F\,\{\Delta'\theta S^2+\Delta\theta' S'^2+(\theta\theta'-3\Delta\Delta')\,SS'\}-\Delta\Delta'^2S^3$$
$$-\Delta'\Delta^2S'^3+\Delta'(2\Delta\theta'-\theta^2)\,S^2S'+\Delta\,(2\Delta'\theta-\theta'^2)\,SS'^2.$$

Storey states that for four real intersections $D$ must be positive, $S_2 > 0$, $S_1 S_3 \geqslant 0$. Now $S_3$, in harmony with the interpretation of (IX.), and on independent grounds, is clearly equal to $J^2$, when $J$ is the Jacobian of $S$ and $S'$, and is, moreover, the exact equivalent of $r$, which we showed to be positive in all cases.

The same reasoning applied to $S_3$ as to $r$ shows that it is also necessarily *positive*. The appearance of $J^2$ is what we should be led to expect from the reciprocal method implied in the use of the above multipliers $(ax+hy+gz)$, ..., and, just as we showed $r = \Gamma^2/\Delta^2$ to be always positive and $\Gamma$ always real, so does $S_3$ remain always positive and $J$ always real. The results of Storey should consequently be modified to read $D$ positive, $S_1 \geqslant 0$, $S_2 > 0$.

### *Gundelfinger's Results.*

By different methods Gundelfinger arrives at the expression $(\Sigma\lambda^2+\phi\lambda+\Sigma')$, and thence deduces Storey's

$$S_1 = (-\theta' S-\theta S'+3F),$$

the equivalent of $p$ in point coordinates. This function he treats as a "combinant" conic $\psi$, gives it a geometrical meaning, and deduces three conditions for four real intersections from the fact that it must represent an imaginary conic in this case, so as to constantly preserve a positive sign for all values of the variables.

These three conditions, however, contain, in addition to the invariants, the specific constants of the two conics $S$ and $S'$, and do not present the results in invariant contravariant or invariant covariant forms appropriate to the general projective problem.

The following alternative criteria have now been established:—

For four real points

(1) $D$ must be positive;

(2) $p' = \begin{array}{ccc} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{array}$ must be positive;

(3) $q' = \frac{1}{4}\{p'^2 - D(\phi^2 - 4\Sigma\Sigma')\}$ must be positive;

for all values of the variables.

For four imaginary points (1) $D$ must be positive, (2) $p'$ or $q'$ at least negative. (X.)

Or for four real points

(1) $D$ must be positive,

(2) $S_1 = (-\theta' S - \theta S' + 3F)$ must be positive or zero,

(3) $S_2 = \{\theta\Delta' S^2 + \theta'\Delta S'^2 + 3F^2 + (\theta\theta' - 3\Delta\Delta')\, SS' - 2\theta' SF - 2\theta S'F\}$ must be positive,

for all values of the variables.

For four imaginary points (1) $D$ must be positive, but not at once $S_1 \geqslant 0$, $S_2 > 0$. (XI.)

Both sets of criteria require alternatives for special cases of intersection, as follows:—

In (X.), if $D = 0$, or if there be contact, the other intersections are real or imaginary, according as $p' >$ or $< 0$. If $D = 0$ and $p' = 0$, $q'$ is also $= 0$, and double contact is easily inferred. The distinction between real and imaginary double contact is given by the sign of $p$ in the $z$ cubic, or more symmetrically thus. The $\lambda$ cubic has equal roots, one common to $\Sigma\lambda^2 + \phi\lambda + \Sigma' = 0$. Two values of $\Sigma\lambda^2 + \phi\lambda + \Sigma'$ vanish, and the third value $(= -p)$ is easily found to be, since the unequal root of $\lambda$ is $\Delta'(2\theta^2 - 6\Delta\theta')/\Delta(2\theta'^2 - 6\Delta'\theta)$,

$$-p = \frac{\Sigma\Delta'^2(2\theta^2 - 6\Delta\theta')^2 + \phi\Delta\Delta'(\theta\theta' - 9\Delta\Delta')^2 + \Sigma'\Delta^2(2\theta'^2 - 6\Delta'\theta)^2}{\Delta^2(2\theta'^2 - 6\Delta'\theta)^2}; \quad \text{(XII.)}$$

and therefore the double contact is real or imaginary, according as $p >$ or $< 0$. The cases of osculation require no criteria.

Similarly in (XI.). For four real intersections $S_1$ must generally be positive for all values of the variables; but in the case of four-pointed osculation it is easily shown to vanish identically, requiring $S_1 > 0$.

If the quantities $p'$ and $q'$ in the $z'$ cubic receive special constant values, *i.e.*, $\sin A$, $\sin B$, $\sin C$ for $u$, $v$, $w$ in $\Sigma$, $\phi$, and $\Sigma'$, these quantities are invariants for projections, such as the linear Cartesian transformation, in which the line at infinity is unaltered, and the criteria also have a metrical significance exhibited in a footnote.*

In conclusion, the criteria for real and imaginary common tangents may now be developed.

If we reciprocate the original conics $S$ and $S'$ with respect to

$$x^2+y^2+z^2 = 0,$$

and apply the foregoing criteria for real intersections to

$$R = Ax^2+By^2+Cz^2+2Fyz, \ldots,$$
$$R' = A'x^2+B'y^2+\ldots,$$

we shall evidently get the criteria for four real common tangents.

The identities analogous to (I.) now assume the types

$$\left.\begin{array}{ll} -\{a\Delta\lambda^2+(BC'+B'C-2FF')\lambda+a'\Delta'\} & = \tfrac{1}{4}(mn'-m'n)^2, \ \ldots \\ -\{f\Delta\lambda^2+(GH'+G'H-AF'-A'F)\lambda+f'\Delta'\} & = \tfrac{1}{4}(lm'-l'm)(nl'-n'l), \ldots \end{array}\right\} \quad \text{(XIII.)},$$

and, if we multiply these six identities on both sides by

$$(Au+Hv+Gw)^2, \ \ldots, \qquad 2(Hu+Bv+Fw)(Gu-Fv+Cw), \ \ldots,$$

and add once more, we shall obtain the new equivalent of

$$-\{\Sigma\lambda^2+\phi\lambda+\Sigma'\}$$

as

$$-\{\Delta^2\Sigma\lambda^2+\Delta(\theta'\Sigma-\Delta\Sigma')\lambda+\Delta'(\Sigma\theta-\Delta\phi)\},$$

---

* If $Q$ represent the area of the quadrilateral formed by the four points of intersection, and

$$z' = -\frac{Q^2(\phi^2-4\Sigma\Sigma')^2}{16\mu^4},$$

where $$\mu^4 = 4T'^2\sin^2 A\sin^2 B\sin^2 C,$$

it can be readily shown that the three values of $Q^2$ are given by the roots of the $z'$ cubic

$$z'^3+p'z'^2+q'z'+r' = 0,$$

a verification of this equation's validity.

For four real points there are three real values of $Q^2$ which are positive, viz., the conventional area $(1234)^2$, $(\Delta 123 \sim \Delta 124)^2$ and $(\Delta 124 \sim \Delta 134)^2$.

For four imaginary points there are two negative values of $Q^2$, and one positive value, corresponding to the one real area which is easily shown to remain for two pairs of conjugate imaginary points. The invariable values of $\Sigma$, $\Sigma'$, and $\phi$ here employed, with line infinity coordinates used for the variables, subside in the case of Cartesians into the well known invariants $(ab-h^2)$, $(ab'+a'b-2hh')$, and $(a'b'-h'^2)$.

where the new cubic determining $\lambda R+R'$ is

$$\Delta^2\lambda^3+\Delta\theta'\lambda^2+\Delta'\theta\lambda+\Delta'^2 = 0.$$

As before, the product of the three values of this new quantity is positive in all cases and is equal to $r\Delta^4\Delta'^2$, and we must simply express that the sum of its three values is positive and the sum of its product pairs positive for four real common tangents.

The sum $= P = (-\theta\Delta'\Sigma-\theta'\Delta\Sigma'+3\Delta\Delta'\phi)$.

The sum of the product pairs

$$= Q = \Delta\Delta'\{\theta'\Delta'\Sigma^2+\theta\Delta\Sigma'^2+3\Delta\Delta'\phi^2+(\theta\theta'-3\Delta\Delta')\Sigma\Sigma'$$
$$-2\theta\Delta'\Sigma\phi-2\theta'\Delta\Sigma'\phi\},$$

and the product $= r\Delta^4\Delta'^2 = \Gamma^2\Delta^2\Delta'^2$,

and the new cubic whose roots are the values of

$$-\{\Delta^2\Sigma\lambda^2+\Delta(\theta'\Sigma-\Delta\Sigma')\lambda+\Delta'(\Sigma\theta-\Delta\phi)\}$$

is
$$\zeta^3+P\zeta^2+Q\zeta+\Gamma^2\Delta^2\Delta'^2 = 0.$$

The discriminant of the new $\lambda$ cubic is now $\Delta^2\Delta'^2D$, where $D$ has its old meaning.

> And therefore for four real common tangents (1) $D$ must be positive, (2) $P$ must be positive, (3) $Q$ must be positive.
>
> For four imaginary common tangents (1) $D$ must be positive, and (2) $P$ or $Q$ at least must be negative. (XIV.)

If $D$ be negative, there are two real and two imaginary common tangents.

These tangent conditions can be thrown into the alternative forms involving $S$, $S'$, and the covariant $F$, by multiplying the identities of (XIII.) by $x^2$, $y^2$, $z^2$, $2yz$, $2zx$, and $2xy$, and then adding.

The result gives $-(\Delta S\lambda^2+F\lambda+\Delta'S')$ as the analogue of the original $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$.

Since this new quantity stands in relation to the cubic

$$\Delta^2\lambda^3+\Delta\theta'\lambda^2+\Delta'\theta\lambda+\Delta'^2 = 0$$

in precisely the same way as $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$ stood with regard to the original $\lambda$ cubic, the new results may be at once inferred from (X.) by writing $\Delta S$, $F$, and $\Delta'S'$ for $\Sigma$, $\phi$, and $\Sigma'$, and making corresponding changes in the invariants.

The criteria appear as follows:—

For four real common tangents

(1) $D$ must be positive,

(2) $$S_1 = \Delta\Delta' \begin{vmatrix} S & F & S' \\ 3\Delta & 2\Delta\theta' & \theta \\ \theta' & 2\Delta'\theta & 3\Delta' \end{vmatrix} \text{ must } > 0,$$

(3) $$S_2 = \tfrac{1}{4}\{S_1^2 - \Delta^2\Delta'^2 D(F^3 - 4\Delta\Delta' S S')\} > 0, \qquad \text{(XV.)}.$$

for all real values of the variables.

For four imaginary common tangents (1) $D$ must $> 0$, (2) not at once $S_1 > 0$, $S_2 > 0$.

The results of (X.) and (XI.), of (XIV.) and (XV.) constitute a complete solution of the problem proposed. While it is obvious that the results may be expressed in an unlimited number of modes by varying the positive symmetrical multipliers of the roots of the cubics discussed, an effort has been made to present them in their simplest and most symmetrical forms.

# ON THE PROJECTION OF TWO TRIANGLES ON TO THE SAME TRIANGLE

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[Read March 9th, 1905.—Received March 17th, 1905.]

## *Abstract.*

There are in the theory of projection many constructions for the solution of a problem which contain one or more arbitrary elements. In such cases interesting results are sometimes obtainable by a consideration of the aggregate of all the possible solutions of a problem.

A simple example is the construction for determining the correspondence of two homographic point rows when three points of the one and the three corresponding points of the other are given.

In this case the line joining two centres of projection, from which both point rows may be projected on' to the same point row, touches a conic.

The problem considered in this paper is the projection of two triangles on to the same triangle, and the results which may be deduced from a consideration of the aggregate of all the possible solutions.

We are not aware that the same problem has been dealt with by any previous writer. Grassmann in the fourth section of his paper entitled "Die stereometrischen Gleichungen dritten Grades, und die dadurch erzeugten Oberflächen" (which section deals with the projective relation between planes and pencils of rays in space) obtains incidentally a construction for projecting any four points in one plane by three projections and three sections on to any four points in another plane (no three of the four points in either plane being in a straight line). Other writers, of whom we believe the first was Seydewitz, discussed the distinct problem of finding the number of ways in which a plane containing any four fixed points can be moved in space and so placed on another plane that the four fixed points are in perspective with four fixed points in the second plane (no three of the four points in either plane being in a straight line). The number of solutions turns out to be four, and is in contrast with the number—six—of solutions of some of the problems which occur in this

paper. But the problem discussed by Grassmann is of a higher order of complexity than the one from which the present investigation arises. The present investigation leads to a construction which solves Grassmann's problem in a way which at one step is of a more general character than that adopted by Grassmann; and its value consists in the light which it throws on that problem.

The first result that is here obtained is this:—The straight line joining two points from which two triangles in space can be projected on to the same triangle is a generator of the regulus of which the three directing lines are the lines joining corresponding vertices of the triangles (Art. 1).

It is next shown that the two triangles can be projected into the same triangle in any plane in space (Art. 2).

From this the construction for the solution of the problem considered by Grassmann, and referred to above, follows easily (Art. 3).

It is then proved that the projective relation between the planes of the two triangles is unaltered if the centres of projection be moved to other positions on the same generator of the regulus referred to in the first article (Art. 4), and in this case the plane on to which the two triangles are projected passes through a fixed point on the line of intersection of the planes of the two triangles. The fixed point is a self-corresponding point of the planes of the two triangles (Art. 5).

A construction for the determination of this fixed point, when the position of the generator of the regulus is known, is then given (Art. 6).

The next two articles (7 and 8) deal with special positions of the plane on to which the two triangles are projected, which are important for the drawing of the figures required.

The regulus of Art. 1 meets each of the planes of the triangles in a conic. The vertices of one triangle and the point in which the generator joining the centres of projection meets one of the conics correspond to the vertices of the other triangle and the point in which the generator joining the centres of projection meets the other conic. In general no fifth point of either conic can correspond to any point of the other conic. If any fifth point on the first conic correspond to a point on the second conic, then all the points on the first conic correspond to points on the second conic, and the projective relation between the two planes is that which is determined by making those points on the conics correspond which lie on the same generator of the regulus (Art. 9).

It is next shown that, if three points $A_1$, $B_1$, $C_1$ in one plane correspond to three points $A_2$, $B_2$, $C_2$ on another plane, then, as the straight line joining the centres of projection describes the regulus, the point $D_2$ in the second plane which corresponds to a fixed point $D_1$ in the first plane describes a nodal cubic passing through $A_2$, $B_2$, $C_2$ (Arts. 10 and 11).

Arts. 12–14 are concerned with the construction of the node and the drawing of the cubic, and the way in which the node arises is explained.

In the last article (15) it is shown that the least number of projections and sections necessary to pass from one set of four points in a plane to another set of four points in a second plane is *in the general case* three. There are, of course, the special cases in which two or one of each are sufficient.

It follows that Grassmann's solution is the simplest.

1. If $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles situated anywhere in space, and if $O_1$ and $O_2$ be two points such that $O_1$ projects $A_1B_1C_1$ and $O_2$ projects $A_2B_2C_2$ into the same triangle $ABC$, to prove that the sole condition which must be satisfied by $O_1$ and $O_2$ is that $O_1O_2$ is a generator of the regulus determined by the three directing lines $A_1A_2$, $B_1B_2$, and $C_1C_2$.

Since $O_1A_1$ meets $O_2A_2$ at $A$, therefore the points $O_1$, $A_1$, $O_2$, $A_2$, $A$ all lie in one plane.

Therefore $O_1O_2$ meets $A_1A_2$.

Similarly $O_1O_2$ meets $B_1B_2$ and $C_1C_2$.

Hence $O_1O_2$ is a generating line of the regulus of which $A_1A_2$, $B_1B_2$, and $C_1C_2$ are the directing lines.

Conversely, if $O_1$ and $O_2$ lie on a generating line of the regulus of which $A_1A_2$, $B_1B_2$ and $C_1C_2$ are the directing lines, then $O_1O_2$ meets $A_1A_2$, and therefore $O_1A_1$ meets $O_2A_2$ in a point which may be called $A$; similarly $O_1B_1$ meets $O_2B_2$ in a point which may be called $B$, and $O_1C_1$ meets $O_2C_2$ in a point which may be called $C$. Therefore $O_1$ projects $A_1B_1C_1$ and $O_2$ projects $A_2B_2C_2$ into the same triangle $ABC$.

Throughout this paper $A_1A_2$, $B_1B_2$, and $C_1C_2$ will be said to belong to the first system of the generators of the hyperboloid on which the above mentioned regulus lies; and $O_1O_2$ will be said to belong to the second system of generators, and will be sometimes denoted by the single letter $g$. The intersection of the two planes $A_1B_1C_1$ and $A_2B_2C_2$ will be called the line $i$.

2. The triangles $A_1B_1C_1$, $A_2B_2C_2$ having any position in space, and any plane $\pi$ being chosen arbitrarily, then it is in general possible to find $O_1$ and $O_2$ so as to project $A_1B_1C_1$ and $A_2B_2C_2$ into a triangle $ABC$ situated in the plane $\pi$.

Let the straight lines $B_1C_1$, $C_1A_1$, $A_1B_1$, $B_2C_2$, $C_2A_2$, $A_2B_2$ meet the plane $\pi$ in the points $J_1$, $K_1$, $L_1$, $J_2$, $K_2$, $L_2$ respectively. (See Fig. 1.)

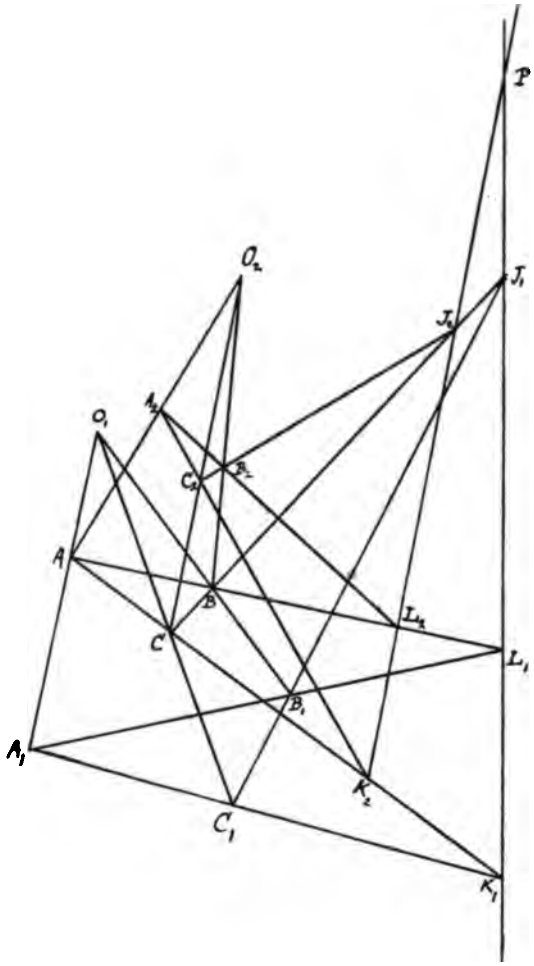

FIG. 1.

The most general case is that in which each of these six points is a single fully determined point and all the six points are distinct.

This case will be taken first.

The points $J_1$, $K_1$, $L_1$ are collinear, because they lie on the intersection of the planes $A_1B_1C_1$ and $\pi$.

Similarly $J_2$, $K_2$, $L_2$ are collinear.

The straight lines $J_1K_1L_1$ and $J_2K_2L_2$ are in general distinct.

Since $J_1K_1L_1$ and $J_2K_2L_2$ are in one plane $\pi$, therefore $K_1K_2$ will meet $L_1L_2$ in a point which may be called $A$.

Similarly let $L_1L_2$ meet $J_1J_2$ at $B$, and let $J_1J_2$ meet $K_1K_2$ at $C$.

Then $BC$ is the line $J_1J_2$, and therefore meets $B_1C_1$ at $J_1$.

Similarly $CA$ meets $C_1A_1$ at $K_1$, and $AB$ meets $A_1B_1$ at $L_1$.

Since $J_1$, $K_1$, and $L_1$ are collinear, therefore the triangles $ABC$ and $A_1B_1C_1$ are in perspective.

Therefore $AA_1$, $BB_1$, and $CC_1$ meet in some point $O_1$.

Similarly $AA_2$, $BB_2$, and $CC_2$ meet in some point $O_2$.

The positions of $O_1$ and $O_2$ have therefore been found which will project $A_1B_1C_1$ and $A_2B_2C_2$ into a triangle $ABC$ in the plane $\pi$.

3. The construction given in the last article renders it possible to determine the projective relation between two planes when four points in the one and the corresponding four points in the other are given. [It will be supposed that no three of either set of four points lie on one straight line.]

The construction about to be given is in the case of one step a little more general than that given by Grassmann (*Crelle*, T. XLIX., S. 55)

Suppose that $A_1$, $B_1$, $C_1$, $D_1$ in the plane $\pi_1$ correspond to $A_2$, $B_2$, $C_2$, $D_2$ in the plane $\pi_2$.

Take any point $O_2$ (distinct from $A_1$ and $A_2$) on the straight line $A_1A_2$.

Through $A_1$ draw any plane $\pi_3$ (distinct from $\pi_1$).

From $O_2$ project $A_2B_2C_2D_2$ into $A_1B_3C_3D_3$ on $\pi_3$.*

Draw any plane $\pi$ (distinct from $\pi_1$ and $\pi_3$) through $A_1$. Then, by means of Art. 2, find the centres $O'_1$ and $O'_3$ which will project $B_1C_1D_1$ and $B_3C_3D_3$ into the same triangle $BCD$ in the plane $\pi$.

Since the point $A_1$ is common to the planes $\pi_1$, $\pi_3$, and $\pi$, it follows that the projections from $O'_1$ and $O'_3$ will both leave it unaltered in position.

Thus $O_2$ projects the points $A_2, B_2, C_2, D_2$ situated on $\pi_2$ into $A_1, B_3, C_3, D_3$ on $\pi_3$.

Then $O'_3$ projects the points $A_1, B_3, C_3, D_3$ situated on $\pi_3$ into $A_1, B, C, D$ on $\pi$.

Then $O'_1$ projects $A_1$, $B$, $C$, $D$ on $\pi$ into $A_1$, $B_1$, $C_1$, $D_1$ on $\pi_1$.

There are, therefore, three projections and three sections, as in Grassmann's method, necessary to establish the projective relation between the two planes when four points in one plane and the corresponding four points in the other plane are given.

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* Grassmann's next step is to draw a plane through $A_1B_1$. It is possible, however, to proceed more generally, as in this paper, by drawing any plane through $A_1$.

4. The projective relation between the planes of the two triangles is fully determined when the position of the generator $g$, on which $O_1$ and $O_2$ lie, has been fixed; in other words, if $O_1$ and $O_2$ be moved to other positions along $g$, the point $E_2$ in the plane $A_2B_2C_2$ which corresponds to a fixed point $E_1$ in the plane $A_1B_1C_1$ remains unaltered.

Let $g$ meet the planes $ABC$, $A_1B_1C_1$, and $A_2B_2C_2$ at $D$, $D_1$, and $D_2$ respectively. Then $O_1$ projects $A_1$, $B_1$, $C_1$, $D_1$ into $A$, $B$, $C$, $D$, which are projected from $O_2$ on to $A_2$, $B_2$, $C_2$, $D_2$.

Hence $A_1$, $B_1$, $C_1$, $D_1$ in the one plane correspond to $A_2$, $B_2$, $C_2$, $D_2$ in the other plane for all positions of $O_1$, $O_2$ on the same generator $g$. Now, when four points in the one plane forming a quadrilateral and the corresponding points in the other plane are given, the projective relation between the two planes is determined by the theorem of which a proof is given in the preceding article.

Hence the point $E_2$ in the plane $A_2B_2C_2$ corresponding to a fixed point $E_1$ in the plane $A_1B_1C_1$ will not be altered if $O_1$ and $O_2$ are moved along the fixed generator $g$.

[It must be noticed, however, that the plane $ABC$ is altered in position when $O_1$ and $O_2$ are moved along the fixed generator $g$.]

5. To prove that, when $O_1$ and $O_2$ lie on a fixed generator $g$, the plane $ABC$ passes through a fixed point on the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$, and that this is a self-corresponding point of the two planes.

Suppose that, for some given positions of $O_1$ and $O_2$ on $g$, the straight lines $J_1K_1L_1$ and $J_2K_2L_2$, both of which lie in the plane $ABC$, meet at $P$ (see Fig. 1). Then $P$ is the intersection of the three planes $ABC$, $A_1B_1C_1$, and $A_2B_2C_2$.

Then $O_1$ projects $P$ in the plane $A_1B_1C_1$ into $P$ in the plane $ABC$, and then $O_2$ projects $P$ in the plane $ABC$ into $P$ in the plane $A_2B_2C_2$.

Therefore $P$ in the plane $A_1B_1C_1$ corresponds to $P$ in the plane $A_2B_2C_2$.

Now, when the generator $g$ is fixed, the projective relation between the two planes is determined.

Hence $P$ in the plane $A_1B_1C_1$ corresponds to $P$ in the plane $A_2B_2C_2$ for all positions of $O_1$ and $O_2$ on $g$.

Hence the plane $ABC$ passes through the point $P$ for all positions of $O_1$ and $O_2$ on $g$.

The point $P$ may be called the self-corresponding point of the two planes when $O_1$ and $O_2$ lie on $g$.

The point $P$ in the intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$ will be said to correspond to the generator $g$.

6. To show how to construct the position of $P$ when that of $g$ is known.

Let $PA_1$ meet $B_1C_1$ at $Q_1$. (See Fig. 2.)

Let $O_1Q_1$ meet $BC$ at $Q$.

Let $O_2Q$ meet $B_2C_2$ at $Q_2$.

Then $Q_1$ in the plane $A_1B_1C_1$ corresponds to $Q_2$ in the plane $A_2B_2C_2$.

Also $O_1Q_1$ and $O_2Q_2$ meet at $Q$. Therefore $Q_1Q_2$ meets $O_1O_2$.

Also $A_1A_2$ meets $O_1O_2$.

Again, $P$, $Q_1$, $A_1$ correspond to $P$, $Q_2$, $A_2$ respectively.

Now $P$, $Q_1$, $A_1$ are collinear; therefore $P$, $Q_2$, $A_2$ are collinear.

Therefore $A_1A_2$ meets $Q_1Q_2$.

We have also proved that $A_1A_2$ meets $O_1O_2$, and $Q_1Q_2$ meets $O_1O_2$.

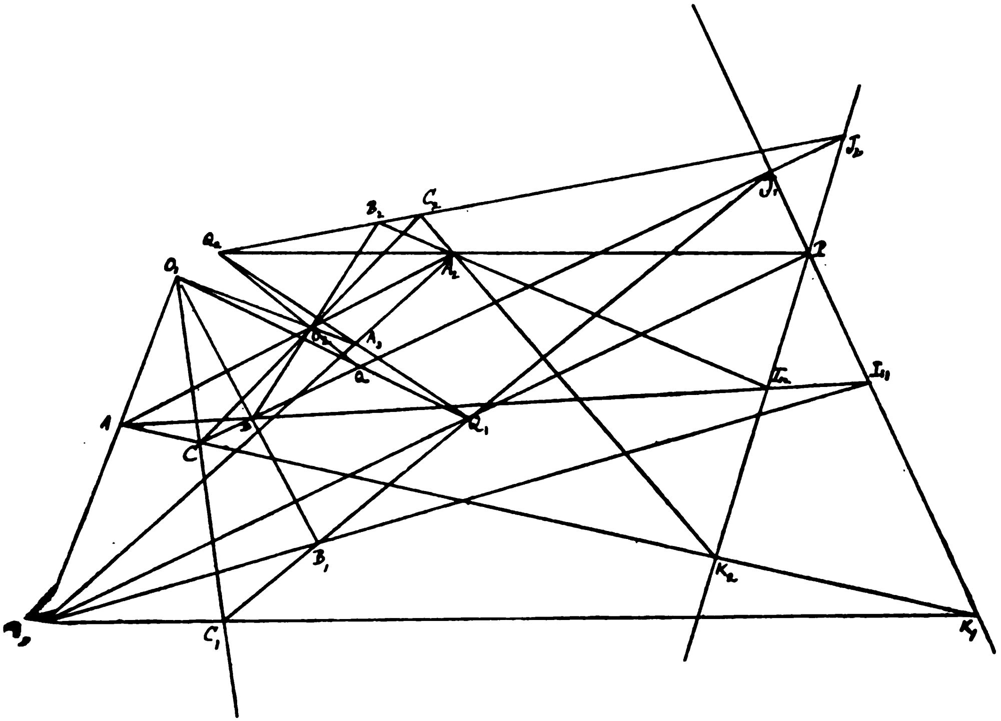

FIG. 2.

Hence there are two alternatives:—

(*a*) The three straight lines $A_1A_2$, $O_1O_2$, and $Q_1Q_2$ all lie in one plane.

In this case $P$ lies in the plane which contains $O_1O_2A_1A_2$.

Similarly it could be proved that $P$ lies in the plane $O_1O_2B_1B_2$ and in the plane $O_1O_2C_1C_2$.

But these planes meet in $O_1O_2$.

Hence $P$ lies on $O_1O_2$, which cannot be generally true.

(*b*) The lines $A_1A_2$, $O_1O_2$, and $Q_1Q_2$ have one point in common.

This being the only alternative left, we take it to hold.

Let $A_1A_2$ meet $O_1O_2$ at $A_3$.

Therefore $Q_1Q_2$ passes through $A_3$.

The line $A_3Q_1Q_2$ meets $O_1O_2$, $B_1C_1$, and $B_2C_2$.

This gives the following construction for the point $P$:—

Let $A_1A_2$ meet the generator $g$, *i.e.* $O_1O_2$, at $A_3$. Through $A_3$ draw the single straight line which meets both $B_1C_1$ and $B_2C_2$. Let it meet $B_1C_1$ at $Q_1$, and $B_2C_2$ at $Q_2$. Then $A_1Q_1$ and $A_2Q_2$ meet at $P$.

Hence $A_3Q_1Q_2$ is a generator of the regulus which has $B_1C_1$, $B_2C_2$, and $A_1A_2$ for directing lines.

Hence the range of $A_3$ on $A_1A_2$ is projective to the range of $Q_2$ on $B_2C_2$.

Now project the range of $Q_2$ on $B_2C_2$ from $A_2$ on to the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$, *i.e.*, on the line $i$.

Then the range of $Q_2$ on $B_2C_2$ is projective with the range of $P$ on the line $i$.

Therefore the range of $A_3$ (*i.e.*, the point in which the selected generator $g$ is met by the fixed line $A_1A_2$) on the fixed line $A_1A_2$ is projective with the range of $P$ on the line $i$.

7. It is necessary now to return to the exceptional case mentioned in Art. 2, viz., that in which the planes $A_1B_1C_1$ and $A_2B_2C_2$ meet the plane $\pi$ in the same straight line, *i.e.*, $J_1K_1L_1$ and $J_2K_2L_2$ coincide. In this case there are the following classes of cases:—

(*a*) $J_1$ and $J_2$ do not coincide, $K_1$ and $K_2$ do not coincide, and $L_1$ and $L_2$ do not coincide.

This is the most general case of this kind, and it arises whenever the plane $\pi$ is drawn through the intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$.

In this case the straight lines $J_1J_2$, $K_1K_2$, and $L_1L_2$ coincide with the line of intersection of the planes; so that the triangle $ABC$ degenerates into a straight line coinciding with the line of intersection of the two planes.

If in this case points $O_1$ and $O_2$ exist such that $O_1A_1$ meets $O_2A_2$ at $A$, $O_1B_1$ meets $O_2B_2$ at $B$, and $O_1C_1$ meets $O_2C_2$ at $C$, where $A$, $B$, $C$ all lie on the line of intersection of the planes, then $A_1A$ lies in the plane $A_1B_1C_1$, and therefore $O_1$ also lies in this plane.

Similarly $O_2$ lies in the plane $A_2B_2C_2$.

As in the general case, $O_1O_2$ meets $A_1A_2$, $B_1B_2$, and $C_1C_2$. Consequently $O_1O_2$ is a generator of the regulus determined by $A_1A_2$, $B_1B_2$, and $C_1C_2$.

Hence $O_1$ is any point on the conic in which this regulus meets the plane $A_1B_1C_1$; and $O_2$ is the point in which the generator $g$ of the second system through $O_1$ meets the plane $A_2B_2C_2$.

Hence in this case the triangles can (in an infinite number of ways) be projected into the same triad of points lying on the line of intersection of the two planes.

[Although $O_1$ is in the plane $A_1B_1C_1$ and $O_2$ is in the plane $A_2B_2C_2$, the points on the plane $A_1B_1C_1$ correspond uniquely to the points on the plane $A_2B_2C_2$; because the projective relation between the planes is the same as when $O_1$ and $O_2$ have any other positions on the generator $g$.]

(*b*) Let $J_1$ and $J_2$ coincide, say at $J$.

Let $K_1$ and $K_2$ not coincide, and let $L_1$ and $L_2$ not coincide. (See Fig. 3.)

In this case $K_1K_2$, which is $AC$, and $L_1L_2$, which is $AB$, coincide with the line $i$.

The point $B$ is the intersection of $J_1J_2$ with $L_1L_2$; hence it is at $J$.

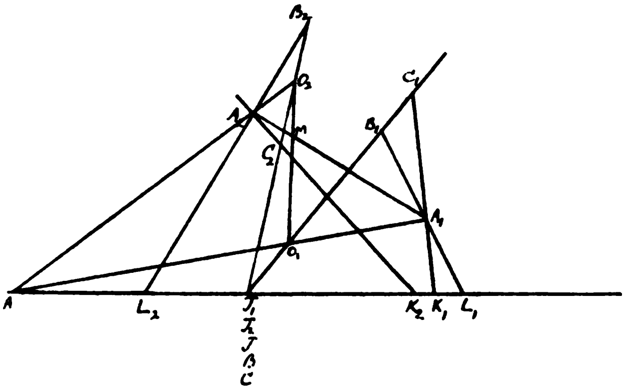

FIG. 3.

The point $C$ is the intersection of $J_1J_2$ with $K_1K_2$; hence it also is at $J$.

The straight line $BC$, which is $J_1J_2$, goes through $J$, but is not otherwise limited; but $B$ and $C$ are not separate points on it. They coincide at $J$.

In this case the straight lines $B_1C_1$ and $B_2C_2$ meet, viz., at $J$.

Hence $B_1B_2$ and $C_1C_2$ meet, and therefore in this case there is no proper regulus* determined by $A_1A_2$, $B_1B_2$, and $C_1C_2$.

Again $O_1$ lies on $BB_1$, *i.e.*, on $JB_1C_1$.

Also $O_2$ lies on $BB_2$, *i.e.*, on $JB_2C_2$.

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* It degrades into a plane. All its generating lines lie in the plane $B_1C_1B_2C_2$ and pass through the point in which $A_1A_2$ meets this plane. By taking $O_1$ and $O_2$ on any generating line it is possible to project $A_1B_1C_1$ and $A_2B_2C_2$ into the same triangle.

Further $O_1 O_2$ meets $A_1 A_2$.

If, therefore, $A_1 A_2$ meet the plane $C_1 B_1 J B_2 C_2$ in $M$, and any line be drawn through $M$ in this plane meeting $B_1 C_1$ in $O_1$ and $B_2 C_2$ in $O_2$, we obtain positions of $O_1$, $O_2$ which project $A_1 B_1 C_1$ and $A_2 B_2 C_2$ into three points $A$, $B$, $C$ on the line $i$; but two of these points, $B$ and $C$, coincide with each other.

In this case there is no proper projection, *i.e.*, so long as the plane $ABC$ is drawn through the line $i$.

(*c*) $J_1$ and $J_2$ coincide, say at $J$; $K_1$ and $K_2$ coincide, say at $K$; but $L_1$ and $L_2$ do not coincide.

Then $BC$, which is $J_1 J_2$, goes through $J$, but is not otherwise limited.

Also $AC$, which is $K_1 K_2$, goes through $K$, but is not otherwise limited.

Whilst $AB$ is the line $L_1 L_2$, which is the same as $JK$.

Hence $A$ is at $K$, $B$ is at $J$, and $C$ is any point in the plane $\pi$.

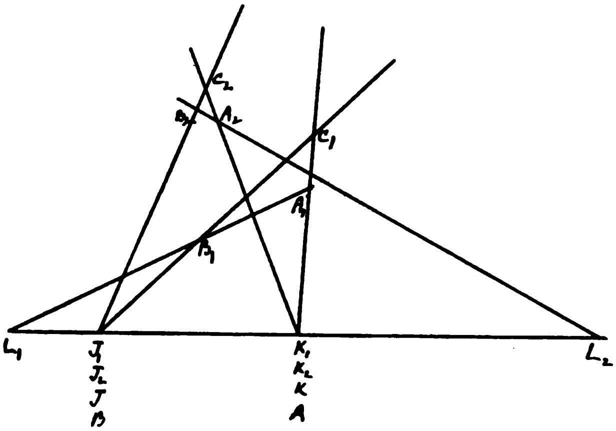

FIG. 4.

In this case $AA_1$, $BB_1$, and $CC_1$ meet at $C_1$.

Hence $O_1$ is at $C_1$, and there is no proper projection of the plane $A_1 B_1 C_1$ from $O_1$.

Also $AA_2$, $BB_2$, and $CC_2$ meet at $C_2$. Hence $O_2$ is at $C_2$.

(*d*) $J_1$ and $J_2$ coincide, $K_1$ and $K_2$ coincide, and $L_1$ and $L_2$ coincide.

In this case the triangles are in perspective, and the points $O_1$ and $O_2$ coincide at the centre of perspective.

8. It is necessary now to consider a particular case of another kind.

In the preceding cases the six points $J_1$, $K_1$, $L_1$, $J_2$, $K_2$, and $L_2$ are all fully determined. Let us now suppose that the straight lines $J_1 K_1 L_1$ and $J_2 K_2 L_2$ are distinct, but let the point $J_2$ in which $B_2 C_2$ meets the plane $\pi$ be not fully determined.

This is equivalent to the case in which the plane $\pi$ is drawn through $B_2C_2$; and then $J_2$ can be regarded as any point on $B_2C_2$. (See Fig. 5.)

Now $C_2A_2$ meets $\pi$ at $K_2$. Therefore $K_2$ is at $C_2$.

Also $A_2B_2$ meets $\pi$ at $L_2$. Therefore $L_2$ is at $B_2$.

The lines $A_1A$, $B_1B$, $C_1C$ meet at $O_1$.

The lines $A_2A$, $B_2B$, $C_2C$ which meet at $O_2$ are now $A_2A$, $L_2B$, $K_2C$ respectively.

These lines meet at $A$; therefore $O_2$ is at $A$. But $O_1$ is on $AA_1$; therefore $O_1A_1O_2$ is a straight line; therefore the generator $O_1O_2$ passes through $A_1$.

In this case there is no proper projection. For the plane $ABC$ passes through $O_2$, which is at $A$.

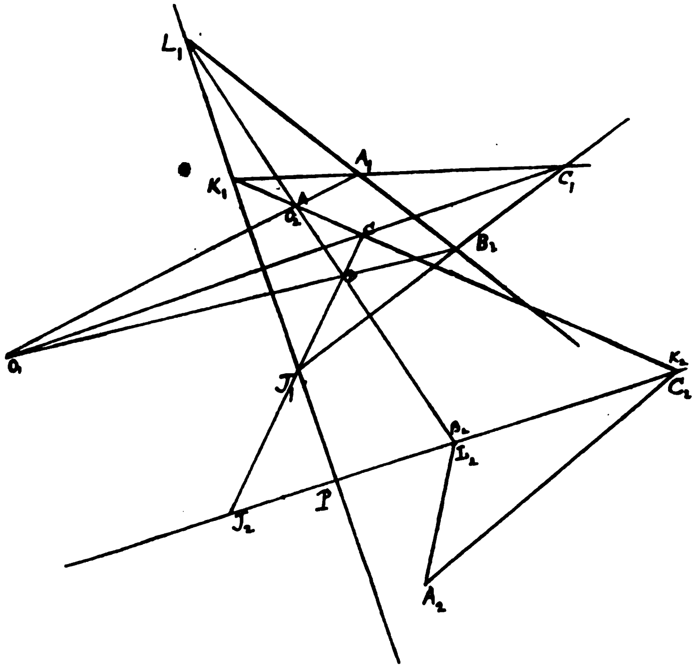

FIG. 5.

The distinguishing feature of this case is that the generator $g$ passes through $A_1$.

Conversely, let $g$ pass through $A_1$. Then the straight line $O_1A_1$ is $g$. In the general case $O_1A_1$ meets $O_2A_2$ at $A$.

Hence in this special case $g$ meets $O_2A_2$ at $A$. But $g$ meets $O_2A_2$ at $O_2$. Hence $O_2$ coincides with $A$.

Now in the general case $A_2B_2C_2$ is projected from $O_2$ on to $ABC$. But in this special case $O_2$ is the point $A$ in the plane $ABC$.

In this case there is no proper projection.

Note that in this case, since $BC$ projects into $B_2C_2$, the plane $ABC$ is the same as the plane $O_2B_2C_2$, and it therefore passes through $B_2C_2$.

Those points of the plane $A_2B_2C_2$ which are situated on $B_2C_2$ are projected on to $BC$, and thence on to $B_1C_1$.

This would result in making the point in which $B_2C_2$ meets the line $i$ not a self-corresponding point unless $J_2$ is taken at that point. This will therefore be supposed to be done in the rest of this section.

All points on the plane $A_2B_2C_2$ not on $B_2C_2$ are projected from $O_2$ (which is $A$) on to $A$, and thence from $O_1$ on to $A_1$.

Hence when $g$ passes through $A_1$ all points on $A_2B_2C_2$ which are not on $B_2C_2$ correspond to $A_1$. The points on $B_2C_2$ correspond to points on $B_1C_1$.

It is required to determine the self-corresponding point when the generator $g$ passes through $A_1$.

The self-corresponding point is the intersection of $J_1K_1L_1$ and $J_2K_2L_2$. Now in this case $J_2K_2L_2$ is $B_2C_2$. Hence the self-corresponding point lies on $B_2C_2$. It also lies on the intersection of the two planes $A_1B_1C_1$ and $A_2B_2C_2$. Hence it is the point in which $B_2C_2$ meets the line of intersection of the two planes.

Hence, if the generator $g$ pass through $A_1$, the self-corresponding point $P$ is the point in which $B_2C_2$ meets the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$.

It should be noticed that in this case the points $B_1$, $C_1$, and $P$ which are not in a straight line correspond to $B_2$, $C_2$, and $P$ which are in a straight line and all points on $A_2B_2C_2$ which are not on $B_2C_2$ correspond to $A_1$. This confirms what was shown before, that there is no proper projection in this case.

There are six cases of this class: viz., when $g$ passes through $A_1$, $B_1$, $C_1$, $A_2$, $B_2$, $C_2$ the self-corresponding points are those in which the line of intersection of the two planes meets the lines $B_2C_2$, $C_2A_2$, $A_2B_2$, $B_1C_1$, $C_1A_1$, $A_1B_1$ respectively.

Although in these six cases there is no proper projection, it is possible to make the relation between the two planes quite definite in each case. As the self-corresponding point $P$ moves along the line of intersection $i$ of the two planes to each point $E_1$ in one plane corresponds a definite point $E_2$ in the other plane for every position of $P$, except when it takes up one of the six exceptional positions noted above. As $P$ approaches one of these exceptional positions $E_2$ approaches a perfectly definite position, and this definite position may be said to correspond to $E_1$ when $P$ occupies the exceptional position. That this is so follows from Art. 10 below.

9. The regulus whose directing lines are $A_1A_2$, $B_1B_2$, $C_1C_2$ cuts the vo planes $A_1B_1C_1$, $A_2B_2C_2$ in two conics.

If $g$ meet these planes at $D_1$ and $D_2$, then $A_1$, $B_1$, $C_1$, $D_1$ on the one nic correspond to $A_2$, $B_2$, $C_2$, $D_2$ respectively on the other conic.

It will now be proved :—

(i.) That *in general* no fifth point of the first conic can correspond a point on the second conic.

(ii.) That, if any fifth point on the first conic correspond to a point n the second conic, then all the points on the first conic correspond to oints on the second conic, and the projective relation between the two lanes is that which is determined by making those points on the conics orrespond which lie on the same generator of the regulus.

To prove (i.), suppose, if possible, that a point $E_1$ of the first conic orresponds to a point $E_2$ of the second conic.

The most general supposition that can be made is that $E_1$ and $E_2$ are istinct points. This will be taken first.

$O_1E_1$ meets $O_2E_2$ in a point $E$ of the plane $ABCD$.

Hence $E_1E_2$ meets $O_1O_2$.

Hence $E_1E_2$ has in common with the hyperboloid three distinct points, iz., $E_1$, $E_2$, and the point where $E_1E_2$ meets $O_1O_2$.

It must therefore be a generator of the same system as $A_1A_2$, $B_1B_2$, nd $C_1C_2$.

Now $O_1$ projects $A_1B_1C_1D_1E_1$ and $O_2$ projects $A_2B_2C_2D_2E_2$ into $BCDE$.

Draw the conics through $ABCDE$, $A_1B_1C_1D_1E_1$, and $A_2B_2C_2D_2E_2$.

Take any point $F$ on the conic $ABCDE$.

Let $O_1F$ cut the first plane at $F_1$; then $F_1$ is on the conic $A_1B_1C_1D_1E_1$.

Let $O_2F$ cut the second plane at $F_2$; then $F_2$ is on the conic ${}_2B_2C_2D_2E_2$.

Also $F_1$ and $F_2$ correspond to each other.

As $F$ may be any point on the conic $ABCDE$, it follows that every oint on the first conic corresponds to a point on the second conic, and he line joining two corresponding points is a generator of the first ystem.

This conclusion is in the general case impossible, because $D_1D_2$, hich joins corresponding points, is a generator of the second system.

The exceptional cases are (*a*) when $D_1D_2$ is a generator of both systems, .*e*., the hyperboloid is a cone; (*b*) when $D_1$ and $D_2$ coincide in one point, o that $D_1$ and $D_2$ both lie on a generator of each system.

Putting aside for the present these exceptional cases, it follows that only the points $A_1B_1C_1D_1$ on the one conic correspond to points on the other conic, viz., to $A_2B_2C_2D_2$.

In the exceptional case (*a*) the points $O_1$ and $O_2$ coincide with the vertex of the cone and the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective.

The second exceptional case (*b*) is important. It brings us, in fact, to the case marked (ii.) above.

Suppose that the hyperboloid meets the line of intersection of the two planes at the points $T$ and $U$. Then $T$ and $U$ lie on both the conics. Suppose that $g$ is taken to pass through $T$. Then $D_1$ and $D_2$ coincide at $T$. In this case $A_1B_1C_1T$ corresponds to $A_2B_2C_2T$. Therefore the projective relation between the two planes is fully determined. It will now be proved that $U$ corresponds to $U$. Let the rays of the regulus containing $A_1A_2$, $B_1B_2$, and $C_1C_2$ which pass through $T$ and $U$ be called $t$ and $u$ respectively. Then the points $T, A_1, B_1, C_1$ on one conic fall on the rays $t$, $A_1A_2$, $B_1B_2$, $C_1C_2$ of the regulus. The conic and the regulus are therefore in perspective, each ray of the regulus cutting the conic in the point corresponding to it.

In like manner the regulus and the second conic are in perspective. Hence the point $U$ of the first conic corresponds to the generator $u$ of the regulus, and this corresponds to the point $U$ of the second conic. Therefore $U$ corresponds to $U$.

It remains only to find the position of the self-corresponding point when $g$ passes through $T$. It will be proved that this is at $U$.

For, if the plane $ABC$ do not pass through $U$ when $g$ passes through $T$, then $U$ will not correspond to itself.

Hence, when $g$ passes through $T$, the self-corresponding point is $U$.

Also, when $g$ passes through $U$, the self-corresponding point is $T$.

Further, these two positions of $g$ determine the same projective relation between the two planes, viz., all the points on the first conic correspond to points on the second conic, corresponding points lying on generators of the first system.

10. Given that $A_1, B_1, C_1$ correspond to $A_2, B_2, C_2$, to prove that as $g$ describes the regulus the point $D_2$ corresponding to a fixed point $D_1$ in the plane $A_1B_1C_1$ describes a cubic curve, having a double point and passing through $A_2, B_2, C_2$.

Suppose that, in one position of $g$, $P$ is the self-corresponding point.

Let $PA_1$ meet $B_1C_1$ at $Q_1$; let $PA_2$ meet $B_2C_2$ at $Q_2$.

Then $Q_1Q_2$ meets $A_1A_2$ at $A_3$, where $A_3$ is the point at which $g$ meets $A_1A_2$.

Let $PA_1$ meet $B_1D_1$ at $R_1$ and $C_1D_1$ at $S_1$.

Since $A_1, B_1, C_1, P$ correspond to $A_2, B_2, C_2, P$ respectively, therefore

$A_1P$, $B_1C_1$ correspond to $A_2P$, $B_2C_2$ respectively; therefore $Q_1$, the intersection of $A_1P$, $B_1C_1$, corresponds to $Q_2$, the intersection of $A_2P$, $B_2C_2$.

Since $A_1$, $Q_1$, $P$ correspond to $A_2$, $Q_2$, $P$, the ranges on $A_1Q_1P$ and $A_2Q_2P$ are in perspective, and the centre is $A_3$ (see Fig. 6).

Hence, to get the points corresponding to $R_1$, $S_1$, it is only necessary to make $A_3R_1$ meet $A_2P$ at $R_2$, and $A_3S_1$ meet $A_2P$ at $S_2$.

Then, since $D_1$ is the intersection of $B_1R_1$ and $C_1S_1$, therefore $D_2$ is the intersection of $B_2R_2$ and $C_2S_2$.

As $P$ moves along the line of intersection of the two planes, $Q_1$, $R_1$, $S_1$ describe respectively the straight lines $B_1C_1$, $B_1D_1$, and $C_1D_1$.

$Q_2$ describes the fixed line $B_2C_2$.

Projecting from the fixed point $A_1$, $(P) \barwedge (Q_1) \barwedge (R_1) \barwedge (S_1)$.*

Projecting from the fixed point $A_2$, $(P) \barwedge (Q_2)$; therefore $(Q_1) \barwedge (Q_2)$.

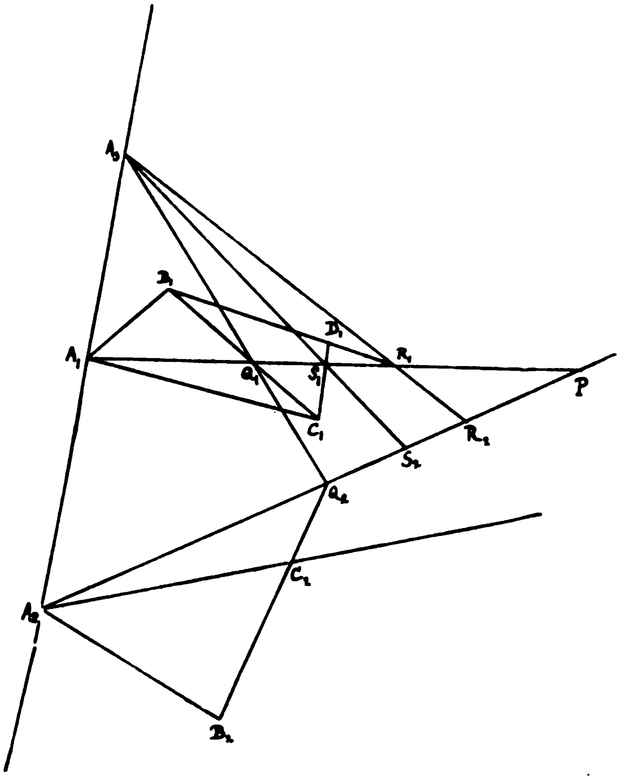

FIG. 6.

Hence $Q_1Q_2$ describes a regulus, having $B_1C_1$, $B_2C_2$, and $A_1A_2$ for directing lines; therefore $(Q_1) \barwedge (Q_2) \barwedge (A_3)$; therefore $(A_3) \barwedge (R_1)$.

---

* The symbol $\barwedge$ is used to denote "projective with" and $(P) \barwedge (Q_1)$ means "any four positions of $P$ are projective with the four corresponding positions of $Q_1$."

Hence $A_3R_1$ is a generator of a regulus having $A_1A_2$ and $B_1D_1$ for directing lines.

This regulus meets the plane $A_2B_2C_2$ in a conic on which $R_2$ lies. This conic passes through $A_2$, because $A_2$ is a possible position of $A_3$. It also passes through the point where $B_1D_1$ meets the line of intersection of the two planes.

Similarly, $(A_3)\barwedge(S_1)$; therefore $A_3S_1$ is a generator of a regulus which has $A_1A_2$ and $C_1D_1$ for directing lines.

This regulus meets the plane $A_2B_2C_2$ in a conic on which $S_2$ lies. This conic passes through $A_2$. It also passes through the point where $C_1D_1$ meets the line of intersection of the two planes.

There are, therefore, two conics in the plane $A_2B_2C_2$, both passing through $A_2$. The points $R_2$ and $S_2$ on these conics are projectively related by the lines drawn from $A_2$. Each such line meets one conic in $R_2$ and the other in $S_2$.

The point $D_2$ is the intersection of $B_2R_2$ and $C_2S_2$.

If, now, we draw the sheaf of planes of the second order, whose centre is $B_2$, and which passes through the generating lines $A_3R_1R_2$, it is projective to the regulus generated by $A_3R_1R_2$, and therefore to the points $R_1$.

Similarly, the sheaf of planes of the second order, whose centre is $C_2$, and which passes through the generating lines $A_3S_1S_2$, is projective to the regulus generated by $A_3S_1S_2$, and therefore to the points $S_1$, and therefore to the points $R_1$, and therefore to the sheaf of planes of the second order whose centre is $B_2$ and which passes through the generating lines $A_3R_1R_2$.

Now the surface generated by the intersections of corresponding planes of two projective sheaves of planes of the second order is, *in general*, a ruled surface of the fourth order having a nodal line, which is a curve in space of the third order (Reye, *Geometrie der Lage* (1886), Zweite Abtheilung, Fünfzehnter Vortrag, S. 115).

The section by any plane is, therefore, a curve of the fourth order with three double points.

In this case, however, there is a simplification.

It will now be proved that the two projective sheaves of planes have a self-corresponding plane. The ruled surface of the fourth order will then reduce to this plane and a cubic surface of the third order.

Consider what happens when the self-corresponding point $P$ is at the point where $B_2C_2$ meets the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$. Then (Art. 8) $g$ passes through $A_1$, and therefore $A_3$ is at $A_1$. In this case both $R_2$ and $S_2$ coincide with $P$.* Thus the locus of $R_2$ is a conic through $A_2$ and the point where $B_2C_2$ meets the line of intersection of the two planes. This is the point $J_2$. In like manner the locus of $S_2$ is a conic through $A_2$ and $J_2$. Since $R_1$ is on $A_1P$, both $R_1$ and $R_2$ are on $A_1P$, and therefore the plane $B_2R_1R_2$ is the plane $B_2A_1P$ which contains $C_2$, since $P$ is on $B_2C_2$. It is therefore the plane $A_1B_2C_2$. In like manner the plane $C_2S_1S_2$ is the plane $A_1B_2C_2$. Hence the two sheaves of planes of the second order $B_2R_1R_2$ and $C_2S_1S_2$ have a self-corresponding plane.

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* It should be observed that, when $R_2$ and $S_2$ coincide at the point where $B_2C_2$ meets the line of intersection of the two planes, it does not follow that $D_2$, which is the intersection of $B_2R_2$ and

Hence the ruled surface of the fourth order reduces to this plane and a cubic surface.

Hence the locus of the intersection of $B_2R_2$ and $C_2S_2$ is a cubic curve.

Since the locus of $R_2$ is a conic through $A_2$ and the locus of $S_2$ a conic through $A_2$, therefore $A_2$ is a point on the cubic locus.

By symmetry $B_2$ and $C_2$ lie on the cubic locus.

Hence the locus of $D_2$ is a cubic curve through $A_2$, $B_2$, $C_2$.

It has a double point at the point which corresponds to $D_1$ when the projective relation is that which is determined by making the regulus containing $A_1A_2$, $B_1B_2$, $C_1C_2$ perspective to the two conics in which it meets the two planes. For, if the regulus meet the line of intersection of the two planes in $T$ and $U$, then we get the same projective relation between the planes when $P$ is at $T$ as we do when $P$ is at $U$. Hence the same point will correspond to $D_1$ when $P$ is at $T$ as when $P$ is at $U$. This point will therefore be a double point on the cubic locus.

11. We might also have completed the proof thus:—The conic which is the locus of $R_2$ and the conic which is the locus of $S_2$ intersect at $A_2$. The straight line $B_2C_2$ passes through one of the intersections of these two conics, viz., the point $J_2$. The points $A_2$, $R_2$, $S_2$ are on a straight line. Therefore $R_2$ on its conic is projectively related to $S_2$ on its conic. So the pencils $B_2R_2$ and $C_2S_2$ are projectively related, and they have a self-corresponding ray, viz., $B_2C_2$ (for when $P$ coincides with the point in which $B_2C_2$ meet the line of intersection of the two planes $R_2$ and $S_2$ coincide with $P$). Hence the locus of the intersection of $B_2R_2$ and $C_2S_2$ is a cubic curve. Hence the locus of $D_2$ is a cubic curve. (Reye, Holgate's Translation, Art. 205; or the German edition, Erste Abtheilung, S. 137.)

12. In order to draw the cubic, a further examination of the loci of $R_2$ and $S_2$ is necessary.

It will be proved in the first place that the locus of $R_2$ touches the locus of $S_2$ at $A_2$.

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$C_2S_2$, also coincides with this point, for $B_2R_2$ and $C_2S_2$ have in this case the same direction, and the limiting point of their intersection may be different from the point at which $B_2C_2$ meets the line of intersection of the planes.

The locus of $R_2$ is the section of the hyperboloid generated by $A_3R_1R_2$ by the plane $A_2B_2C_2$.

The tangent plane to the hyperboloid at $A_2$ is the plane which contains $A_1A_2$ and the position of $A_3R_1R_2$ when $A_3$ reaches $A_2$.

But, when $A_3$ is at $A_2$, $P$ is at the point where $B_1C_1$ cuts the line $i$. This is the point we have previously called $J_1$.

Hence $R_1$ is at the intersection of $A_1J_1$ and $B_1D_1$, say at $R_1'$.

Hence the tangent plane at $A_2$ is the plane containing $A_1A_2$ and the line joining $A_2$ to the intersection of $A_1J_1$ and $B_1D_1$. But this is the plane $A_1A_2J_1$.

Hence the tangent at $A_2$ to the locus of $R_2$ is $A_2J_1$.

By similar reasoning the tangent at $A_2$ to the locus of $S_2$ is also $A_2J_1$.

Hence the two conics touch at $A_2$.

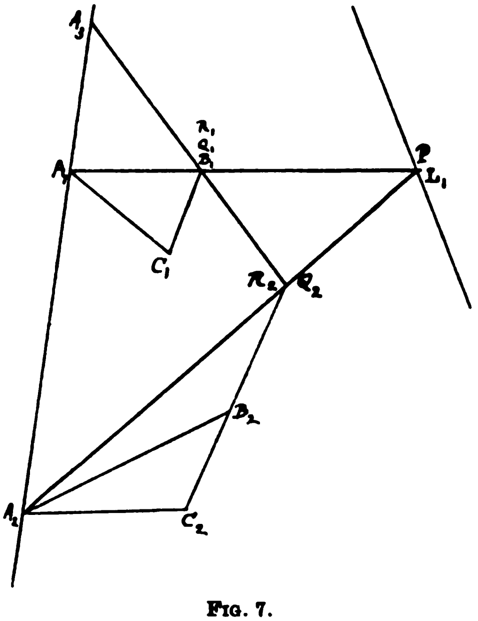

FIG. 7.

The points in which $B_2C_2$ meets each conic will next be found.

Take $P$ (see Fig. 7) where $A_1B_1$ meets the line $i$, *i.e.*, take $P$ at $L_1$.

Then $PA_1$, *i.e.*, $A_1L_1$, meets $B_1C_1$ at $Q_1$, and $B_1D_1$ at $R_1$ in the general case.

Hence $Q_1$ and $R_1$ coincide with $B_1$.

The point $Q_2$ is the intersection of $B_2C_2$ and $A_2L_1$.

Hence $Q_1Q_2$ is $B_1Q_2$, and this meets $A_1A_2$ in $A_3$.

Now $A_3R_1$ meets $A_2P$ in $R_2$ in general.

But $R_1$ coincides with $Q_1$.

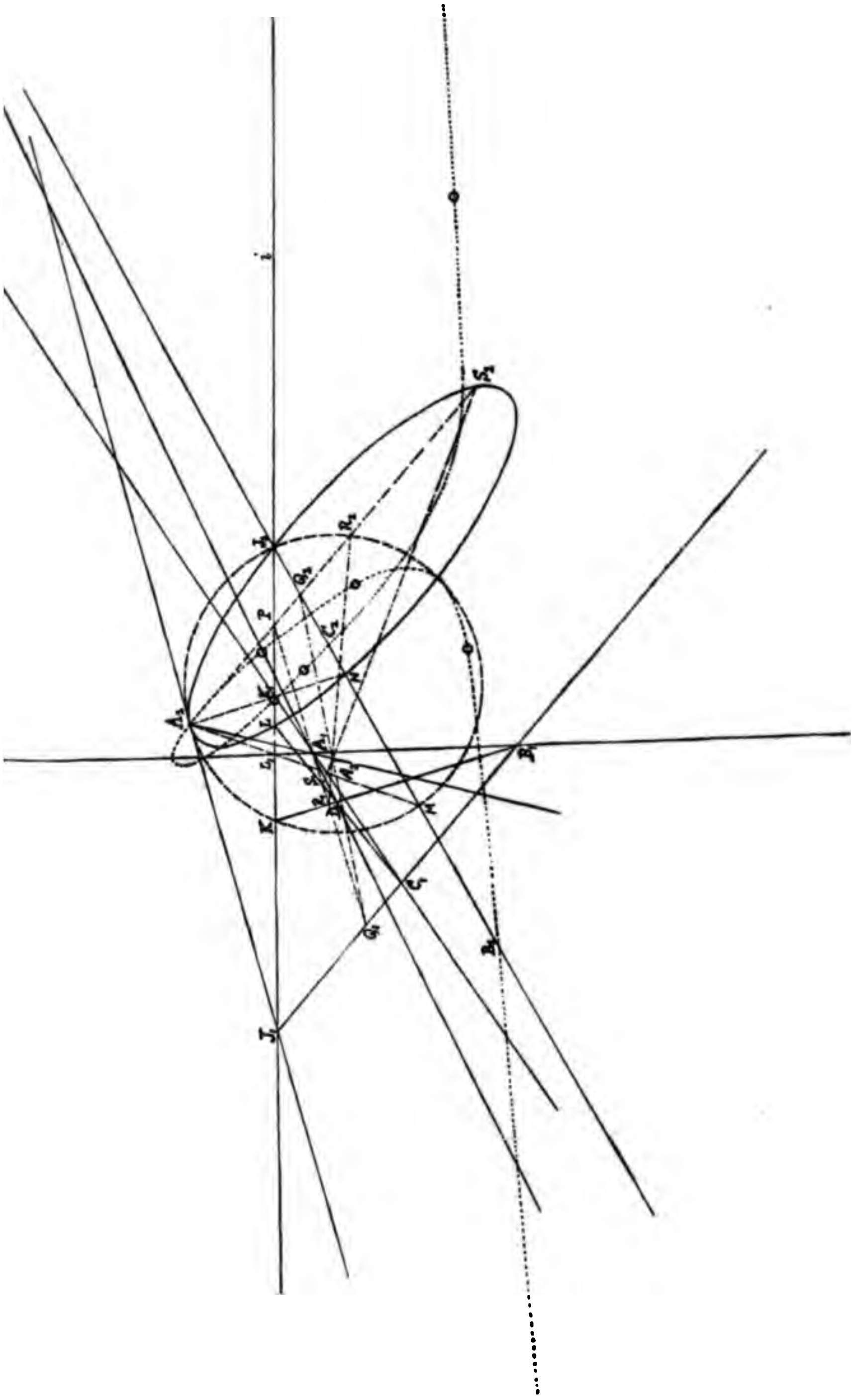

FIG. 8.

Consequently $R_2$ coincides with $Q_2$.

Hence one position of $R_2$ is the intersection of $A_2L_1$ and $B_2C_2$.

Similarly one position of $S_2$ is the intersection of $A_2K_1$ and $B_2C_2$.

The loci of $R_2$ and $S_2$ are now completely determined.

(1) Each locus passes through $A_2$ (Art. 10).

(2) Each locus is touched by $A_2J_1$ at $A_2$ (Art. 12).

(3) Each locus passes through $J_2$ (Art. 10).

(4) The locus of $R_2$ passes through the intersection of $B_1D_1$ and the line $i$ (Art. 10). Call this the point $K$. (See Fig. 3.)

The locus of $S_2$ passes through the intersection of $C_1D_1$ and the line $i$ (Art. 10). Call this the point $L$.

(5) The locus of $R_2$ passes through the intersection of $B_2C_2$ and $A_2L_1$. Call this point $M$.

The locus of $S_2$ passes through the intersection of $B_2C_2$ and $A_2K_1$. Call this point $N$.

13. The conics are determined when the positions of $A_1$, $B_1$, $C_1$, $D_1$, $A_2$, $B_2$, $C_2$ and the line $i$ are given.

But the converse is not true.

The conics are determined when the positions of $J_1$, $K_1$, $L_1$ on the line $i$; the position of $J_2$ on the line $i$; the direction of the straight line $B_2C_2$ through $J_2$ (the positions of $B_2$ and $C_2$ on this line need not be known); the position of $A_2$ and the position of $D_1$ are known.

If the positions of the points are assumed arbitrarily, it is often very difficult to obtain a sufficient length of the cubic to show its characteristics.

It is, however, possible to construct the cubic in a convenient manner by first drawing two conics having a real point of contact and two real intersections, and then to construct positions of $A_1$, $B_1$, $C_1$, $D_1$, $A_2$, $B_2$, $C_2$, and $i$ which will lead to these two conics.

Draw two conics touching at $A_2$ and having two real intersections. Call either of these $J_2$.

Draw two arbitrary lines through $J_2$. Call one of these $i$, and the other $B_2C_2$ ($B_2$ and $C_2$ may be anywhere on this line).

Let $J_2B_2C_2$ meet the locus of $R_2$ in $M$ and the locus of $S_2$ in $N$.

Let the line $i$ meet the locus of $R_2$ in $K$ and the locus of $S_2$ in $L$.

Then on referring to Art. 12 it is seen that it is necessary to take $J_1$ at the intersection of the common tangent at $A_2$ to the conics with the line $i$, $K_1$ at the intersection of $A_2N$ with $i$, $L_1$ at the intersection of $A_2M$ with $i$.

It is possible now to draw any three lines through $J_1$, $K_1$, $L_1$, viz., $J_1B_1C_1$, $K_1C_1A_1$, $L_1A_1B_1$, which will give the positions of $A_1$, $B_1$, $C_1$.

Then $D_1$ must be taken at the intersection of $B_1K$ and $C_1L$.

It will now be proved that the positions assigned to $A_1$, $B_1$, $C_1$, $D_1$, $A_2$, the line $i$, and the line $B_2C_2$ lead uniquely to the conics selected.

For (1) both conics pass through $A_2$;

(2) both conics touch $A_2J_1$;

(3) both conics pass through $J_2$;

(4) the locus of $R_2$ passes through the point $K$, which is the intersection of $B_1D_1$ and $i$, and the locus of $S_2$ passes through the point $L$, which is the intersection of $C_1D_1$ and $i$;

(5) the locus of $R_2$ passes through the point $M$, which is the intersection of $B_2C_2$ and $A_2L_1$; and the locus of $S_2$ passes through $N$, which is the intersection of $B_2C_2$ and $A_2K_1$.

Hence the conics are uniquely determined by the positions assigned to $A_1$, $B_1$, $C_1$, $D_1$, $A_2$ and the lines $B_2C_2$ and $i$.

In order to draw the cubic, all that is necessary is to draw the conics; then any line through $A_2$ meets the conics in corresponding points $R_2$ and $S_2$.

Then $B_2R_2$ and $C_2S_2$ meet on the cubic.

## 14. *Construction for the Double Point of the Cubic.*

Take any point $R_2$ on the locus of $R_2$; let $A_2R_2$ cut the locus of $S_2$ in $S_2$.

Let $B_2R_2$ cut the locus of $R_2$ in $R_2'$, and let $A_2R_2'$ cut the locus of $S_2$ in $S_2'$.

Then, if $S_2S_2'$ passes through $C_2$, the double point of the cubic is the intersection of $R_2R_2'$ and $S_2S_2'$.

For call this point of intersection $\delta$. Then, when $A_2R_2S_2$ occupies its first position, $B_2R_2$ and $C_2S_2$ intersect at $\delta$, and therefore $\delta$ is a point on the curve.

Now let $A_2R_2S_2$ move into the position $A_2R_2'S_2'$. Then $B_2R_2'$ and $C_2S_2'$ intersect at $\delta$, and therefore the curve passes a second time through $\delta$, and therefore $\delta$ is the double point of the cubic.

It is necessary to find a construction for the double point $\delta$.

Draw any straight line through $A_2$ meeting the conics at $R_2$ and $S_2$. Then $(R_2) \barwedge (S_2)$.

Let $B_2R_2$ meet the locus of $R_2$ in $R_2'$. Then the points $R_2$, $R_2'$ are corresponding points of an involution on the locus of $R_2$. Therefore $(R_2) \barwedge (R_2')$.

Let $A_2R_2'$ cut the locus of $S_2$ in $S_2'$. Then $(R_2) \barwedge (S_2')$. Therefore $(S_2) \barwedge (S_2')$.

The points $S_2'$ are determined uniquely from the points $S_2$, and the construction is reversible.

Hence the points $S_2$ and $S_2'$ form an involution on the conic.

Hence $S_2S_2'$ always passes through a fixed point $W$. (See Fig. 9.)

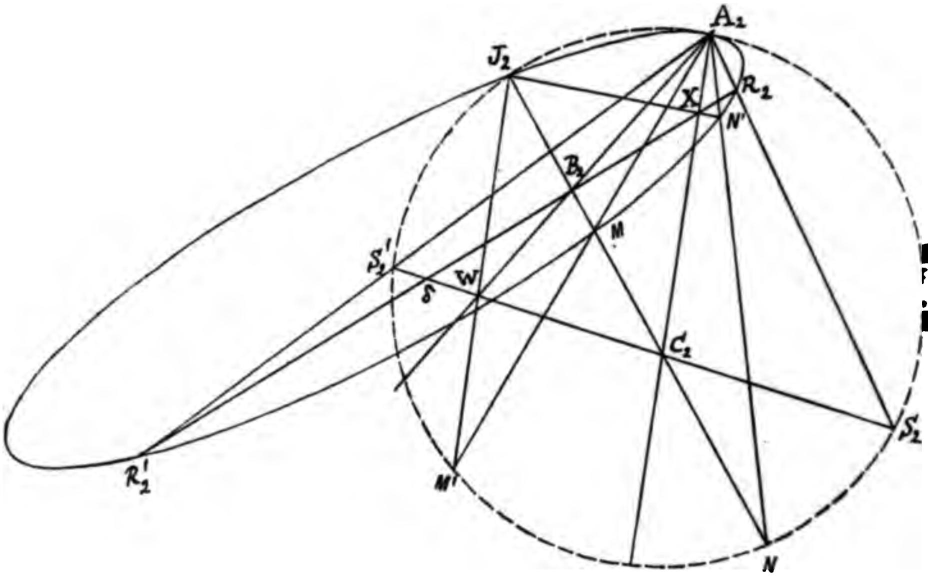

Fig. 9.

Hence $C_2W$ will meet the locus of $S_2$ in corresponding points $S_2$ and $S_2'$, such that the line joining the corresponding positions of $R_2$ and $R_2'$ will pass through $B_2$. Then $R_2R_2'$ meets $C_2W$ in the double point $\delta$.

This construction can be completed if $C_2W$ meets the locus of $S_2$ in real points, but, if $C_2W$ meets the locus of $S_2S_2'$ in imaginary points, then it is not immediately apparent how to construct the corresponding positions $R_2$, $R_2'$.

It has, however, been shown that the double point $\delta$ lies on the straight line $C_2W$.

It is now possible to get another line which passes through $\delta$ thus:— Draw through $C_2$ any straight line meeting the locus of $S_2$ in $S_2$, $S_2'$. Let $A_2S_2$ and $A_2S_2'$ cut the locus of $R_2$ in $R_2$ and $R_2'$. Then $R_2$, $R_2'$ are corresponding points of an involution on the locus of $R_2$.

Therefore $R_2R_2'$ passes through a fixed point $X$ (see Fig. 9), and the double point lies on $B_2X$.

The double point is therefore the intersection of the fully determined straight lines $B_2X$ and $C_2W$, and the construction is one which does not fail when $C_2W$ cuts the locus of $S_2$ in imaginary points.

To find the position of $W$, take $S_2$ at $A_2$. Then $R_2$ is also at $A_2$.

Therefore $R_2'$ is the other point in which $A_2B_2$ cuts the locus of $R_2$, and $S_2'$ is the other point in which $A_2B_2$ cuts the locus of $S_2$. Hence in this position $S_2S'$ coincides with $A_2B_2$.

Hence $W$ lies on $A_2B_2$.

Next take $S_2$ at $J_2$. Then $R_2$ is also at $J_2$. Then $R_2'$ is at the other point in which $B_2J_2$, *i.e.*, $B_2C_2$, cuts the locus of $R_2$. Call this point $M$. Then $A_2M$ cuts the locus of $S_2$ in $S_2'$. Call this point $M'$. Then $S_2S_2'$ is now in the position $J_2M'$. Hence $W$ lies on $J_2M'$, and, as it also lies on $A_2B_2$, it is fully determined.

Similarly $X$ can be determined. In the first place, by taking $R_2$ at $A_2$, $S_2$ coincides with $A_2$. Therefore $S_2'$ is the other point in which $A_2C_2$ cuts the locus of $S_2$, and $R_2'$ is the other point in which $A_2C_2$ cuts the locus of $R_2$. Hence in this position $R_2R_2'$ coincides with $A_2C_2$. Hence $X$ lies on $A_2C_2$.

Take $R_2$ at the point $J_2$ in which $B_2C_2$ meets both conics. Then $S_2$ is also at $J_2$. Then $S_2'$ is at the other point where $C_2J_2$, *i.e.*, $B_2C_2$, meets the locus of $S_2$. Call this point $N$. Then $R_2'$ is at the other point in which $A_2Z$ meets the locus of $R_2$. Call this point $N'$. Hence $R_2R_2'$ is now $J_2N'$. Hence $X$ lies on $J_2N'$, and also on $A_2C_2$. It is therefore fully determined.

15. It has been shown that the points $A_1$, $B_1$, $C_1$ may be projected on to the points $A_3$, $B_3$, $C_3$ in an infinite number of ways by two projections and two sections. The locus of the point $D_3$ in the second plane corresponding to a fixed point $D_1$ in the first plane has been shown to be a cubic.

Hence it is in general impossible to project four arbitrary points $A_1$, $B_1$, $C_1$, $D_1$ in one plane into four arbitrary points $A_3$, $B_3$, $C_3$, $D_3$ in another plane by only two projections and two sections.

Hence it follows that the fewest number of projections and sections necessary to project four arbitrary points $A_1$, $B_1$, $C_1$, $D_1$ in one plane on to four arbitrary points $A_3$, $B_3$, $C_3$, $D_3$ in another plane is three, which is the number given by Grassmann.

Suppose now that a third plane is drawn, and on it three points $A_2$, $B_2$, $C_2$ are taken arbitrarily.

Now let $A_1B_1C_1$ be projected by two projections and two sections on to $A_2B_2C_2$. Then the locus of the projection of $D_1$ will be a cubic through $A_2$, $B_2$, $C_2$.

Now let $A_3B_3C_3$ be projected by two projections and sections on to $A_2B_2C_2$. Then the locus of the projection of $D_3$ is another cubic through $A_2$, $B_2$, $C_2$. These two cubics intersect on $A_2B_2C_2$ and six other points.

Let $D_2$ be one of these six points. Then $A_1B_1C_1D_1$ are projected by two projections and two sections into $A_2B_2C_2D_2$, and $A_2B_2C_2D_2$ are projected into $A_3B_3C_3D_3$ in the same way.

Hence $A_1$, $B_1$, $C_1$, $D_1$ can be projected into $A_3B_3C_3D_3$ by four projections and four sections in six ways, as there are six positions of $D_2$.

This case is illustrated by a comparison of Figs. 8 and 10.

Fig. 8 shows the projection of $A_1$, $B_1$, $C_1$, $D_1$ on to $A_2$, $B_2$, $C_2$, $D_2$; whilst Fig. 10 shows the projection of $A_3$, $B_3$, $C_3$, $D_3$ on to the same four points $A_2$, $B_2$, $C_2$, $D_2$. There are then six points any one of which may be taken as $D_2$. These are marked on each figure by a circle, but are not lettered, and the two figures 8 and 10, which are drawn separately, must be superposed. The points $A_1$, $B_1$, $C_1$, $D_1$, $J_1$, $K_1$, $L_1$ of Fig. 8 correspond to $A_3$, $B_3$, $C_3$, $D_3$, $J_3$, $K_3$, $L_3$ of Fig. 10 respectively.

In Fig. 11 the position of the points is such that the cubic has a cusp. This is due to the fact that corresponding positions of $B_2R_2$, $C_2S_2$ simultaneously touch the conics. This figure was constructed by drawing a ray $A_2R_2S_2$. Then tangents were drawn to the conics at $R_2$ and $S_2$ to meet the line taken for $B_2C_2$ in $B_2$, $C_2$. The intersection of the tangents gives the cusp at $\delta$.

In Fig. 12 the position of the points is such that the cubic has an acnode at $\delta$. The figure is obtained thus:—The conics and the positions of $J_2$ and $B_2$ are first selected. $J_2B_2$ meets the locus of $R_2$ in $M$, $A_2M$ meets the locus of $S_2$ in $M'$, $J_2M'$ meets $A_2B_2$ in $W$. Then $C_2$ is so chosen that $C_2W$ does not cut the locus of $S_2$ in real points. Then $X$ and therefore $\delta$ are determined. $T$ and $U$ are conjugate imaginary points on the line of intersection of the two planes.

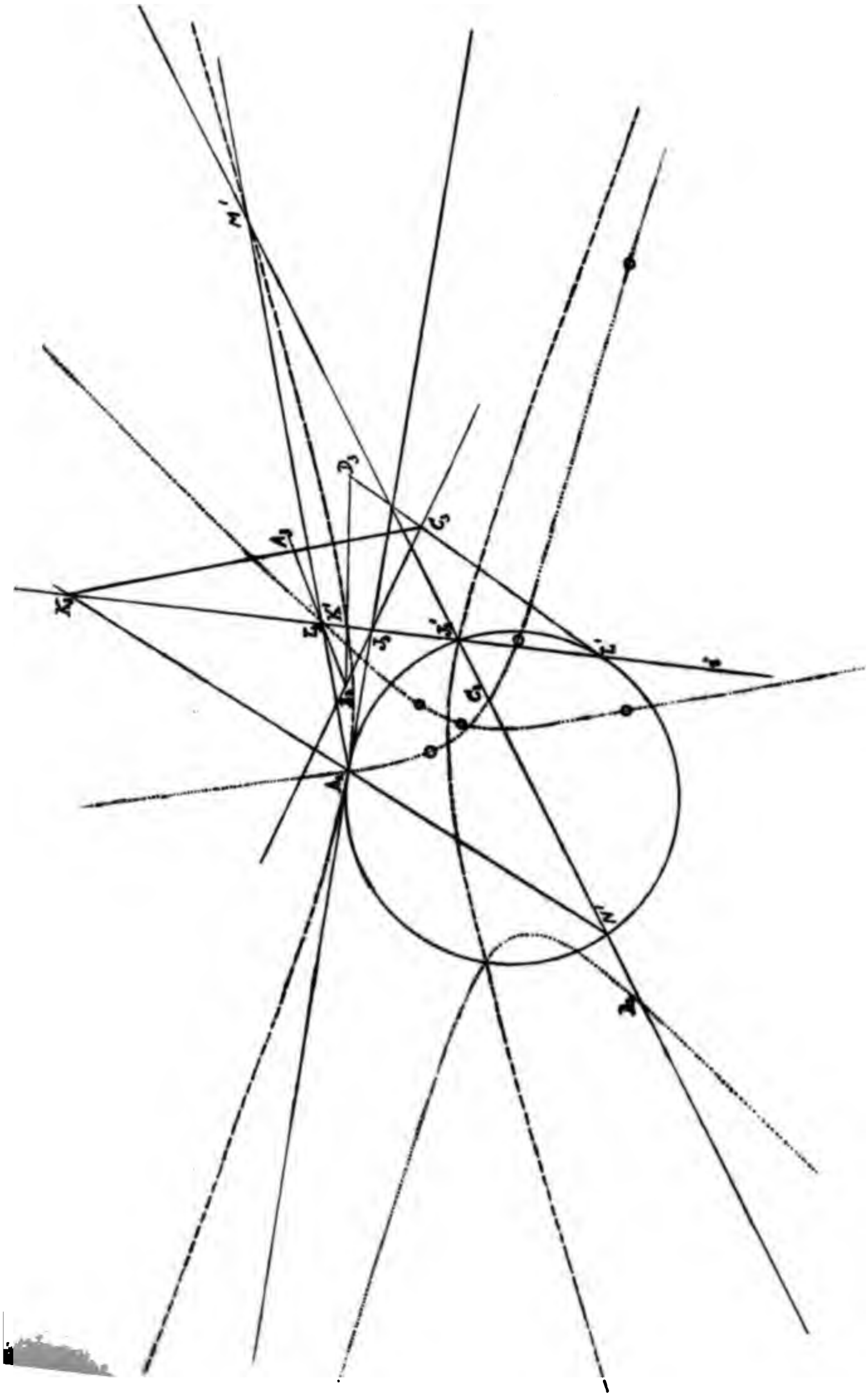

FIG. 10.

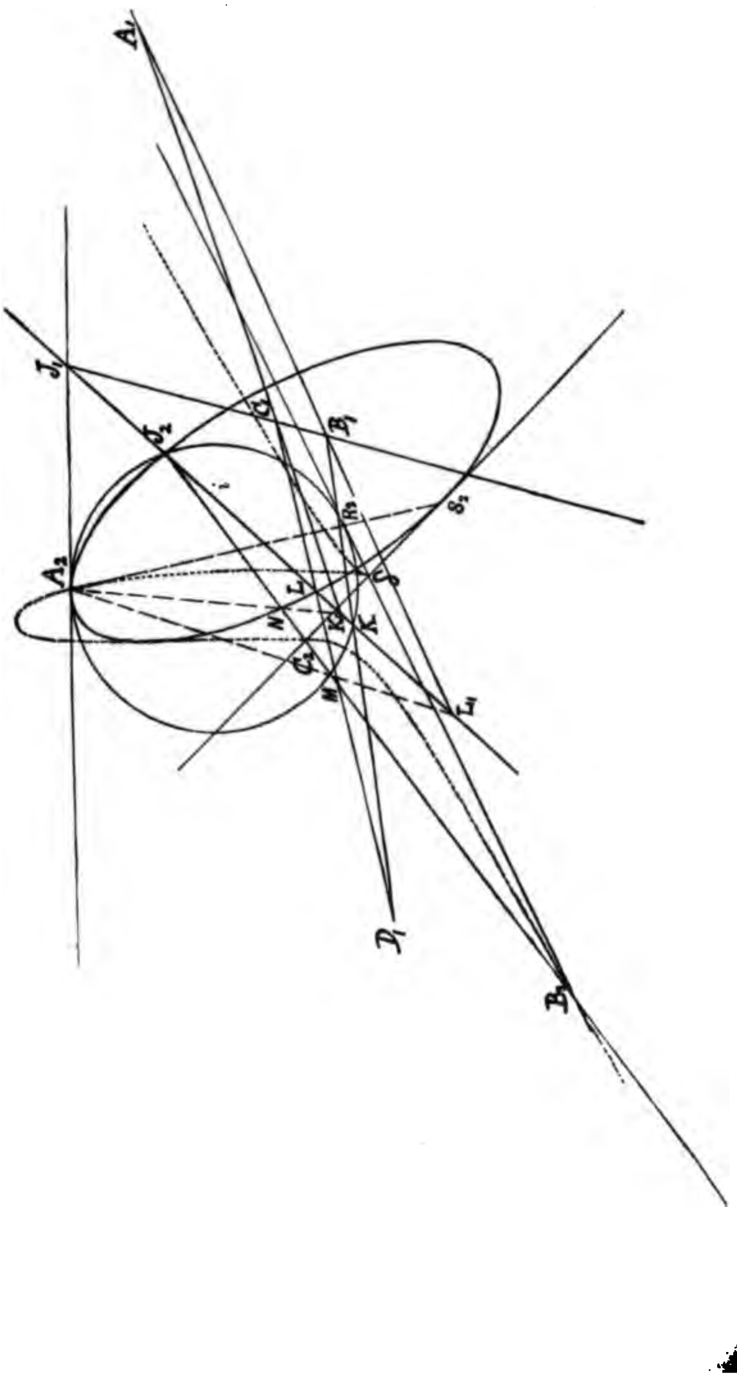

FIG. 11.

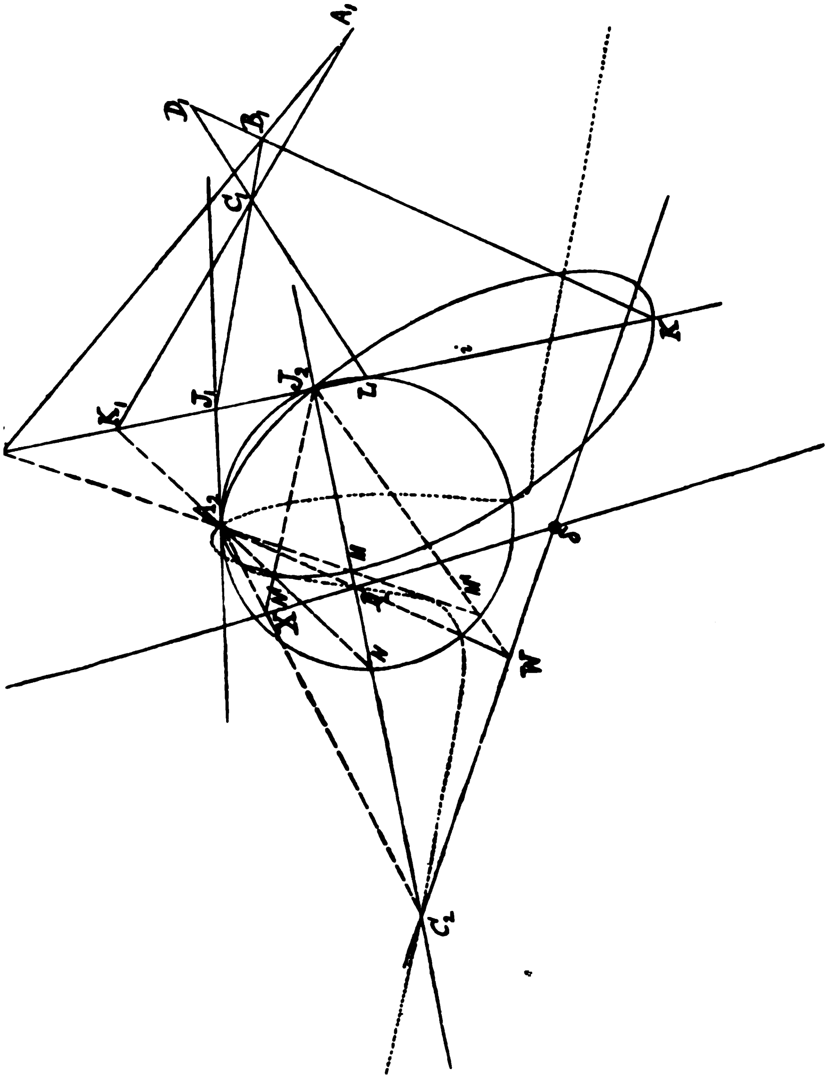

FIG. 12.

# ON THE CONDITION OF REDUCIBILITY OF ANY GROUP OF LINEAR SUBSTITUTIONS

### *By* W. BURNSIDE.

[Received June 3rd, 1905.—Read June 8th, 1905.]

LET
$$\left.\begin{aligned} x_i' &= \sum_1^n \alpha_{ij} x_j \\ x_i' &= \sum_1^n \beta_{ij} x_j \\ x_i' &= \sum_1^n \gamma_{ij} x_j \end{aligned}\right\} \quad (i = 1, 2, \ldots, n)$$

be the substitutions $A, B, C$ of a group $G$ of linear substitutions, Then, if
$$AB = C,$$
where, in the product $AB$, the symbols are to be read from left to right,
$$\gamma_{ij} = \sum_k \beta_{ik} \alpha_{kj}. \tag{i}$$

If in these equations the $\alpha$'s are regarded as the original variables and the $\gamma$'s as the transformed variables, then they define a group of linear substitutions in $n^2$ variables simply isomorphic with $G$, the substitution given being that corresponding to $B$. Moreover this group is reducible, transforming each set of $n$ variables
$$\alpha_{kj} \quad (k = 1, 2, \ldots, n)$$
among themselves exactly as $G$ transforms the $x$'s.

If $G$ is reducible, it is possible to choose new variables
$$\xi_i = \Sigma A_{ij} x_j \quad (i = 1, 2, \ldots, n)$$
so that the first $r$ $(< n)$ of them are transformed among themselves by every substitution of the group. The transformed group so set up may be represented by
$$\xi_i' = \sum_1^n \alpha_{ij}' \xi_j \quad (i = 1, 2, \ldots, n),$$
the substitution given corresponding to $A$; and in this form
$$\alpha_{ij}' = 0 \quad (i = 1, 2, \ldots, r;\ j = r+1, r+2, \ldots, n)$$

for every substitution of the group. But in this transformed form of the group

$$a'_{ij} = \sum_{st} B_{st} a_{st},$$

where the $B$'s are constants depending on the equations defining the new variables.

Hence, if $G$ is reducible, so that a set of $r$ new variables may be chosen which are transformed only among themselves, then the coefficients of every substitution of $G$ must satisfy a system of $r(n-r)$ independent linear homogeneous equations.

Conversely, suppose that the coefficients of $G$ satisfy a system of $s$ independent linear equations

$$\Sigma C_{ij}^{(k)} \beta_{ij} = 0 \quad (k = 1, 2, \dots, s). \qquad \text{(ii)}$$

If in the equations (i) the $n^2$ variables are replaced by $n^2$ independent linear functions of themselves, of which the first $s$ are

$$A_k = \Sigma C_{ij}^{(k)} a_{ij} \quad (k = 1, 2, \dots, s),$$

the new form of the group defined by them will be

$$A'_i = \sum_{k=1}^{k=n^2} B_{ik} A_k \quad (i = 1, 2, \dots, n^2),$$

where the $B$'s are linear functions of the $\beta$'s with constant coefficients depending on the $C$'s. If in any given set of these equations, specifying a particular substitution of the group, the $a$'s which occur in the definition of the $A$'s are replaced by the coefficients of an arbitrarily chosen substitution of $G$, the equations become identities. Now, when this is done,

$$A_1 = A_2 = \dots = A_s = A'_1 = \dots = A'_s \equiv 0.$$

Hence the first $s$ equations become

$$\sum_{k=s+1}^{k=n^2} B_{ik} A_k = 0 \quad (i = 1, 2, \dots, s). \qquad \text{(iii)}$$

Now, if for any given substitution of the group on the $n^2$ variables the $B$'s of these equations are not identically zero, these $s$ equations are equivalent to a further set of linear homogeneous equations connecting the coefficients of every substitution of $G$.

Hence, if the set of $s$ equations (ii) are the only linear independent equations connecting the coefficients of every substitution of $G$, the $B$'s in the equations (iii) must be zero for every substitution of the group on the $n^2$ variables, and the $s$ symbols

$$A_1, A_2, \dots, A_s$$

are transformed among themselves by every substitution of this group.

Now it has been seen above that each set of $n$ $a$'s which have the same second suffix are transformed among themselves by every substitution of the group. Hence for each $j$ the set of $s$ linear functions

$$\sum_{i=1}^{i=n} C_{ij}^{(k)} a_{ij} \quad (k = 1, 2, \ldots, s)$$

are transformed among themselves by every substitution of the group. If for any second suffix $j$ which actually occurs this set of $s$ functions is linearly equivalent to $r\ (< n)$, then a set of $r$ linear functions of

$$a_{1j},\ a_{2j},\ \ldots,\ a_{nj}$$

exists which are transformed among themselves; and therefore there is a corresponding set of $r$ linear functions of the $x$'s which are transformed among themselves by the substitutions of $G$, *i.e.*, $G$ is reducible.

If for each second suffix that occurs the $s$ functions were equivalent to $n$ (in which case $s$ must be a multiple of $n$), successive sets of symbols with the same second suffix might be eliminated from equations (ii) till there remain only $n$ equations. These may be brought to the form

$$\left.\begin{array}{l} \beta_{1j_1} + B_{1j_2} + \ldots = 0 \\ \beta_{2j_1} + B_{2j_2} + \ldots = 0 \\ \ldots \quad \ldots \quad \ldots \\ \beta_{nj_1} + B_{nj_2} + \ldots = 0 \end{array}\right\}, \qquad \text{(iv)}$$

where $B_{1j_2}, B_{2j_2}, \ldots, B_{nj_2}$ are $n$ linearly independent functions of

$$\beta_{1j_2},\ \beta_{2j_2},\ \ldots,\ \beta_{nj_2}.$$

Now, in the group on the $n^2$ variables, symbols with the same second suffix are transformed among themselves. Hence, if $A_{rj_2}$ is the same function of $a_{1j_2}, a_{2j_2}, \ldots, a_{nj_2}$ that $B_{rj_2}$ is of the corresponding $\beta$'s, then

$$\begin{array}{l} a_{1j_1} + A_{1j_2} + \ldots, \\ a_{2j_1} + A_{2j_2} + \ldots, \\ \ldots \quad \ldots \quad \ldots \\ a_{nj_1} + A_{nj_2} + \ldots \end{array}$$

are transformed among themselves, and therefore $a_{1j_1}, a_{2j_1}, \ldots, a_{nj_1}$ and $A_{1j_2}, A_{2j_2}, \ldots, A_{nj_2}$ undergo the same transformation for every substitution of the group.

Hence also $\qquad a_{1j_2},\ a_{2j_2},\ \ldots,\ a_{nj_2}$

and $\qquad A_{1j_2},\ A_{2j_2},\ \ldots,\ A_{nj_2}$

undergo the same transformation for every substitution of the group. This is the same as the statement that every substitution on

$$a_{1j_2},\ a_{2j_2},\ \ldots,\ a_{nj_2}$$

is permutable with

$$a'_{1j_2} = A_{1j_2},\quad a'_{2j_2} = A_{2j_2},\quad \ldots,\quad a'_{nj_2} = A_{nj_2},$$

and therefore either that

$$A_{rj_2} = ka_{rj_2}\quad (r = 1,\, 2,\, \ldots,\, n),$$

where $k$ is a constant, or that the group on

$$a_{1j_2},\ a_{2j_2},\ \ldots,\ a_{nj_2}$$

is reducible.

Now, if for each set of symbols, such as $B_{rj_2}$, which occur in the equations (iv), $B_{rj_2} = k\beta_{rj_2}$, the equations would be

$$\begin{aligned} &\beta_{1j_1}+k\beta_{1j_2}+l\beta_{1j_3}+\ldots = 0,\\ &\beta_{2j_1}+k\beta_{2j_2}+l\beta_{2j_3}+\ldots = 0,\\ &\ldots\quad \ldots\quad \ldots\quad \ldots \end{aligned}$$

implying that the determinant of the substitution is zero. This is not the case. Hence the group in a set of symbols with the same second suffix, and therefore also the group on the $x$'s, is reducible.

The result may be stated as follows :—

*Theorem.*—The necessary and sufficient condition that a group of linear substitutions on a finite number of symbols should be reducible is that one or more homogeneous linear equations should be satisfied by the coefficients of every substitution of the group.

It follows that, if a group of linear substitutions in $n$ variables has a set of $n^2$ substitutions $A_1, A_2, \ldots, A_{n^2}$, such that

$$\begin{vmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{nn}^{(1)} \\ a_{11}^{(2)} & a_{12}^{(2)} & \ldots & a_{nn}^{(2)} \\ \ldots & \ldots & & \ldots \\ a_{11}^{(n^2)} & a_{12}^{(n^2)} & \ldots & a_{nn}^{(n^2)} \end{vmatrix} \neq 0,$$

the group is irreducible ; and, conversely, if the group is irreducible, there must be such a set of $n^2$ substitutions.

If, in analogy with groups of finite order, the sum of the coefficients in the leading diagonal of a substitution $A$, viz.,

$$a_{11}+a_{22}+\ldots+a_{nn},$$

is called the characteristic of the substitution and is denoted by $\chi_A$, the preceding condition can be expressed in terms of the characteristics of the products, two and two, of the set of $n^2$ substitutions. In fact, apart from sign, the square of the preceding determinant is the determinant for which the element in the $k$-th row and $l$-th column is

$$\sum_{ij} a^{(k)}_{ij} a^{(l)}_{ji}.$$

But this is the sum of the coefficients in the leading diagonal of $A_k A_l$.

The necessary and sufficient condition, then, that a group of linear substitutions in $n$ variables should be irreducible is that it should be possible to choose from it a set of $n^2$ substitutions for which the determinant

$$| \chi_{A_k A} |$$

does not vanish.

# ON CRITERIA FOR THE FINITENESS OF THE ORDER OF A GROUP OF LINEAR SUBSTITUTIONS

*By* W. BURNSIDE.

[Received June 3rd, 1905.—Read June 8th, 1905.]

1. The coefficients in the substitutions of a group of homogeneous linear substitutions are in general complex quantities of the form $\alpha+\beta\sqrt{(-1)}$, where $\alpha$ and $\beta$ are real numbers. If the group is one of finite order, there is a finite number of coefficients, and there must therefore be a finite positive number $M$, such that for each coefficient

$$|\alpha| < M, \qquad |\beta| < M. \tag{i}$$

Similarly there must be another positive number $m$, such that for each coefficient either

$$\left.\begin{aligned} \alpha = 0 \quad \text{or} \quad |\alpha| > m; \\ \text{and} \qquad \beta = 0 \quad \text{or} \quad |\beta| > m \end{aligned}\right\}. \tag{ii}$$

The existence of these two numbers $M$ and $m$ is a necessary condition if the group is a group of finite order.

I propose to shew that, if two positive numbers $M$ and $m$ can be assigned so that each coefficient in every substitution of a group satisfies the system of relations (i) and (ii), then the group is a group of finite order.

The equations
$$x_i' = \sum_{j=1}^{j=n} a_{ij} x_j \quad (i = 1, 2, \ldots, n),$$
defining a linear substitution of a group $G$, imply the equations
$$\bar{x}_i' = \sum_{j=1}^{j=n} \bar{a}_{ij} \bar{x}_j \quad (i = 1, 2, \ldots, n),$$
where $x$ and $\bar{x}$ denote conjugate imaginaries. These $2n$ equations are equivalent to

$$\left.\begin{aligned} x_i'+\bar{x}_i' &= \sum_{j=1}^{j=n} \tfrac{1}{2}(a_{ij}+\bar{a}_{ij})(x_j+\bar{x}_i) \\ &\quad - \sum_{j=1}^{j=n} \frac{1}{2\sqrt{(-1)}}(a_{ij}-\bar{a}_{ij}) \frac{1}{\sqrt{(-1)}}(x_j-\bar{x}_j) \\ \frac{1}{\sqrt{(-1)}}(x_i'-\bar{x}_i') &= \sum_{j=1}^{j=n} \frac{1}{2\sqrt{(-1)}}(a_{ij}-\bar{a}_{ij})(x_j+\bar{x}_j) \\ &\quad + \sum_{j=1}^{j=n} \tfrac{1}{2}(a_{ij}+a_{ij}) \frac{1}{\sqrt{(-1)}}(x_j-\bar{x}_j) \end{aligned}\right\} \quad (i = 1, 2, \ldots, n),$$

which define a linear substitution with real coefficients on the $2n$ variables

$$x_i+\bar{x}_i, \qquad \frac{1}{\sqrt{(-1)}}\,(x_i-\bar{x}_i), \quad (i = 1, 2, \ldots, n).$$

It may be easily verified by direct calculation that the system of linear substitutions thus formed from all the substitutions of $G$ have the group property and constitute a group $G'$ simply isomorphic with $G$.

Moreover, if the coefficients of $G$ satisfy the system of relations (i) and (ii), then the real coefficients of $G'$ are such that each one of them is either zero or numerically (*i.e.*, apart from sign) less than $M$ and greater than $m$.

As the succeeding argument does not depend in any way on the number of variables being even, their number will still be represented by $n$, and any substitution of $G'$ by

$$x_i = \sum_{j=1}^{j=n} a_{ij}x_j \quad (i = 1, 2, \ldots, n),$$

where now $a_{ij}$ is a real number, such that either

$$a_{ij} = 0$$

or

$$M > |\,a_{ij}\,| > m.$$

The sum of the squares of the coefficients of any substitution is certainly less than $n^2M^2$. Hence, if $a_{11}, a_{12}, \ldots, a_{nn}$ be regarded as the rectangular co-ordinates of a point $P$ in space of $n^2$ dimensions, the point is restricted to be within a sphere of radius $nM$ with the origin as centre. Consider now the aggregate of points $P$ which correspond to the totality of the substitutions of $G'$. If the aggregate is a finite one, the number of substitutions of $G'$, *i.e.*, its order, is finite. If the aggregate is not finite, then, since the points are restricted to be within a certain sphere of finite radius, the aggregate must have at least one limit point. Let

$$A_{11}, A_{12}, \ldots, A_{nn}$$

be the coordinates of a limit-point. Then, if $\frac{1}{2}\epsilon$ is any given small positive quantity, an infinite number of points of the aggregate lie within a sphere of radius $\frac{1}{2}\epsilon$ described about the limit-point as centre. If

$$a_{11}, a_{12}, \ldots, a_{nn}$$

be an assigned one of these points, and

$$b_{11}, b_{12}, \ldots, b_{nn}$$

be any other, then

$$\sum_{ij} (a_{ij}-b_{ij})^2 < \epsilon^2,$$

and therefore for all suffixes $|\,a_{ij}-b_{ij}\,| < \epsilon$.

Let $A$ and $B$ be the two substitutions of $G'$ which correspond to these points, and denote by $C$ with coefficients $c_{ij}$ the substitution $A^{-1}B$. Then, since

$$b_{ij} = a_{ij}+\eta,$$

where $\eta$ lies between $-\epsilon$ and $+\epsilon$,

$$| c_{ii}-1 | < nM\epsilon,$$

$$| c_{ij} | < nM\epsilon \quad (i \neq j);$$

and $G'$ has an infinite number of substitutions whose coefficients satisfy these inequalities, however small $\epsilon$ may be. If, for every one of them $c_{ij}$ $(i \neq j)$ were zero, every one of them would replace each of the symbols by a real multiple of itself. Since there is an infinite number of these substitutions, the multipliers cannot all be either $+1$ or $-1$. But, if any multiplier is different from $+1$ and $-1$, a sufficiently high power of the corresponding substitution would have a coefficient which is either greater than $M$, or without being zero less than $m$. If, on the other hand, some $c_{ij}$ is not zero, then, by making $\epsilon$ small enough, it will be numerically less than $m$. In either case the supposition that the aggregate of points $P$ is not finite leads to a contradiction. It follows therefore that, with the limitation of the coefficients given by the relations (i) and (ii), the order of $G'$, and necessarily also the order of $G$, is finite.

It is perhaps worth remarking that either relation (i) by itself, or relation (ii) by itself, is not sufficient to ensure that the order shall be finite. In fact the group

$$x' = \alpha x+\beta y,$$

$$y' = \gamma x+\delta y,$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are real integers such that

$$\alpha\delta-\beta\gamma = 1$$

is not of finite order, and has no coefficient other than zero numerically less than unity. Again, the group

$$x' = \cos n\theta x-\sin n\theta y,$$

$$y' = \sin n\theta x+\cos n\theta y,$$

where $n$ is an integer and $\pi/\theta$ is not rational, is not of finite order and has no coefficient numerically greater than unity.

2. If the order of a group of linear substitutions is finite, the order of every substitution of the group is a finite number equal to or less than the order of the group. I propose to show that the converse of this theorem may be proved in the form :—

If, in a group of linear substitutions on a finite number $n$ of symbols, the order of every substitution is equal to or less than a finite number $m$, then the group is a group of finite order.

As above, any substitution $A$ of the group is given by the system of equations

$$x_i' = \sum_{j=1}^{j=n} a_{ij} x_j \quad (i = 1, 2, \ldots, n).$$

The substitution $A$ being one of finite order, equal to or less than $m$, its characteristic

$$a_{11} + a_{22} + \ldots + a_{nn}$$

is the sum of $n$ roots of unity, the index of each root that occurs being equal to or less than $m$. Such a set of roots of unity can only be chosen in a finite number of ways, and therefore there is only a finite number of distinct characteristics. Now the characteristic of $AB$, where the coefficients of $B$ are $\beta_{ij}$, is

$$\sum_{ij} a_{ij} \beta_{ji},$$

and therefore, whatever substitution of the group $B$ may be, its coefficients satisfy the equation

$$\sum_{ij} a_{ij} \beta_{ji} = \chi,$$

where $\chi$ is one of the finite number of distinct characteristics.

Suppose now that it is possible to choose $n^2$ distinct substitutions of the group, $A_1, A_2, \ldots, A_{n^2}$, such that

$$\begin{vmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{nn}^{(1)} \\ a_{11}^{(2)} & a_{22}^{(2)} & \ldots & a_{nn}^{(2)} \\ \ldots & \ldots & & \ldots \\ a_{11}^{(n^2)} & a_{12}^{(n^2)} & \ldots & a_{nn}^{(n^2)} \end{vmatrix} \neq 0;$$

then the coefficients of any other substitution $B$ satisfy a system of $n^2$ linear equations of non-vanishing determinant, the right-hand sides of which can take only a finite number of distinct values. In this case then the number of substitutions, and therefore the order of the group, is finite.

If no such set of $n^2$ substitutions as

$$A_1, A_2, \ldots, A_{n^2}$$

exists, there must be one or more homogeneous linear equations

$$\Sigma B_{ij} \beta_{ij} = 0$$

satisfied by the coefficients of every substitution of $G$. This, however, is known to imply that the group is reducible. Hence under the conditions assumed the group is either reducible, or, if irreducible, of finite order.

From this it immediately follows that it must be of finite order in any case. Suppose, in fact, that of the $n$ variables $r$ are transformed irreducibly among themselves. So far as it affects these $r$ symbols the group is of finite order, and therefore it has a self-conjugate sub-group of finite index for which each of the $r$ symbols is unchanged. The substitutions on the remaining $n-r$ symbols, if in the equations defining them the first $r$ symbols are all made zero, give rise to a group with the properties of the original one in a smaller number of symbols, which may or may not be reducible. If it is not, it again is a group of finite order; and $G$ has a self-conjugate sub-group of finite index, whose substitutions are of the form

$$x'_s = x_s \qquad (s = 1, 2, \ldots, r),$$

$$x'_{r+s} = x_{r+s} + \sum_1^r a_{st} x_t \qquad (s = 1, 2, \ldots, n-r).$$

Unless the coefficients $a_{st}$ are all zero, so that the sub-group reduces to the identical operation, the substitutions of this sub-group are clearly not of finite order.

If the group on the $n-r$ last symbols is reducible, it may be treated as the original group. Hence, finally, in any case, whether reducible or not, the group is necessarily of finite order.

3. A group of linear substitutions of finite order has a finite number of distinct sets of conjugate substitutions. The converse of this theorem is also true in the form :—

If a group of linear substitutions on a finite number of symbols has a finite number of distinct sets of conjugate substitutions, the order of the group is finite.

The characteristics of two conjugate substitutions (*i.e.*, the sums of the coefficients in the leading diagonals) are the same. Hence, if the number of conjugate sets is finite, the number of characteristics is finite. When the group is irreducible, the argument of the preceding paragraph then shows that the number of substitutions is also finite. When the group is reducible, the same argument shows that the group must have a self-conjugate sub-group $H$ of finite index, whose substitutions are of the form

$$x'_s = x_s \qquad (s = 1, 2, \ldots, r_1),$$

$$x'_{r_1+s} = x_{r_1+s} + \sum_{t=1}^{t=r_1} a_{st} x_t \qquad (s = 1, 2, \ldots, r_2),$$

$$x'_{r_1+r_2+s} = x_{r_1+r_2+s} + \sum_{u=1}^{u=r_2} \beta_{su} x_{r_1+u} + \sum_{v=1}^{v=r_1} \gamma_{sv} x_v \qquad (s = 1, 2, \ldots, r_3).$$

... ... ... ... ... ...

If this sub-group actually exists, *i.e.*, if it does not consist of the identical substitution only, it is necessarily *not* of finite order.

Now, so far as the first $r_1+r_2$ symbols are concerned, it is an Abelian group, whose substitutions can only be conjugate in respect of the substitutions of $G$ not contained in $H$. Of these there are only a finite number of classes, and therefore unless all the $a$'s are zero the group would contain an infinite number of sets of conjugate substitutions. But, if the $a$'s are all zero, the group, so far as it affects the first $r_1+r_2+r_3$ symbols, is Abelian; and the same reasoning shows that all the $\beta$'s and $\gamma$'s must vanish. Hence, if the group contains only a finite number of distinct conjugate sets, $H$ must consist of the identical substitution only, and $G$ must be a group of finite order.

# ON A CLASS OF ANALYTIC FUNCTIONS

*By* G. H. HARDY.

[Received May 17th, 1905.—Read June 8th, 1905.—Received in revised form*
August 3rd, 1905.]

## I.

1. The distinction between integral functions whose increase is regular (*fonctions à croissance régulière*) and those whose increase is irregular was first explicitly formulated by M. Borel.† According to him a function is *à croissance régulière* if

$$\lim_{r=\infty} \frac{\log\log M(r)}{\log r}$$

is determinate, $M(r)$ denoting the greatest value of the modulus of the function on a circle of radius $r$. A function is then *à croissance irrégulière* if we can determine two constants $\alpha$, $\beta$ $(\alpha<\beta)$ such that for an infinity of values of $r$ tending to infinity

$$M(r) < e^{r^\alpha},$$

and for a similar infinity of values of $r$

$$M(r) > e^{r^\beta}.$$

The numerous extensions which have been given lately to the theory of integral functions render it necessary to give a greater degree of precision to this definition. We shall say that $f(x)$ is a function whose increase is irregular if we can determine $\beta_\nu$, $\gamma_\nu$ $(\beta_\nu < \gamma_\nu)$ so that for two infinities of values of $r$ tending to infinity $M(r) < e^{V_1(r)}$ and $M(r) > e^{V_2(r)}$ respectively, where

$$V_1(r) = A_1 r^{\alpha_0} (\log_1 r)^{\alpha_1} \dots (\log_{\nu-1} r)^{\alpha_{\nu-1}} (\log_\nu r)^{\beta_\nu}.$$

and

$$V_2(r) = A_2 r^{\alpha_0} (\log_1 r)^{\alpha_1} \dots (\log_{\nu-1} r)^{\alpha_{\nu-1}} (\log_\nu r)^{\gamma_\nu},$$

$\log_1 r$, $\log_2 r$, ... denoting as usual $\log r$, $\log\log r$, .... It is evident that this definition applies only to functions of finite order (*genre*), but it is easy to frame similar definitions for functions of higher order, the general principle being obvious.

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* The contents of the paper have been considerably altered in revision, and the title has been changed.

† *Leçons sur les fonctions entières*, pp. 107 *et seq.*

In this paper I propose to consider certain classes of series of the general type

$$\sum_{\nu=1}^{\infty} \frac{c_\nu}{\sin \nu\pi\lambda} x^\nu, \tag{1}$$

where $\lambda$ is any quantity which is not *real and rational.*[*] The peculiar interest of these functions lies in the fact that for the same set of values of the numbers $c_\nu$ the series may

(i.) be convergent for all values of $x$, and represent an integral function whose increase is either regular or irregular;

(ii.) have a finite circle of convergence which is a critical line for the function represented by the series;

(iii.) diverge for all values of $x$.

All these peculiarities, for example, present themselves for different values of $\lambda$ in the case in which $c_\nu = 1/\nu!$, to which I shall devote particular attention.

I wish to point out that the series (1) is not an instance of a series artificially constructed in order to provide an illustration of certain theoretical possibilities. On the contrary, the series

$$\sum \frac{x^\nu}{\nu! \sin \nu\lambda\pi} \tag{2}$$

presents itself naturally and inevitably when we attempt to determine the behaviour of the function represented by the simple definite integral

$$\int_0^\infty \frac{e^{-u^\lambda} du}{u+x}, \tag{3}$$

and is therefore (in some cases) an instance of a *fonction à croissance irrégulière* quite unlike those devised by M. Borel.[†]

2. There is another point of view from which the series (1) may be regarded. Let us suppose that

$$f(x) = \sum c_\nu x^\nu, \tag{4}$$

and that $\lambda$ has such a value that (1) is convergent for at any rate some values of $x$. The function defined by (1) is then the most obvious solution of the difference equation

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[*] The series is obviously meaningless if $\lambda$ is real and rational.

[†] *Loc. cit.*

$$F(xe^{\lambda\pi i})-F(xe^{-\lambda\pi i}) = 2if(x), \tag{5}$$

an equation which is fundamentally the same as the classical difference equation

$$\Phi(x+a)-\Phi(x) = \phi(x), \tag{6}$$

to which it may easily be reduced by means of the substitutions

$$\left\{\begin{array}{l} x = e^{(\omega+\lambda)\pi i}, \quad F(e^{\omega\pi i}) = \Phi(\omega), \\ 2if\{e^{(\omega+\lambda)\pi i}\} = \phi_1'(\omega), \quad 2\lambda = a. \end{array}\right. \tag{7}$$

### *The Series* $\sum\limits_{\nu=1}^{\infty} \dfrac{c_\nu x^\nu}{\sin \nu\lambda\pi}$.

3. I shall now consider directly what is the nature of the function defined by the series (1) in those cases in which it has a radius of convergence other than zero. I shall suppose for simplicity that

$$|c_\nu| \geqslant |c_{\nu+1}| \tag{8}$$

for all values of $\nu$, and that $\lim c_\nu^{1/\nu}$ is determinate.

4. (i.) If $\lambda$ is complex and equal to $\lambda_1+i\lambda_2$,

$$|\sin \nu\lambda\pi| = \tfrac{1}{2}e^{\nu\pi|\lambda_2|}(1+\epsilon), \tag{9}$$

where $\epsilon$ is small when $\nu$ is large. The radius of convergence of (1) is then $\rho e^{\pi|\lambda_2|}$, $\rho$ being the radius of convergence of (4). This case possesses no particular feature of interest.

(ii.) If $\lambda$ is real, we may without loss of generality suppose it *positive, irrational, and less than unity*. The region of convergence will then depend upon the arithmetical nature of $\lambda$. The radius of convergence of (1) is certainly not greater than $\rho$. I shall prove first *that, if* $\lambda$ *is an algebraic number, the radius of convergence of* (1) *is equal to* $\rho$.

For suppose that $\lambda$ is algebraic and of degree $m$, *i.e.*, the root of an irreducible equation

$$x^m+a_1x^{m-1}+\ldots+a_m = 0, \tag{10}$$

where $a_1, \ldots, a_m$ are integers. Then, by a well known property of algebraic numbers,*

(11)
$$\left|\frac{\mu}{\nu}-\lambda\right| > \frac{K_m}{\nu^{m+1}},$$

where $K_m$ is a constant (depending only on $m$) for all integral values of $\mu$ and $\nu$. Hence

(12)
$$|\mu-\nu\lambda| > K_m\nu^{-m},$$

from which it obviously follows that

(13)
$$|\operatorname{cosec} \nu\lambda\pi| < K'_m\nu^m$$

for all values of $\nu$. Hence the radius of convergence of (1) is not less than that of $\Sigma c_\nu \nu^m x^\nu$, *i.e.*, than $\rho$, and therefore it is equal to $\rho$. In particular, if $\rho = \infty$, the series (1) represents an integral function of $x$.

In the second place, *values of* $\lambda$ *can be found such that the radius of convergence of* (1) *is any quantity* $R$, *where*

(14)
$$0 \leqslant R < \rho.$$

In order to prove this and the further results which we shall establish later on, we must consider certain properties of simple continued fractions.†

5. Let us suppose that $\lambda$ is expressed as a simple continued fraction

(15)
$$\frac{1}{a_0+}\,\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots,$$

and that $p_\nu/q_\nu$ $(\nu = 0, 1, \ldots)$ are the successive convergents, while

(16)
$$\frac{p_0}{q_0}=\frac{0}{1},\quad \frac{p_1}{q_1}=\frac{1}{a_0},\quad \frac{p_2}{q_2}=\frac{a_1}{a_0a_1+1};$$

and let us denote the complete quotient

(17)
$$a_\nu+\frac{1}{a_{\nu+1}+}\,\frac{1}{a_{\nu+2}+}\cdots$$

by $a'_\nu$. Let $m$ be any positive integer such that

(18)
$$q < m < q_{\nu+1}.$$

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* Borel, *Leçons*, I., p. 27. This inequality is employed in a very similar manner by H. J. S. Smith, "On some Discontinuous Series considered by Riemann," *Mess. of Math.*, Vol. XI., pp. 1-11.

† The purpose of §§ 5, 6 is simply to establish the equations (30) and (31) at the end of § 6.

We can form successively the equations

$$(19)\qquad \begin{cases} m = k_1 q_\nu + m_1 & (0 \leqslant k_1 \leqslant a_\nu, \quad 0 \leqslant m_1 < q_\nu), \\ m_1 = k_2 q_{\nu-1} + m_2 & (0 \leqslant k_2 \leqslant a_{\nu-1}, \quad 0 \leqslant m_2 < q_{\nu-1}, \\ \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ m_{\nu-2} = k_{\nu-1} q_2 + m_{\nu-1} & (0 \leqslant k_{\nu-1} \leqslant a_2, \quad 0 \leqslant m_{\nu-1} < q_2), \\ m_{\nu-1} = k_\nu q_1 + m_\nu & (0 \leqslant k_\nu \leqslant a_1, \quad 0 \leqslant m_\nu < q_1), \end{cases}$$

and in one way only. Then

$$(20)\qquad m = \sum_{s=1}^{\nu} k_s q_{\nu-s+1} + m_\nu .$$

Now

$$q_{\nu-s+1}\lambda = q_{\nu-s+1} \frac{a_{\nu-s+1} p_{\nu-s+1} + p_{\nu-s}}{a'_{\nu-s+1} q_{\nu-s+1} + q_{\nu-s}} = p_{\nu-s+1} + \frac{(-1)^{\nu-s+1}}{a_{\nu-s+1} q_{\nu-s+1} + q_{\nu-s}},$$

since

$$p_{\nu-s} q_{\nu-s+1} - q_{\nu-s} p_{\nu-s+1} = (-1)^{\nu-s+1}.$$

Hence

$$(21)\qquad m\lambda = I_m + m_\nu \lambda + (-)^\nu S_m,$$

where

$$(22)\qquad I_m = \sum_{s=1}^{\nu} k_s p_{\nu-s+1},$$

which is integral, and

$$(23)\qquad S_m = \sum_{s=1}^{\nu} \frac{(-)^{s-1} k_s}{a'_{\nu-s+1} q_{\nu-s+1} + q_{\nu-s}} .$$

In what follows I shall suppose $\nu$ *odd*. The work in the case in which $\nu$ is even is strictly analogous. And I shall write

$$q'_{r+1} = a'_r q_r + q_{r-1};$$

so that

$$(23')\qquad S_m = \sum_{s=1}^{\nu} \frac{(-)^{s-1} k_s}{q'_{\nu-s+2}} .$$

In the first place we can prove that, if $k_r$ is the last $k$ which does not vanish, $S_m$ has the sign of $(-)^{r-1}$, *i.e.*, the sign of its last term. In fact the last term is numerically greater than the sum of all preceding terms of opposite sign, that is to say,

$$(24)\qquad \frac{k_r}{q'_{\nu-r+2}} > \frac{k_{r-1}}{q'_{\nu-r+3}} + \frac{k_{r-3}}{q'_{\nu-r+5}} + \dots .$$

Suppose, for example, that $r$ is even. The most unfavourable case is obviously that in which

$$k_r = 1, \quad k_{r-1} = a_{\nu-r+2}, \quad k_{r-3} = a_{\nu-r+4}, \quad \dots, \quad k_1 = a_\nu ,$$

and all the other $k$'s vanish. This case cannot actually occur. For, if it could, we should have

$$m-m_\nu = a_\nu q_\nu + a_{\nu-2} q_{\nu-2} + \ldots + a_{\nu-r+2} q_{\nu-r+2} + q_{\nu-r+1},$$

*i.e.*, $= q_{\nu+1}$. But, even if it could, the inequality (24) would still be valid. For we should have

$$q_{\nu+1}\lambda = I_m - S_m = p_{\nu+1} - S_m.$$

Now, as $\nu$ is odd, $$p_{\nu+1} < q_{\nu+1}\lambda.$$

Hence $S_m < 0$; *i.e.*, $S_m$ has the sign of $(-)^{r-1}$. *A fortiori* this is true in any case which can actually occur. Moreover, since at least one $k$ must differ from the value assigned to it in the most unfavourable case, the excess of the left-hand side of (24) over the right-hand side must be at least

$$\frac{1}{q'_{\nu+1}}, \tag{25}$$

and therefore the modulus of $S_m$ must be at least equal to the same quantity.

If, on the other hand, $r$ is odd, the most unfavourable case is given by taking

$$k_r = 1, \quad k_{r-1} = a_{\nu-r+2}, \quad \ldots, \quad k_2 = a_{\nu-1},$$

$$m-m_\nu = a_{\nu-1} q_{\nu-1} + a_{\nu-3} q_{\nu-3} + \ldots + a_{\nu-r+2} q_{\nu-r+2} + q_{\nu-r+1},$$

*i.e.*, $= q_\nu$. But then we should have

$$q_\nu \lambda = p_\nu - S_m,$$

and, as $p_\nu > q_\nu \lambda$, $S_m$ would be positive, from which we can draw the same conclusions. Thus an inferior limit for $|S_m|$ is given by the quantity (25).

We shall also require a superior limit for $|S_m|$. If $S_m$ is positive, it is certainly less than $S'_m$, deduced from $S_m$ by taking

$$k_1 = a_\nu, \quad k_2 = 0, \quad k_3 = a_{\nu-2}, \quad \ldots, \quad k_\nu = a_1,$$

$$m-m_\nu = a_\nu q_\nu + a_{\nu-2} q_{\nu-2} + \ldots + a_1 q_1 = q_{\nu+1} - q_0 = q_{\nu+1} - 1.$$

Then $$q_{\nu+1}\lambda = I'_m - S'_m + \lambda,$$

and, as $$q_{\nu+1}\lambda = p_{\nu+1} + \frac{1}{q_{\nu+2}}$$

and $$I'_m = p_{\nu+1},$$

$$S'_m = \lambda - \frac{1}{q'_{\nu+2}}, \tag{26}$$

which is less than $\lambda$, and nearly equal to $\lambda$ when $\nu$ is large.

Now $S_m$ can be equal to $S'_m$ if, and only if, $m = q_{\nu+1} - 1$, $m_\nu = 0$. In this case $S_m < \lambda$; in all other cases

$$S_m < \lambda - \frac{1}{q'_{\nu+1}}, \tag{26'}$$

since at least one $k$ must differ from the value assigned to it in the formation of $S'_m$.

If, on the other hand, $S_m$ is negative, it is numerically less than $-S''_m$, deduced from $S_m$ by taking

$$k_1 = 0, \quad k_2 = a_{\nu-1}, \quad k_3 = 0, \quad \dots, \quad k_\nu = 0,$$

$$m - m_\nu = a_{\nu-1} q_{\nu-1} + a_{\nu-3} q_{\nu-3} + \dots + a_2 q_2 = q_\nu - q_1 = q^{\nu} - a_0.$$

Then $$q_\nu \lambda = I''_m - S''_m + a_0 \lambda,$$

and, as $$q_\nu \lambda = p_\nu - \frac{1}{q'_{\nu+1}}$$

and $$I''_m = p_\nu - 1,$$

$$-S''_m = 1 - a_0 \lambda - \frac{1}{q'_{\nu+1}}, \tag{27}$$

which is less than $$1 - a_0 \lambda = \frac{1}{a_0 a'_1 + 1},$$

and nearly equal to it when $\nu$ is large.

To sum up, we have obtained the following limits for $S_m$ :—

(i.) If $S_m > 0$,

$$\frac{1}{q_{\nu+1}} < S_m < \lambda - \frac{1}{q'_{\nu+1}}.^{*} \tag{28}$$

(ii.) If $S_m < 0$,

$$\frac{1}{q'_{\nu+1}} < -S_m < 1 - a_0 \lambda - \frac{1}{q'_{\nu+1}}. \tag{28'}$$

6. Now

$$|\sin m\lambda\pi| = |\sin (m_\nu \lambda - S_m)\pi|. \tag{29}$$

There are two cases to consider:

(*a*) Suppose $m_\nu = 0$. Then

$$|\sin m\lambda\pi| = |\sin S_m \pi| > \frac{K}{q'_{\nu+1}},$$

where $K$ is a constant.

(*b*) Suppose $1 \leqslant m_\nu \leqslant a_0 - 1$. If $S_m > 0$, it is clear that

$$m_\nu \lambda - S_m < (a_0 - 1)\lambda < 1 - \lambda.$$

On the other hand, $$m_\nu \lambda - S_m \geqslant \lambda - S_m > \frac{1}{q'_{\nu+1}}.$$

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[*] One exception to the second of these inequalities was noted above. In this exceptional case the inequality $S_m < \lambda$ will be sufficient for our purpose.

If $S_m < 0$, $m_\nu\lambda - S_m > \lambda$, and, on the other hand,

$$m_\nu\lambda - S_m \leqslant (a_0 - 1)\lambda - S_m < 1 - \lambda.$$

And *in all cases*

$$(30) \qquad |\sin m\lambda\pi| > \frac{K}{q'_{\nu+1}} \quad (q_\nu < m < q_{\nu+1}).$$

On the other hand, $|\sin q_{\nu+1}\lambda\pi| = \sin\dfrac{\pi}{q_{\nu+2}}$,
so that for large values of $\nu$

$$(31) \qquad |\sin q_{\nu+1}\lambda\pi| = \frac{\pi(1+\epsilon_\nu)}{q'_{\nu+2}},$$

where $\epsilon_\nu$ is small when $\nu$ is large. The two formulæ (30) and (31) will form the basis of the succeeding argument.

7. We return now to the series (1). In all that follows I shall suppose that

$$a_1 < a_2 < a_3 < \dots ;$$

so that for large values of $\nu$

$$a_\nu, \quad q_\nu, \quad q_{\nu+1}/q_\nu$$

are large and

$$a'_\nu/a_\nu, \quad q'_\nu/q_\nu$$

nearly equal to unity.

Suppose first that the radius of convergence of $\Sigma c_\nu x^\nu$ is finite; we may without loss of generality suppose it equal to unity. And let $R = 1/a < 1$.

It is easy to see that, if we are given any sequence of ascending integers $Q_\nu$ such that $\lim Q_{\nu+1}/Q_\nu = \infty$, we can find a continued fraction such that

$$\lim q_\nu/Q_\nu = 1.$$

We can therefore find a value of $\lambda$ such that

$$(32) \qquad \lim q_{\nu+1}/a^{q_\nu} = 1.$$

The radius of convergence of the series

$$(33) \qquad \Sigma \frac{c_{q_\nu} x^{q}}{\sin q_\nu\lambda\pi}$$

is the same as that of

$$\Sigma\, q_{\nu+1} c_{q_\nu} x^{q_\nu},$$

and is evidently $R$.

The remainder of the series (1) is $\sum\limits_{\nu=1}^{\infty} u_\nu$, where

$$u_\nu = \sum_{q_\nu+1}^{q_{\nu+1}-1} \frac{c_n x^n}{\sin n\lambda\pi}.$$

Now $$|u_\nu| < K_{q_{\nu+1}} |c_{q_\nu} x^{q_\nu}|/(1-|x|),$$

and so the radius of convergence of the series $\Sigma u_\nu$ is *at least* $R$. Hence the radius of convergence of (1) is $R$.

In this case the circle $|x| = R$ is a *coupure* for the function

$$F(x) = \Sigma \frac{c_\nu x^\nu}{\sin \nu\lambda\pi}. \tag{34}$$

For $$F(xe^{\lambda\pi i}) - F(xe^{-\lambda\pi i}) = 2if(x),$$

where $f(x) = \Sigma c_\nu x^\nu$. There is at least one singularity of $F(x)$ of the type $x = Re^{i\theta}$, and it is plain that, if $k$ is any integer,

$$x = Re^{i(\theta + 2k\lambda\pi)}$$

is also a singularity; and these points are everywhere dense on the circle $|x| = R$.

If $\rho$ is infinite, we can still without loss of generality suppose $R = 1/a < 1$, and we can determine $\lambda$ so that

$$\lim (c_{q_\nu} q_{\nu+1})^{1/q_\nu} = a.$$

The argument is then substantially the same. And it is obvious that by supposing $q_{\nu+1}$ a function of $q_\nu$ whose increase is sufficiently rapid, we can in any case ensure that the series (1) shall be divergent for all values of $x$ other than zero.

8. In order to justify the assertions made in § 1, it remains only to prove that, when $f(x)$ is an integral function, values of $\lambda$ can be found for which $F(x)$ is an integral function and its increase irregular.

I shall consider, for simplicity, the particular case in which $c_\nu = 1/\nu!$, which enables us to illustrate adequately the different cases which may occur.

In the first place, the increase of $F(x)$ *may* be regular. Suppose, for example, that $\lambda$ is an algebraic number of degree $m$. By a well known theorem $M(r)$ is greater than the modulus of the greatest term in the series (1), and so certainly

$$> Ke^r r^{-\frac{1}{2}}.\text{[*]}$$

On the other hand, the terms of (1) are less than those of the series

$$K \Sigma \nu^m \frac{r^\nu}{\nu!}$$

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[*] Since, by Stirling's theorem, $\frac{n^n}{n!} = \frac{e^n}{\sqrt{2\pi n}}(1 + \epsilon_n)$ when $n$ is large.

(by § 4), and so $$M(r) < Ke^r r^m.$$
Thus, if $$M(r) = e^{F(r)},$$
$$r - \tfrac{1}{2}\log r + K < F(r) < r + m\log r + K,$$
and therefore the increase of $F(x)$ is regular.

9. I propose now to find inferior and superior limits for $M(r)$ when $r = q_\nu$, and to deduce that $F(x)$ may be an integral function whose increase is irregular. I shall write $u_n$ for the general term of the series (1), and $U_\nu$ for
$$\sum_{q_\nu}^{q_{\nu+1}-1} u_n.^*$$

Assuming for the moment that $F(x)$ is an integral function, an *inferior* limit to $M(r)$ is given by the fact that $M(r)$ is greater than the modulus of the numerically greatest term in the series. Selecting the $q_\nu$th term, we find that

(36) $$M(r) > q_{\nu+1} q_\nu^{q_\nu}/q_\nu!.$$

Again, since $|x^n/n!|$ is greatest when $n = q_\nu$,

(37) $$\left|\sum_1^{q_\nu-1} u_n\right| < Kq_\nu^2\,\frac{q_\nu^{q_\nu}}{q_\nu!},$$

and, if the $q_\nu$'s are so chosen that

(38) $$\lim q_{\nu+1}/q_\nu^2 = \infty,$$

this is certainly small in comparison with (36).

On the other hand, if $\mu > \nu$,

(39) $$|U_\mu| < Kq_{\mu+1}^2\,\frac{q_\nu^{q_\mu}}{q_\mu!} = U'_{\nu,\mu}.$$

(40) Now $$\frac{U'_{\nu,\nu+1}}{U'_{\nu,\nu}} < K\,\frac{q_{\nu+2}^2}{q_{\nu+1}^2}\,\frac{q_\nu!}{q_{\nu+1}!}\,q_\nu^{q_{\nu+1}-q_\nu} < K\left(\frac{eq_\nu}{q_{\nu+1}}\right)^{q_{\nu+1}} q_{\nu+2}^2$$

by an easy application of Stirling's theorem, and so, by (38),

(41) $$< Kq_{\nu+2}^2\,q_{\nu+1}^{-\frac{1}{2}q_{\nu+1}}.$$

If the $q$'s are such that

(42) $$\lim q_{\nu+1}^2/q_\nu^{\frac{1}{2}q_\nu} = 0,$$

the quantity on the right hand of (39) will be exceedingly small. Similarly we can prove that $U'_{\nu,\mu+1}/U'_{\nu,\mu}$ is exceedingly small if $\mu > \nu$. From this fact, in conjunction with (39) and (37), it follows that the series is convergent for $x = q_\nu$ and that the modulus of its sum is less than

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* In the notation of § 7, $U_\nu =$ [illegible]

$$K^2_{q_\nu+1} \frac{q_\nu^{q_\nu}}{q_\nu!}.$$

The series (1) therefore represents an integral function $F(x)$, and we have the inequalities

$$q_{\nu+1} \frac{q_\nu^{q_\nu}}{q_\nu!} < M(r) < Kq^2_{\nu+1} \frac{q_\nu^{q_\nu}}{q_\nu!}. \tag{42}$$

But, since $q_\nu = r$,

$$\frac{q_\nu^{q_\nu}}{q_\nu!} \sim \frac{e^r}{\sqrt{2\pi r}}.$$

Hence

$$q_{\nu+1} \frac{e^r}{\sqrt{2\pi r}} < M(r) < Kq^2_{\nu+1} \frac{e^r}{\sqrt{2\pi r}}, \tag{43}$$

for $r = q_\nu$, provided the conditions (38) and (41) are satisfied.

Now let us suppose that when $\nu$ is odd

$$q_{\nu+1} \sim q_\nu^p \quad (p > 2),$$

and when $\nu$ is even

$$q_{\nu+1} \sim q_\nu^{aq_\nu} \quad (a < \tfrac{1}{2}).$$

Then when $r = q_\nu$ and $\nu$ is odd

$$M(r) < Kr^{2p-\frac{1}{2}}e^r,$$

and when $r = q_\nu$ and $\nu$ is even

$$M(r) > Kr^{-\frac{1}{2}}e^{ar \log r + r}.$$

Hence the increase of $F(x)$ is irregular.

10. In a precisely similar manner we could, by taking

$$c_\nu = 1/(\nu!)^\beta \quad \text{or} \quad c_\nu = 1/\Gamma(\beta\nu+1),$$

construct functions $F(x)$ such that, for an infinity of values of $r$, $M(r)$ is (roughly) of order $e^{r^{1/\beta}}$ and, for another infinity of values of $r$, $M(r)$ is (roughly) of order

$$e^{ar^{1/\beta}\log r}.$$

We can also find functions of *infinite* order which possess similar peculiarities. For example, by taking

$$c_\nu = \log 2 . \log 3 \dots \log \nu,$$

we can define a function whose maximum modulus is of the order of

$$e^{e^r/r}$$

for one infinity of values of $r$, and of order

$$e^{ae^r \log r}$$

for another. And the whole of the preceding analysis might be made considerably more precise, as we have generally left a considerable margin in our inequalities.

The preceding method, however, does not (as might at first sight be expected) enable us, for a *given* set of coefficients $c_\nu$, such as $c_\nu = 1/\nu!$, to determine $\lambda$ so that the increase of $F(x)$ shall be *arbitrarily* great. For, since we must for convergence have $q_{\nu+1} < q_\nu!$, $|w_{q_\nu}|$ is never, when $r = q_\nu$, of order substantially greater than $e^{r \log r}$.

## II.

11. It might well be thought that functions such as those which I have considered in the first part of this paper were merely examples of an artificial character constructed in order to illustrate theoretical possibilities. This is far from being the case, as I shall proceed to show.

Let us consider the function

$$F_{\lambda, a}(x) = \int_0^\infty \frac{e^{-u^\lambda} u^{a-1}}{u+x}\, du, \tag{1}$$

$x$ being a complex variable whose variation is restricted by a cut along the negative real axis, $\lambda$ and $a$ being any real or complex quantities subject to certain restrictions which will be defined later, and $u^\lambda$ and $u^{a-1}$ having their principal values. This function includes as particular cases a number of well known functions. For instance,

(i.) if $\lambda = a = 1$, and we suppose for a moment that $x$ is real and positive,

$$F_{1,1}(x) = e^x \int_x^\infty \frac{e^{-w}}{w}\, d\omega = -e^x\, li(e^{-x}) = e^x \left\{ \sum_1^\infty \frac{(-)^{n-1} x^n}{n.n!} - \gamma - \log x \right\}, \tag{2}$$

this expansion defining the behaviour of $F_{1,1}(x)$ for all values of $x$.

(ii.) If $\lambda = 1$, $a = \frac{1}{2}$, and $x = \xi^2$ is real and positive,

$$F_{1,\frac{1}{2}}(x) = 2\int_0^\infty \frac{e^{-\omega^2} d\omega}{\omega^2+\xi^2} \doteq \frac{2\sqrt{\pi}}{\xi}\, e^{\xi^2} \int_\xi^\infty e^{-t^2} dt^* \tag{3}$$

$$= \pi \left\{ x^{-\frac{1}{2}} e^x - \sum_0^\infty \frac{x^n}{\Gamma(n+\frac{3}{2})} \right\},$$

after some transformations which will easily be supplied. This expansion again defines the behaviour of $F_{1,\frac{1}{2}}(x)$ for all values of $x$.

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[*] G. F. Meyer's edition (1871) of Dirichlet's *Lectures on Definite Integrals*, § 98, p. 285; Arendt's edition (1904), p. 208.

(iii.) If $\lambda = \frac{1}{2}$, $a = \frac{1}{2}$, and $x = \xi^2$ as before,

$$F_{\frac{1}{2},\frac{1}{2}}(x) = 2\int_0^\infty \frac{e^{-\omega}\,d\omega}{\omega^2+\xi^2} = \frac{2}{\xi}\left(\cos\xi \int_1^\infty \frac{\sin \xi t}{t}\,dt - \sin\xi \int_1^\infty \frac{\cos\xi t}{t}\,dt\right).^*$$

From this we can deduce, by some simple transformations,

$$(4) \qquad F_{\frac{1}{2},\frac{1}{2}}(x) = \frac{2\cos\sqrt{x}}{\sqrt{x}}\left\{\tfrac{1}{2}\pi - \sqrt{x}\sum_0^\infty \frac{(-)^n x^n}{(2n+1)(2n+1)!}\right\}$$
$$+\frac{2\sin\sqrt{x}}{\sqrt{x}}\left\{\gamma + \tfrac{1}{2}\log x + \sum_1^\infty \frac{(-)^n x^n}{2n\,.\,2n!}\right\},$$

a formula which again defines the behaviour of $F_{\frac{1}{2},\frac{1}{2}}(x)$ for all values of $x$.

12. Let us now consider for what values of $\lambda$ and $a$ the general integral is convergent.

(i.) If $\lambda$ is *complex*, we may without loss of generality suppose its real part positive, since the transformation of the integral by the substitution $u = \frac{1}{v}$ gives

$$F_{\lambda,a}(x) = \frac{1}{x}F_{-\lambda,1-a}\left(\frac{1}{x}\right).$$

If $\lambda = \mu + i\nu$, $u^\lambda$ has its principal value, and so

$$|e^{-u^\lambda}| = e^{-u^\mu \cos(\nu \log u)},$$

and it is easy to see that the integral is divergent.

(ii.) If $\lambda$ is *purely imaginary*, the integral is convergent if

$$0 < R(a) < 1,$$

and may be expanded in the form

$$(5) \qquad F_{i\nu,a}(x) = \sum_0^\infty \frac{(-)^n}{n!}\int_0^\infty \frac{u^{ni\nu+a-1}}{u+x}\,du = \pi x^{a-1}\sum_0^\infty \frac{x^{ni\nu}}{n!\sin(a+ni\nu)\pi},$$

where $x^{a-1}$ has its principal value. This series represents an integral function of $x^{i\nu}$.

(iii.) If $\lambda$ is real, we may suppose $\lambda \geqslant 0$ after what precedes. The case of $\lambda = 0$ is trivial, $F_{0,a}(x)$ reducing to a constant multiple of $x^{a-1}$.

(iv.) We need, therefore, only consider the case in which $\lambda$ is *real and positive*. In this case the integral is convergent, if $R(a) > 0$, for all values of $x$ save real and negative or zero values, and it represents a branch of a function of $x$ whose only singularities are 0 and $\infty$.[†]

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[*] Dirichlet (Arendt), p. 209. The transformation really dates from Cauchy's "Mémoire sur les Intégrales Définies" (*Œuvres*, t. I., p. 377).

[†] Although the integral ceases to converge when $x$ is real and negative, it is easy to see that no finite negative value of $x$ is really a singularity.

13. I shall now consider the integral

$$P\int_{-\infty i}^{\infty i}\Gamma(-t)e^{-at}\,dt, \tag{6}$$

the contour of integration being the imaginary axis in the plane of $t$, $t = \xi+i\eta$, and $P$ denoting the principal value.* Since

$$|\Gamma(-i\eta)| = e^{-\frac{1}{2}\pi|\eta|}\,|\eta|^{-\frac{1}{2}}\sqrt{2\pi}(1+\epsilon_\eta),$$

the integral is convergent, provided the imaginary part of $a$ is numerically less than $\frac{1}{2}\pi$.

By a simple application of Cauchy's theorem we obtain the formula

$$P\int_{-\infty i}^{\infty i}\Gamma(-t)\,e^{-at}\,dt = 2\pi i\,(e^{-e^{-a}}-\tfrac{1}{2}). \tag{7}$$

This may also be put in the forms

$$P\int_{-\infty}^{\infty}\Gamma(i\eta)\,e^{ai\eta}\,d\eta = 2\pi\,(e^{-e^{-a}}-\tfrac{1}{2}), \tag{8}$$

$$P\int_{-\infty}^{\infty}\Gamma(i\eta)\,u^{-\lambda i\eta}\,d\eta = 2\pi\,(e^{-u^{\lambda}}-\tfrac{1}{2}). \tag{9}$$

If $u = re^{i\theta}$, we must, in (9), have $|\lambda\theta| < \frac{1}{2}\pi$. The formulæ (7) and (9) are particular and slightly exceptional cases of the formulæ

$$\int_{-\infty i-\kappa}^{\infty i-\kappa}\Gamma(-t)\,e^{-at}\,dt\dagger = 2\pi i e^{-e^{-a}} \quad (\kappa > 0),$$

$$\int_{-\infty}^{\infty}\Gamma\left(i\eta+\frac{s-1}{\lambda}\right)u^{-\lambda i\eta}\,d\eta = 2\pi u^{s-1}e^{-u^{\lambda}},$$

where $\lambda$ and $s-1$ are real and positive.

14. Now multiply (9) by $\dfrac{u^{a-1}}{x+u}$, where $0 < R(a) < 1$, and integrate from $u = 0$ to $u = \infty$. Since

$$\int_0^{\infty}\frac{u^{a-1-\lambda i\eta}}{x+u}\,du = x^{a-1-\lambda i\eta}\,\frac{\pi}{\sin(a-\lambda i\eta)\,\pi},$$

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* $\Gamma(-t)$ becomes infinite for $t = 0$ like $-1/t$. I have considered the theory of "principal values" in great detail in three papers in these *Proceedings* (Vol. xxxiv., p. 16; Vol. xxxiv., p. 55; and Vol. xxxv., p. 81).

† This formula is due to Mellin, *Acta Soc. Fennicæ*, Vol. xxix., 4, p. 41.

where $x^{a-1-\lambda i\eta}$ has its principal value, we find*

$$2F_{\lambda,a}(x) = \frac{\pi x^{a-1}}{\sin a\pi} + x^{a-1} P\int_{-\infty}^{\infty} \frac{\Gamma(i\eta)\, x^{-\lambda i\eta}}{\sin(a-\lambda i\eta)\pi}\, d\eta. \tag{10}$$

This is our fundamental equation. It has been proved on the assumptions that $\lambda > 0$, $-\pi < am.x < \pi$, and $0 < R(a) < 1$.

15. Another easy application of Cauchy's theorem shows that

$$P\int_{-\infty}^{\infty} \frac{\Gamma(i\eta)\, x^{-\lambda i\eta}}{\sin(a-\lambda i\eta)\pi}\, d\eta = \frac{1}{i} P\int_{-\infty i}^{\infty i} \frac{\Gamma(-t)\, x^{\lambda t}}{\sin(a+\lambda t)\pi}\, dt \tag{11}$$

$$= 2\pi \left\{ \tfrac{1}{2} \operatorname{cosec} a\pi - \lim_{R=\infty} S_R \right\},$$

where $S_R$ denotes the sum of the residues of the subject of integration for which the real part of $t$ is positive and $|t|$ less than $R$, and $R$ tends to infinity in such a way that the circle $|t| = R$ never passes at less than a certain fixed distance from any pole of the subject of integration. Hence

$$F_{\lambda,a}(x) = \pi x^{a-1} (\operatorname{cosec} a\pi - \lim_{R=\infty} S_R). \tag{12}$$

15. Let us suppose first that $\lambda$ is *rational*; and we may without loss of generality suppose it *integral*. For, if $\lambda = \alpha/\beta$, $\alpha$ and $\beta$ being integral, we find on transforming the integral which expresses $F_{\lambda,a}(x)$ by the substitution $u = w^{\beta}$, and splitting up $1/(w^{\beta}+x)$ into partial fractions, that $F_{\alpha/\beta,a}(x)$ may be expressed as the sum of a finite number of functions of the type $A_s F_{\alpha,a\beta}(x_s)$.

We may remark further that the formulæ of § 14 were proved on the assumption $0 < R(a) < 1$. The case in which $R(a) = 1$ will be considered later. Those cases in which $R(a) > 1$ may be reduced to these two cases by means of the formula

$$F_{\lambda,a}(x) = \frac{1}{\lambda} \Gamma\left(\frac{a-1}{\lambda}\right) - xF_{\lambda,a-1}(x). \tag{13}$$

We shall therefore at present confine ourselves to the case in which $\lambda$ is an integer and $0 < R(a) < 1$.

16. There can be no *double* poles of the subject of integration in (11), for this would require $0 < a < 1$ and $p = a + \lambda q$ for integral values of $p$

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* The inversion of the order of integration is easily justified.

and $q$, which is manifestly impossible. Hence we find that

$$(14)\qquad S_R = \Sigma \frac{(-)^{m-1} x^{\lambda m}}{m!\,\sin(a+\lambda m)\pi} + \frac{1}{\lambda\pi}\Sigma(-)^n\Gamma\left(-\frac{n-a}{\lambda}\right)x^{n-a},$$

and the two series are clearly convergent separately when prolonged to infinity. Hence

$$(15)\quad F_{\lambda,a}(x) = \pi x^{a-1}\sum_{m=0}^{\infty}\frac{(-)^m x^{\lambda m}}{m!\,\sin(a+\lambda m)\pi} + \frac{1}{\lambda}\sum_0^\infty(-)^n\Gamma\left(-\frac{n+1-a}{\lambda}\right)x^n.$$

Here $-\pi < am.x < \pi$, and $x^{a-1}$ has its principal value. It can easily be verified that, if $\lambda = \frac{1}{2}$, $a = \frac{1}{2}$, this equation reduces to (8) of § 11.

17. It is easy to see that, if $a = 1-\epsilon+i\sigma$ ($\sigma \neq 0$), each of the two sides of (15) is continuous for $\epsilon = 0$. Hence (15) holds for $a = 1+i\sigma$.

If $\sigma = 0$, this is still true, but we can no longer simply substitute the limiting value of $a$ in the right-hand side. We find after a little reduction that

$$(16)\qquad F_{\lambda,1}(x) = -\log x \,.\, e^{(-)^{\lambda+1}x^\lambda} + \frac{1}{\lambda}\Sigma'(-)^n\Gamma\left(-\frac{n}{\lambda}\right)x^n$$
$$-\frac{1}{\lambda}\sum_0^\infty\frac{(-)^{(\lambda+1)p}x^{\lambda p}}{p!}\left(\gamma-1-\frac{1}{2}-\ldots-\frac{1}{p}\right),$$

the dash over the sign of summation denoting that it extends to all values of $n$ except multiples of $\lambda$.

If, *e.g.*, $\lambda = 1$,

$$(17)\qquad F_{1,1}(x) = -e^x\log x - \sum_0^\infty\frac{x^p}{p!}\left(\gamma-1-\frac{1}{2}-\ldots-\frac{1}{p}\right).$$

Comparing this with (17), we are led to the conclusion that

$$(18)\qquad \sum_0^\infty\left(1+\frac{1}{2}+\ldots+\frac{1}{p}\right)\frac{x^p}{p!} = e^x\sum_1^\infty\frac{(-)^{n-1}x^n}{n.n!},$$

which is easily verified with the help of the well-known identity*

$${}^pC_1-\tfrac{1}{2}\,{}^pC_2+\tfrac{1}{3}\,{}^pC_3-\ldots = 1+\frac{1}{2}+\ldots+\frac{1}{p}.$$

18. We may now proceed to consider the much more interesting case in which $\lambda$ is *irrational*. Before doing so we may summarize our conclusions as regards the rational case as follows:—

(i.) If $\lambda$ is positive and rational, and the real part of $a$ positive, the function $F_{\lambda,a}(x)$, which is partially represented by the integral (1), has two singular points, viz., 0 and $\infty$, and no others.

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* Chrystal's *Algebra*, Vol. II., p. 19, Ex. 18.

(ii.) If the $x$-plane is cut along the negative real axis, then in the rest of the plane

$$F_{\lambda,a}(x) = x^{a-1}G_1(x^\lambda)+G(x),$$

where $G$ and $G_1$ are integral functions, and $x^{a-1}$ and $x^\lambda$ have their principal values, except in the special case in which $\lambda = \alpha/\beta$ and $a\beta$ is a positive integer, when the expansion contains logarithmic terms. In this equation $F_{\lambda,a}(x)$ is of course the principal branch of the function partially represented by the integral (1).

We may observe further that, if $\xi$ is positive and $\eta$ small, the values of $F_{\lambda,a}(-\xi+i\eta)$ and $F_{\lambda,a}(-\xi-i\eta)$ differ by a quantity whose limit for $\eta = 0$ is

$$2\pi i e^{-\xi^\lambda}\xi^{a-1},$$

and the value of

$$P\int_0^\infty \frac{e^{-u^\lambda}u^{a-1}}{u-\xi}\,du$$

is the arithmetic mean of these two values; and, finally, that every branch of the function tends to zero when $x$ approaches infinity by any path which does not wind an infinite number of times round the origin.

19. When $\lambda$ is irrational the condition for double poles is as before $p = a+\lambda q$, and this is in general impossible: it is always impossible, for example, if $a$ is complex or real and rational, and in any case cannot occur for more than one pair of values of $p$ and $q$.[*] We shall therefore exclude this possibility; the modifications necessary if it should occur present no difficulty.

Using Cauchy's theorem precisely as before, we find

$$(19)\quad F_{\lambda,a}(x) = \lim_{R=\infty}\left\{\pi x^{a-1}\sum_0^{[R]}\frac{(-)^m x^{\lambda m}}{m!\sin(a+\lambda m)\pi} + \frac{1}{\lambda}\sum_0^{[\lambda R+R(a)-1]}(-)^n\Gamma\left(-\frac{n+1-a}{\lambda}\right)x^n\right\},$$

where $[k]$ denotes as usual the greatest integer contained in $k$. The limit on the right-hand side is certainly determinate, but it will be clear from the first part of this paper that the series

$$(20)\qquad \sum^\infty\frac{(-)^m x^{\lambda m}}{m!\sin(a+\lambda m)\pi},\qquad \sum^\infty(-)^n\Gamma\left(-\frac{n+1-a}{\lambda}\right)x^n$$

are not necessarily convergent.

---

[*] If, *e.g.*, $\lambda = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$, $a = \frac{1}{\sqrt{2}}$, it occurs for $p = 1$, $q = 2$, and no other pair of values.

20. If one series converges, so does the other. We can prove, as i[illegible] the first part, that—

(i.) each series is convergent if $a$ is complex, or if $a$ is real an[illegible] rational and $\lambda$ *algebraic ;*

(ii.) that for suitably chosen values of $\lambda$ the series may have an[illegible] radius of convergence, and represent functions for which the circle o[illegible] convergence is a critical line ;

(iii.) that, if the series represent integral functions of $x^\lambda$ and [illegible] respectively, the increase of these functions may be regular or irregular.

21. If $a = 1$, the form of the equation (19) must be modified much as in § 17. We find

$$(21)\quad F_{\lambda,1}(x) = -\left(\log x + \frac{\gamma}{\lambda}\right)$$
$$+\lim_{R=\infty}\left\{\pi\sum_{1}^{[R]}\frac{(-)^m x^{\lambda m}}{m!\sin\lambda m\pi} + \frac{1}{\lambda}\sum_{1}^{[\lambda R]}(-)^n\Gamma\left(-\frac{n}{\lambda}\right)x^n\right\}.$$

The series
$$\Sigma\frac{(-)^m x^{\lambda m}}{m!\sin\lambda m\pi}$$

is one of those to which the first part of this paper was devoted. The case in which its radius of convergence is finite is particularly interesting, it being not a little curious to find that the sum of two functions of $x$ and $x^\lambda$ respectively, whose region of existence is bounded by the circle $|x| = R$, should be a function of $x$, which exists all over the plane.

Many other pairs of series can be constructed possessing similar properties. For example,
$$\Sigma\frac{(-)^m x^{m\lambda}}{\sin m\lambda\pi} + \frac{1}{\lambda}\Sigma\frac{(-)^n x^n}{\sin\dfrac{m\pi}{\lambda}}$$

is such a pair. In this case the radius of convergence of either series cannot possibly be greater than 1, and is equal to 1 if $\lambda$ is algebraic ; and the circle of convergence (whatever its radius) is a *coupure* for the function represented by either series. But the two series taken together in a manner analogous to that indicated by (21) in any case converge throughout the interior of the circle $r = 1$ (supposed cut along the negative real axis).

22. When the series in (21)—or in (19)—are not separately convergent, we have an interesting example of the phenomenon of a "pair

of series" which only converge when taken together in the manner specified in those equations. Such "pairs of series" have been considered before by Prof. Lerch and myself,[*] but, so far as I know, not in connection with the question of the representation of an analytic function in the neighbourhood of a singular point.

When the series diverge separately for *all* values of $x$ the equation (21) serves none the less to determine the behaviour of $F_{\lambda,1}(x)$ near the origin.[†] Thus $F_{\lambda,1}(x)$ behaves like

$$-\left(\log x+\frac{\gamma}{\lambda}\right), \tag{22}$$

and we can determine the way in which any given differential coefficient for $F_{\lambda,1}(x)$ behaves near $x=0$. For instance, $F_{\lambda,1}^{(n)}(x)$ behaves like

$$\frac{(-)^n n!}{\lambda}\Gamma\left(-\frac{n}{\lambda}\right)+\frac{(-)^n(n-1)!}{x^n} \tag{23}$$
$$+\pi\sum_1^p\frac{(-)^m}{m!}\,\frac{\lambda m(\lambda m-1)\dots(\lambda m-n+1)}{\sin m\lambda\pi}\,x^{\lambda m-n},$$

where $p$ is the greatest integer such that $\lambda p-n<0$. In fact the information furnished in this case is really not less complete than in the case in which the series are separately convergent.

22. We can write (21) in the form

$$F_{\lambda,1}(x)=-\left(\log x+\frac{\gamma}{\lambda}\right)$$
$$+\lim\left\{\pi\sum_1\frac{(-)^m x^{\lambda m}}{m!\sin\lambda m\pi}-\frac{\pi}{\lambda}\sum_1\frac{(-)^n x^n}{\Gamma\left(1+\frac{n}{\lambda}\right)\sin\frac{n\pi}{\lambda}}\right\}$$
$$(-\pi\leqslant am.\,x\leqslant\pi).$$

The function $F_{\lambda,1}(x)$ therefore satisfies the two difference equations

$$\left\{\begin{aligned}F(\xi e^{\pi i})-F(\xi e^{-\pi i})&=-2\pi i\,e^{-\xi^\lambda},\\ F(\xi e^{\pi i/\lambda})-F(\xi e^{-\pi i/\lambda})&=-\frac{2\pi i}{\lambda}E_{1/\lambda}(-\xi),\end{aligned}\right. \tag{24}$$

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[*] See a paper "On certain Series of Discontinuous Functions connected with the Modular Functions" (*Quart. Jour. of Math.*, Vol. XXXVI., p. 93), where references are given.

[†] Our previous conclusions as to its behaviour elsewhere are still valid.

where $E_a(\xi)$ is Mittag-Leffler's function[*]

$$E_a(\xi) = \sum_0^\infty \frac{\xi^n}{\Gamma(an+1)}.$$

The latter equation may be verified by means of the expression of $E_a(\xi)$ as a contour-integral given by Mittag-Leffler.[†]

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[*] *Acta Math.*, t. xxix., p. 101.

[†] A full investigation of the properties of this most interesting function will be found in Mittag-Leffler's memoir quoted above and in two memoirs by A. Wiman in the same volume of the *Acta Mathematica*. I may mention incidentally (though it has no connection with the subject of this paper) that the function $E_a(x)$ gives (for suitable values of $a$ and $s$) an interesting generalisation of Heine's contour integral for the gamma function, viz.,

$$\frac{i}{2\pi}\int E_a(-u)(-u)^{s-1}\,du = \frac{1}{\Gamma(1-as)},$$

the contour being the same as in Heine's formula, to which the above equation reduces for $a = 1$.

# LINEAR CONTENT OF A PLANE SET OF POINTS

*By* W. H. YOUNG.

[Received June 15th, 1905.—Read November 9th, 1905.]

IF $G$ be a measurable set of points in a plane, in particular if $G$ be a closed set, it has a definite *plane content*, which may be found by enclosing each point of $G$ in a small region of span* less than some assigned norm $d$, calculating the content $F(d)$ of this set of regions,† and proceeding to the limit when $d$ is indefinitely decreased.

If the set $G$ lies on a straight line, this content is always zero: the content, however, calculated by the rule for the straight line, and which may be called the *ordinary or linear content of the set with respect to the straight line*, may be quite other than zero. In the same way, if $G$ lies on a circle, or on some other curve whose arcs can be measured, and which is of zero plane content, the plane content of $G$ will be zero, but, enclosing each point of $G$ in a small arc of length less than some assigned norm $d$, the content $A(d)$ of this set of arcs may have a limit quite other than zero when $d$ is indefinitely decreased, and it is clear that this limit will play an important part in the theory of such sets, a part exactly corresponding to the ordinary or linear content of a set of points on a straight line. We naturally call this *the linear content of the set with respect to the curve*.

Given a plane set $G$, however, it is not always easy to decide whether or no it lies on a Jordan curve, or indeed whether it lies on any curve at all, still less whether the arcs of such a curve can be measured, and how. If it lies on a curve, it may again lie on various curves, and the doubt arises whether the linear content with respect to all the curves is the same, and, if not, to which curve we must give the preference. We want a definition of linear content from the point of view of the plane itself, and to develop the theory of linear content in the plane independently.

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* The span of a region is the upper limit of the distance between two points internal to the region.

† Any region without its rim has a definite content, and any set of regions without rims has a definite content, viz., the sum of the contents of the equivalent set of non-overlapping regions without rims.

The rim of the region is the locus of those points in the neighbourhood of which there are always both ordinary internal and ordinary external points of the region. ("Regions and Sets of Regions," *Quarterly Journal of Math.*, No. 145, 1905.)

There is another way in which we may approach the matter.

It is clear that the mere fact that the plane content of a certain closed set $G$ is zero is not of itself sufficient to justify our putting all such sets in one class without further sub-divisions. In fact, denoting by $F(d)$ the content of the small regions corresponding to the norm $d$, to say that the plane content of $G$ is zero is merely to assert that $F(d)$ has the limit zero when $d$ is indefinitely decreased. The question arises: What is the order of smallness of $F(d)$? In actual calculations this might be of importance, and in theory it cannot be overlooked. It may be that $F(d)$ is of the same order as $d$, or of some higher or lower order. Examples of this are familiar on the straight line, where a set of zero content (linear content) is of order $d^2$ in general, or at least of the order $d^{1+x}$, where $x$ is positive, while a set of positive content (linear content) is of order $d$.

We want a classification of sets in the plane whose content (plane content) is zero, and it is clear that this theory must be based on the mode in which $F(d)$ vanishes. To attempt in detail the discussion of all possible modes in which $F(d)$ can vanish is by no means desirable as a basis of classification; but the analogy with ordinary geometry will lead us, in the first instance at any rate, to confine ourselves in the straight line to one kind of content, in the plane to two kinds of content, in space to three kinds of content, and so on, just as in ordinary geometry of three dimensions we have length, area, and volume.

So far as I know, however, no more has been attempted in this department than the formulation of definitions which would occur to the merest tyro. The theory of the length of curves, which is closely connected with that of linear content, has indeed been the subject of investigation by more or less varied methods from old times; but even this theory has not been attacked from a broad standpoint, and few general theorems in it have been discovered.

The theory of curves, however, is not a mere part of the theory of sets of zero content. One of the most important contributions to this theory with which I am acquainted as having been made of late years is that of Prof. Osgood in what may be called the reverse direction to that in which I am at present proceeding. It relates to curves of infinite length. Osgood showed that the points of a Jordan curve without double points, although nowhere dense, may, so to speak, have an area, or, more technically, may have a positive plane content. Every arc of a curve which is not of infinite length has necessarily zero plane content; it is not difficult, however, to construct curves of infinite length, or which, in the ordinary sense of the word, cannot be said to have a length at all, and yet of zero plane content.*

We want in the first instance a definition of linear content. One such definition has already been referred to.† It presents itself naturally, and we may assert with confidence that it must have occurred independently to all workers in the subject; it is given explicitly in § 2. In the following articles I trace out some of the consequences of this definition, and show that it leads to surprising and even paradoxical results:—

(1) Even the linear content of such a simple set as a countably infinite set of concentric circular arcs, with no other limiting point except the centre, is not necessarily the sum of the lengths of the arcs, but may be greater than this.

(2) The linear content of a closed countable set of points in the plane is not necessarily zero. It may even be infinite.

The question naturally arises: Is it possible to give a different definition of linear content, one, in fact, more consistent with what our knowledge of sets on the straight line would lead us to expect? In the concluding articles in this paper I return to this point. I have not been able to frame such a definition, and careful consideration suggests that such a definition is impossible. Content, whatever its dimensions, is a property of a set with respect to the fundamental region $R$. If the set lies in a fundamental region $R'$ of lower dimensions $m$, there is every reason to suppose that the $m$-dimensional content of the set with respect to $R$ will depend not only on the $m$-dimensional content with respect to $R'$, but also on the lie of $R'$. In particular, the linear content of a plane closed set of points lying on a curve whose arcs can be measured may or may not be the same as the linear content of the set with respect to the curve, according to the form of the curve.

I may add, finally, that the particular example I have constructed of a closed countable set with a *finite* linear content has a simple closed expression for its content, viz., $2\pi \log(\sqrt{2}+1)$. This, I think, is worth noting in view of the fact that, even for sets on the straight line, closed expressions for the content are not easily found, and have, as far as I know, never been given.

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* Osgood, *Trans. of the Amer. Math. Soc.*, Vol. IV., p. 1, 1903; Grace Chisholm Young, *Quarterly Journal of Math.*, No. 145, 1905.

† Cp. Lebesgue, *Leçons sur l'Intégration*, 1904, p. 37, footnote. In this note there is a hint of the possibility of such a definition, and a reference to Minkowski's use of a similar idea in space of three dimensions to define the length of a curve or the area of a surface.

2. The definition of the linear content of a plane set of points is as follows :—

*Let small regions be described round the points of a closed set $G$, as in the process of finding the plane content, and let them be chosen to be circles of diameter $d$ with the points of the set as centres, and let $F(d)$ be their content. Let $f(d) = F(d)/d$; then, if $f(d)$ has a definite limit $I$ when $d$ is indefinitely decreased, $I$ is called the linear content of $G$.*

It is easy to prove that in the case when $G$ is a closed set of points lying on a straight line the linear content so defined is the same as the ordinary or linear content of the set with respect to the straight line. It is easy to show also that when $G$ is a closed portion of a polygon, or a closed arc of a circle, ellipse, or other ordinary curve, the linear content defined as above is the length of the arc.

3. The linear content of a set of intervals on a straight line is the sum of their contents (lengths), and the same is true of circular arcs on a single circle or on a finite number of circles. It might be supposed that this property would hold for any countable set of measurable arcs, in particular for a countable set of circular arcs.

The following example, simple as it is in geometrical character, shows that this is not the case, and that even circular arcs may be so arranged in the plane that their linear content differs from the sum of their lengths (that is, the sum of their linear contents).

It is clear that, for this to be the case, the number of arcs must be infinite. This necessitates the presence of at least one limiting point. A simpler case of an infinite set of arcs is inconceivable than such a set as is given below in which the arcs are countable and have only one limiting point. Yet even in this simple case it is shown that the linear content of the arcs is more than the sum of their lengths, and depends therefore not only on this latter sum, but also on the lie of the arcs.

4. Ex. 1.—Take a set of concentric circles of radii

$$1,\ \tfrac{1}{2},\ \ldots,\ 1/n,\ \ldots,$$

having $O$ as centre. On the circle of radius $1/k$ place $4(k+1)$ points at equal distances all round the circle (Fig. 1).

The arc between two consecutive points of the set on the $k$-th circle is of length

$$\frac{1}{4(k+1)}\,\frac{2\pi}{k}.$$

We can therefore make each point the middle point of an arc of length $x_k$, where

$$4(k+1)\,x_k = \frac{1}{2^{k+1}}, \tag{1}$$

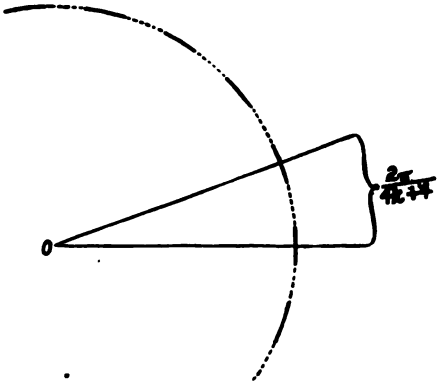

FIG. 1.

without these arcs abutting or overlapping at all, since

$$\frac{2\pi}{k} > \frac{1}{2^{k+1}}.$$

In this way we get for all positive integral values of $k$ a countable set of circular arcs, the sum of whose lengths is

$$\sum_{k=1}^{k=\infty} 4(k+1)\,x_k = \sum_{k=1}^{k=\infty} \frac{1}{2^{k+1}} = \tfrac{1}{2}.$$

These arcs therefore may be said to have length $\frac{1}{2}$. The only limiting point is the centre $O$; so that the arcs together with $O$ form a closed set of points in the plane. I shall now show that the linear content of this set is, owing to the lie of the arcs, certainly greater than the sum of the lengths of the arcs.

Let $y_k$ denote the length of one of the free arcs on the circle of radius $k$. Then

$$y_k = \frac{1}{4(k+1)}\left\{\frac{2\pi}{k} - \frac{1}{2^{k+1}}\right\} = \frac{\pi}{2}\,\frac{1}{k(k+1)} - \frac{1}{2^{k+3}(k+1)}.$$

Therefore, when $k$ is sufficiently large, $y_k$ is greater than $\frac{1}{k(k+1)}$, which is the distance between the $k$-th and the $(k+1)$-th circles. It follows that, if we describe small circles of diameter $d$ with the points of the chosen arcs $x_k$ as centres, these will form bean-shaped regions with semicircular ends, containing each one of the arcs $x_k$, for the smaller values of $k$ (Fig. 2). When, however, $k$ reaches the value $l$, defined as a positive integer by the following inequality—

$$\frac{1}{l(l+1)} < d \leqslant \frac{1}{l(l-1)}, \tag{2}$$

the bean-shaped regions will begin to overlap.

The area of such a bean-shaped region is

$$\tfrac{1}{2}kx_k\left\{\left(\frac{1}{k}+\frac{d}{2}\right)^2-\left(\frac{1}{k}-\frac{d}{2}\right)^2\right\}+\frac{\pi d^2}{4}$$

$$= dx_k+\tfrac{1}{4}\pi d^2.$$

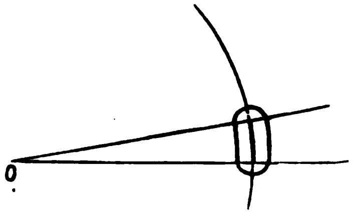

FIG. 2.

Thus, by (1), the sum of the areas of these bean-shaped regions on the $k$-th circle $(0 < k < l)$ is

$$4(k+1)\,dx_k+(k+1)\,\pi d^2 = \frac{d}{2^{k+1}}+(k+1)\,\pi d^2.$$

The sum of the areas of all these bean-shaped regions outside the $l$-th circle is therefore

$$d\left\{\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^l}\right\}+\pi d^2(2+3+\ldots+l)$$

$$= d\left\{\frac{1}{2}-\frac{1}{2^l}\right\}+\frac{\pi}{2}d^2(l-1)(l+2).$$

This last quantity is therefore less than the content $F(d)$ of all the small circles of diameter $d$.

Dividing by $d$, we get therefore

$$\frac{F(d)}{d} = f(d) > \frac{1}{2}-\frac{1}{2^l}+\frac{\pi}{2}d(l-1)(l+2).$$

Now, by (1), $$\operatorname*{Lt}_{d=0} l^2 d = 1.$$

Therefore $$\operatorname*{Lt}_{d=0} f(d) \geqslant \frac{1}{2}+\frac{\pi}{2}.$$

That is, the linear content of the set of arcs, or of the closed set consisting of the arcs with their single limiting point, is greater than the sum of the lengths (linear contents) of the arcs, the linear content being not less than $\tfrac{1}{2}+\tfrac{1}{2}\pi$, while the sum of the lengths is only $\tfrac{1}{2}$.

5. One of the most characteristic properties of the content on a straight line, or of the plane content in the plane, or the $n$-dimensional content in space of $n$ dimensions, is that the content of a countable closed set is zero. On this theorem are based some of the most important

theorems about the content, *e.g.*, the theorem that the content of a closed set is the same as that of any one of its derived and deduced sets. I am about to show that this property of the content is not true of the linear content in space of dimensions higher than one, although the linear content of a plane countable closed set may in special cases be zero. The following are, in fact, examples of such sets having positive linear content.

6. Ex. 2.—*A Plane Countable Closed Set of Positive Linear Content.*—Describe round the origin $O$ as centre a countably infinite series of concentric circles of radii 1, ½, ⅓, …, and place $k'$ points at equal distances

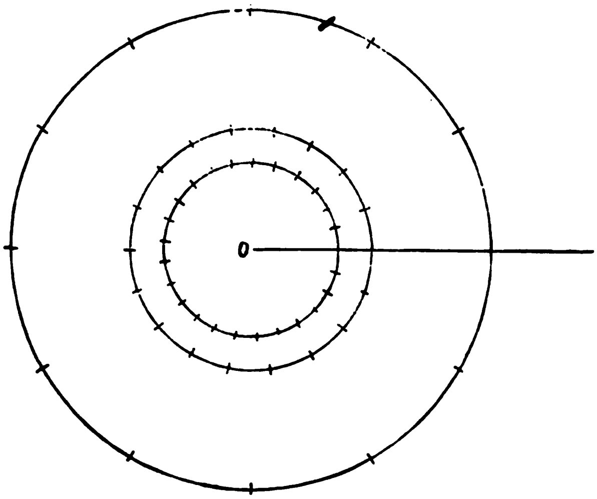

FIG. 3.

round the circumference of the $k$-th circle, for all positive integral values of $k$, where $k'$ is defined as an integer by the inequality

$$2\pi(k+1) < k' < 2\pi(k+1)+1, \tag{1}$$

and let the points be so placed that on a fixed straight line $OX$ through $O$ there is one point of the set on each of the circles (Fig. 3).

*These points together with the origin form a countable closed set whose linear content is about 5½.* Now, since (Fig. 4)

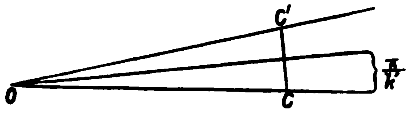

FIG. 4.

$$2\frac{1}{k}\sin\frac{\pi}{k'} = \left(\frac{k'}{\pi}\sin\frac{\pi}{k'}\right)\left(\frac{2\pi(k+1)}{k'}\right)\frac{1}{k(k+1)} < \frac{1}{k(k+1}, \tag{2}$$

the distance between consecutive points of the set on the $k$-th circle (represented by the first of these expressions) is less than the distance from the $k$-th to the $(k+1)$-th circle (represented by the last of these expressions).

If therefore we describe, round the points of the set as centres, circles of diameter $d$, these will, for the earlier values of $k$. be quite separate, but will begin to overlap, in such a way as to form rings covering over the circumferences of the successive circles, as soon as $k$ reaches the value $l$, defined by the following inequality:—

$$2\frac{1}{l}\sin\frac{\pi}{l'} < d \leqslant 2\frac{1}{k}\sin\frac{\pi}{k'} \quad (k < l). \tag{1}$$

Let $OX$ be the fixed radius, $C$ the point in which it meets the $k$-th circle $(k \geqslant l)$, $C'$ the point in which it meets the $(k+1)$-th circle, and $C_1$ the point on the $k$-th circle consecutive to $C$ when we move clockwise on

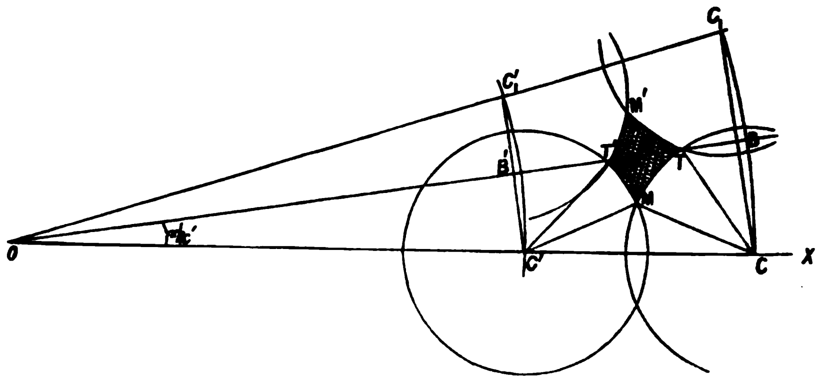

FIG. 5.

the circle. Fig. 5 shows the part of the plane between the $k$-th and $(k+1)$-th circles cut out by the radii $OX$ and $OC_1$.

The point in this part in which the circles of radius $d$ and centres $C$ and $C_1$ meet is $T$, and $OT$ meets the circle of centre $C'$ in $T'$, and the chord $CC_1$ and the parallel chord $C'C'_1$ in $B$ and $B'$ respectively.

The circles of radius $d$ and centres $C$ and $C'$ meet, if, and only if, $d$ is greater than $CC'$, that is, if

$$d > \frac{1}{k(k+1)}.$$

Also, since $CC'$ is the minimum distance between a point of the set on the $k$-th circle and one on the $(k+1)$-th circle, no one of the circles whose centres are on the $k$-th circle can overlap with one of the circles whose centres are on the $(k+1)$-th circle unless those with centres $C$ and $C'$ do so. Thus the rings corresponding to values of $k$ lying between $l$ and $(m-1)$, both inclusive, will not overlap, where the integer $m$ is defined by the following inequality:—

$$\frac{1}{m(m+1)} < d \leqslant \frac{1}{m(m-1)}. \tag{4}$$

The circles whose centres are on the $k$-th circle ($k \geqslant l$) entirely cover over a simple circular ring bounded by two circles of centre $O$ and passing through the points of intersection of consecutive small circles whose centres are on the $k$-th circle; one of these points is $T$ in Fig. 5. Denoting the length of the common chord of the small circles of centres $C$ and $C'$ in Fig. 5 by $t_k$,

$$BT = \tfrac{1}{2}t_k,$$

and $t_k$ is the width of the simple circular ring in question.

On the $(k+1)$-th circle there are at least $k'$ points of the set, and therefore the distance apart of consecutive points of the set on the $(k+1)$-th circle is less than it was on the $k$-th circle. Therefore

$$t_{k+1} > t_k.$$

It follows that the simple circular rings round the $k$-th and $(k+1)$-th circles will certainly overlap, provided $k$ is not less than $n$, where the integer $n$ is defined by the following inequality:—

$$\frac{1}{n(n+1)} < t_n \leqslant \frac{1}{n(n-1)}. \tag{5}$$

Thus the whole area of the $n$-th circle (at least) is tiled over by the small circles.

Now $$BT^2 + BC^2 = CT^2,$$

that is, $$\tfrac{1}{4}t^2 + \left(\frac{1}{k}\sin\frac{\pi}{k'}\right)^2 = \tfrac{1}{4}d^2,$$

or $$\left(\frac{t_k}{d}\right)^2 + \left(\frac{2}{kd}\sin\frac{\pi}{k'}\right)^2 = 1. \tag{6}$$

Now, as $d$ approaches the value zero, $l, m, n, l', m', n'$ all approach infinity. Moreover, since, by (1),

$$\frac{2\pi(k+1)}{k'} < 1 \leqslant \frac{2\pi(k+1)}{k'} + \frac{1}{k'},$$

for all values of $k$, we have, for $l$ and $l'$,

$$\operatorname*{Lt}_{d=0} \frac{2\pi(l+1)}{l'} = 1,$$

the same equation holding for $m$ and $m'$, and for $n$ and $n'$.

Again, since, by (3), $\frac{2}{kd} \sin \frac{\pi}{k'}$ is greater than 1 when $k$ is $l$, and less than 1 when $k$ is $(l-1)$, we have

$$\operatorname*{Lt}_{d=0} \frac{2}{ld} \sin \frac{\pi}{l'} = 1,$$

that is, $$\operatorname*{Lt}_{d=0} \left(\frac{l'}{\pi} \sin \frac{\pi}{l'}\right)\left(\frac{2\pi(l+1)}{l'}\right)\left(\frac{l}{l+1}\right)\frac{1}{l^2d} = 1,$$

whence, using (4), $$\operatorname*{Lt}_{d=0} l^2d = 1 = \operatorname*{Lt}_{d=0} m^2d. \tag{7}$$

Similarly, using (5),

$$\operatorname*{Lt}_{d=0} \frac{2}{nd} \sin \frac{\pi}{n'} = \operatorname{Lt.} \left(\frac{n'}{\pi} \sin \frac{\pi}{n'}\right)\left(\frac{2\pi(n+1)}{n'}\right)\left(\frac{n}{n+1}\right)\frac{1}{n^2d} = \operatorname{Lt} \frac{1}{n^2d} = \operatorname{Lt} \frac{t_n}{d} -$$

hence, by (6), each of these limits has the value $1/\sqrt{2}$; so that

$$\operatorname*{Lt}_{d=0} n^2d = \sqrt{2}. \tag{8}$$

The content of the small circles is best calculated in three parts:

$$F(d) = L+M+N, \tag{9}$$

where $L$ is the content of all those small circles whose centres lie outside the $l$-th circle, $M$ of those of the remaining circles whose centres lie outside the $m$-th circle, and $N$ is the content of the rest of the circles. Thus

$$L = \sum_{k=1}^{k=l-1} k' \frac{\pi d^2}{4};$$

therefore, by (1),

$$\frac{\pi d^2}{4} 2\pi \sum_{k=1}^{k=l-1} (k+1) < L < \frac{\pi d^2}{4} 2\pi \sum_{k=1}^{k=l-1} (k+2),$$

that is, $$\frac{\pi^2 d^2}{4}(l-1)(l+2) < L < \frac{\pi^2 d^2}{4}(l-1)(l+4),$$

whence, by (7), $$\operatorname*{Lt}_{d=0} \frac{L}{d} = \operatorname{Lt} \frac{\pi^2 l^2 d}{4} = \frac{\pi^2}{4}. \tag{10}$$

The ring formed by the small circles round the $k$-th circle, when $k > l$, is less in area than the ring between the two circles with centre $O$ touching the ring all round, that is, it is less than

$$\pi\left(\frac{1}{k}+\frac{d}{2}\right)^2+\pi\left(\frac{1}{k}-\frac{d}{2}\right)^2 < 2\pi d\frac{1}{k}.$$

Thus,* using (7), $$\underset{d=0}{\mathrm{Lt}}\frac{M}{d} < \underset{d=0}{\mathrm{Lt}}\,\pi\sum_{k=l}^{k=m-1}\frac{1}{k}=0. \tag{11}$$

To calculate $N$, we will first calculate the content of the parts of the plane which are covered by the small circles between the circumferences of the $k$-th and $(k+1)$-th circles, where $m \leqslant k < n$.

By (1) the number of points of the set on the $(k+1)$-th circle is less than $k'+8$. Thus, if we replace the points of the set on the $(k'+1)$-th circle by $k'$ points at equal distances, beginning on the fixed radius $OX$, we shall alter the area covered between the $k$-th and the $(k+1)$-th circles by at most $8(\frac{1}{4}\pi d^2)$ or $2d^2\pi$.†

Thus, if we calculate the content between the $k$-th and $(k+1)$-th circles on the supposition that there are only $k'$ points of the set on the $(k+1)$-th circle, and then sum from $k=m$ to $k=n-1$, we shall introduce an error in $N$ of at most $2\pi(n-m)d^2$, and therefore an error in the linear content of at most $\underset{d=0}{\mathrm{Lt}}\,2\pi(n-m)d=0$. Thus it is allowable to calculate $N$ in this manner. We have then $k'$ small uncovered areas between the $k$-th and $(k+1)$-th circles, all equal, and each bounded by four circular arcs, as in Fig. 5, where one of these areas, $TMT'M'$,

---

* Here we use a particular case of the general theorem

$$\underset{l=\infty}{\mathrm{Lt}}\sum_{k=l}^{k=m-1}\frac{1}{k}=\underset{l=\infty}{\mathrm{Lt}}\log\frac{m-1}{l}.$$

To prove this we have $$\sum_{k=}^{k=m-1}\frac{1}{k}=\Sigma\frac{\Delta x}{1+x},$$

where $\Delta=\frac{1}{l}$, and $x$ has the values $0, \frac{1}{l}, \frac{2}{l}, \ldots, \frac{m-l-1}{l}$; thus, if $l$, $m$ are made infinite in the ratio $1:t$, we get as the limit of the summation

$$\int_0^{t-1}\frac{dx}{1+x}=\log t.$$

† Let $R_k$ denote the content of the circles whose centres lie on the $k$-th circle, and $K'_k$ the content of $(k'-q)$ circles of diameter $d$ placed evenly round the circumference of the $k$-th circle. Then

$$R_k = k'\frac{\pi d^2}{4}-2k'\frac{d^2}{4}(\theta-\sin\theta\cos\theta), \qquad \cos\theta=\frac{2}{kd}\sin\frac{\pi}{k'},$$

$$K'_k=(k'-q)\frac{\pi d^2}{4}-2(k'-q)\frac{d^2}{4}(\theta'-\sin\theta'\cos\theta'), \qquad \cos\theta'=\frac{2}{kd}\sin\frac{\pi}{k'-q}>\cos\theta.$$

Therefore $$\theta'<\theta$$

and $$K'_k>(k'-q)\frac{\pi d^2}{4}-2k'\frac{d^2}{4}(\theta-\sin\theta\cos\theta),$$

whence $$R_k-K'_k<q\frac{\pi d^2}{4}.$$

is shaded. Now

$\frac{1}{2}$ area $TMT'M'$

$= \text{quadrilateral} BCC'B' - \text{triangles} BCT, B'C'T', CC'M - \text{sectors} TCM, T'C'$[illegible]

$$= \tfrac{1}{2}\left\{\frac{1}{k^2} - \frac{1}{(k+1)^2}\right\} \sin\frac{\pi}{k'} \cos\frac{\pi}{k'} - \tfrac{1}{8}d^2 \left\{\sin\theta_k \cos\theta_k + \sin\theta_{k+1}\cos\theta_{k+1}\right.$$

$$\left. + 2\sin\psi_k\cos\psi_k + \frac{\pi}{2} - \frac{\pi}{k'} - \theta_k - \psi_k + \frac{\pi}{2} + \frac{\pi}{k'} - \theta_{k+1} - \psi_k\right\}$$

where $\theta_k$ is $BCT$ and $\psi_k$ is $MCC'$; so that

$$\cos\theta_k = \frac{2\sin\dfrac{\pi}{k'}}{kd}, \qquad \cos\psi_k = \frac{1}{k(k+1)d}.$$

Thus, neglecting small quantities which vanish in the limit,

$$\frac{N}{d} = \frac{\pi}{m^2 d} - \sum_{k=m}^{k=n-1} \frac{2k'}{d}\left[\tfrac{1}{2}\left\{\frac{1}{k^2} - \frac{1}{(k+1)^2}\right\} \sin\frac{\pi}{k'} \cos\frac{\pi}{k'}\right.$$

$$\left. - \tfrac{1}{8}d^2\{\pi - \theta_k - \theta_{k+1} - 2\psi_k + \sin\theta_k\cos\theta_k + \sin\theta_{k+1}\cos\theta_{k+1} + 2\sin\psi_k\cos\psi_k\}\right]$$

Now, writing $\cos\phi = 1/k^2 d$, we see that $\theta_k$, $\psi_k$, and $\phi_k$ all have the same limit. Thus, writing $\pi$ for $k'\sin\pi/k'\cos\pi/k'$, $2\pi k$ for $k'$, and $\phi_k$ for $\theta_k$, $\theta_{k+1}$, and $\psi_k$, we introduce no error in the limit,

$$\underset{d=0}{\text{Lt}}\frac{N}{d} = \underset{d=0}{\text{Lt}}\left(\frac{\pi}{m^2 d} - \sum_{k=m}^{k=n-1}\left[\left\{\frac{1}{k^2} - \frac{1}{(k+1)^2}\right\}\frac{\pi}{d} - \frac{\pi^2}{2}dk\right.\right.$$

$$\left.\left. + 2\pi dk\,\{\phi_k - \sin\phi_k\cos\phi_k\}\right]\right)$$

$$= \underset{d=0}{\text{Lt}}\left(\frac{\pi}{m^2 d} - \left[\frac{\pi}{m^2 d} - \frac{\pi}{n^2 d} - \frac{\pi^2 d}{4}(n^2 - m^2)\right.\right.$$

$$\left.\left. + \pi\sum_{k=m}^{k=n-1} 2kd(\phi_k - \sin\phi_k\cos\phi_k)\right]\right)$$

$$= \underset{d=0}{\text{Lt}}\left(\frac{\pi}{n^2 d} + \frac{\pi^2}{4}(n^2 - m^2)d - \pi\sum_{k=m}^{k=n-1} 2kd(\phi_k - \sin\phi_k\cos\phi_k)\right).$$

Now $\qquad \sec\phi_{k+1} - \sec\phi_k = 2kd + d^2 = \Delta(\sec\phi_k);$

thus, neglecting under the sign of summation small quantities whose sum vanishes in the limit, we may write $\Delta(\sec\phi_k)$ for $2kd$, and sum with respect to $\phi$ from $+0$ to $\frac{1}{2}\pi - 0$: in the limit the summation becomes the integral.

Now

$$\int(\phi - \sin\phi\cos\phi)\,d\sec\phi = (\phi - \sin\phi\cos\phi)\sec\phi - \int\sec\phi\,(1 - \cos 2\phi)\,d\phi$$

$$= (\phi - \sin\phi\cos\phi)\sec\phi - 2\int(1 - \cos^2\phi)\sec\phi\,d\phi.$$

Thus $\int^{\frac{1}{4}\pi}(\phi-\sin\phi\cos\phi)\,d\sec\phi = \left(\frac{\pi}{4}-\frac{1}{2}\right)\sqrt{2}-2\log(\sqrt{2}+1)+2\frac{1}{\sqrt{2}}$

$$= \left(\frac{\pi}{4}+\frac{1}{2}\right)\sqrt{2}-2\log(\sqrt{2}+1),$$

whence

$$\operatorname*{Lt}_{d=0}\frac{N}{d} = \frac{\pi\sqrt{2}}{2}+\frac{\pi^2}{4}(\sqrt{2}-1)-\frac{\pi^2}{4}\sqrt{2}-\frac{\pi\sqrt{2}}{2}+2\pi\log(\sqrt{2}+1)$$

$$= -\frac{\pi^2}{4}+2\pi\log(\sqrt{2}+1).$$

Now, by (10), $$\operatorname*{Lt}_{d=0}\frac{L}{d} = \frac{\pi^2}{4}.$$

Thus $I = 2\pi\log(\sqrt{2}+1) = 5{\cdot}588$ approximately.

7. Ex. 3.—*A Plane Closed Countable Set of Infinite Linear Content.* —Take a set of concentric circles of radii 1, $\frac{1}{2}$, $\frac{1}{3}$, ..., $1/k$, .... On each circle, of radius say $1/k$, place $2^k$ points at equal distances, beginning on a fixed radius, the same for all values of $k$. *The points so constructed, together with the centre of the circles, form a closed countable set of infinite linear content.*

To prove this, consider that, since, for all values of $k$ greater than 5,

$$\frac{2}{k}\sin\frac{\pi}{2^k} < \frac{1}{k(k+1)},$$

the distance between consecutive points of the set on the $k$-th circle (represented by the former of these expressions) is less than the least distance between points of the set on the $k$-th and on the $(k+1)$-th circles (represented by the latter of the two expressions). Thus, if $d$ be sufficiently small, small circles of diameter $d$, with the points of the set as centres, will be distinct from one another for small values of $k$, and will first begin to overlap when $k = l$, where the integer $l$ is defined by the following inequality:—

$$\frac{1}{l}\sin\frac{\pi}{2^l} < \frac{d}{2} \leqslant \frac{1}{l-1}\sin\frac{\pi}{2^l}-1. \tag{1}$$

From the $l$-th to the $n$-th circle, where $n$ is defined by the inequality

$$\frac{1}{n(n+1)} < d \leqslant \frac{1}{n(n-1)}, \tag{2}$$

the small circles form rings round the circumference of each of the circles of radii $1/k$, but the rings are distinct from one another; as soon as we get to the $n$-th circle, however, the rings begin to overlap. It is only necessary to consider the content $N$ of these rings in order to prove the content to be infinite.

The area of the ring round the $k$-th circle is greater than that of the ring bounded by the two concentric circles through the points of intersection of the small circles; that is, it is greater than

$$\pi\left(\frac{1}{k}+\tfrac{1}{2}t_k\right)^2-\pi\left(\frac{1}{k}-\tfrac{1}{2}t_k\right)^2=\frac{2\pi}{k}t_k \tag{3}$$

where

$$\tfrac{1}{4}t_k^2=\tfrac{1}{4}d^2-\left(\frac{1}{k}\sin\frac{\pi}{2^k}\right)^2. \tag{4}$$

Let $m$ be the integer defined by the following inequality:—

$$\frac{1}{m}\sin\frac{\pi}{2^m}<\tfrac{1}{4}d\leqslant\frac{1}{m-1}\sin\frac{\pi}{2^{m-1}}; \tag{5}$$

Then, since

$$\frac{1}{m}\sin\frac{\pi}{2^m}<\tfrac{1}{4}d<\tfrac{1}{2}d,$$

$m$ is not less than $l$. Also, since

$$\frac{1}{m(m+1)}=\left\{\frac{4}{m-1}\,\frac{\pi}{2^{m-1}}\right\}\frac{2^{m-3}}{\pi}\,\frac{m-1}{m(m+1)},$$

which, if $m$ is sufficiently large, is greater than

$$\frac{4}{m-1}\,\frac{\pi}{2^{m-1}}>\frac{4}{m-1}\sin\frac{\pi}{2^{m-1}},$$

it follows that

$$\frac{1}{m(m+1)}>d;$$

so that $m$ is less than $n$.

By (4) and (5), for all values of $k$ greater than $m$, $t_k^2$ is greater than $\frac{3}{4}d^2$; and therefore $t_k$ is greater than $\frac{1}{2}d$. Thus, after $k=m-1$, the area of each of the rings is, by (3), greater than $\pi/kd$. Thus

$$\underset{d=0}{\mathrm{Lt}}\frac{N}{d}\geqslant\underset{d=0}{\mathrm{Lt}}\frac{1}{d}\sum_{l}^{n-1}2\pi t_k\frac{1}{k}\geqslant\underset{d=0}{\mathrm{Lt}}\,\pi\sum_{m}^{n-1}\frac{1}{k}\geqslant\underset{d=0}{\mathrm{Lt}}\,\pi\log\frac{n-1}{m}$$

$$\geqslant\tfrac{1}{2}\underset{d=0}{\mathrm{Lt}}\,\pi\log\frac{n(n+1)}{(m-1)^2}.$$

Now, by (2) and (5),

$$\frac{n(n+1)}{(m-1)^2}>\frac{1}{(m-1)^2d}>\frac{1}{(m-1)\,4\sin\frac{\pi}{2^{m-1}}}>\frac{1}{(m-1)\frac{\pi}{2^{m-3}}}>\frac{2^{m-3}}{\pi(m-1)},$$

which can be made as large as we please by sufficiently decreasing $d$. Thus

$$\underset{d=0}{\mathrm{Lt}}\frac{N}{d}=\infty;$$

*a fortiori* the linear content of the whole closed countable set of points is infinite.

8. It is not clear from the definition of linear content given in § 2 that the linear content in the case of a closed set always exists. In other words, it is conceivable that $f(d)$ may oscillate between certain positive limits of variation. In the case of the plane content in the plane or the linear content on the straight line this difficulty did not exist, since $F(d)$ is clearly a monotone function of $d$ which is continuous. In the examples given in §§ 6 and 7 the linear content was definite, but it cannot be assumed without proof that this is always the case. It is, however, important to devise a definition by which the content of a closed set is always definite, since this would enable us to extend the theory of linear content to open sets, precisely as was done in the case of the ordinary content. If it could be shown that the linear content as defined in § 2 is not always definite for a closed set, it might be convenient to take the lower and upper limits of variation of the limit as measures of the content.

There is another point which cannot be passed over. In the case of the ordinary content (whether linear content on the straight line, plane content in the plane, or $n$-dimensional content in space of $n$ dimensions) it can be shown that, if regions of span* less than $d$ be described in any manner so as to enclose all the points of the closed set, and the content of these regions be $\Phi(d)$, then $\Phi(d)$ approaches a definite limit as $d$ is indefinitely decreased, independent of the form of the regions. Thus the choice of circles with the points of the set as centres and of equal diameters $d$ is merely apparently arbitrary, and the ordinary content as originally defined by Cantor by means of circles (or spheres) can be found by the rule given at the commencement of this paper.

In the case of the linear content, however, it may be shown that, if $\Phi(d)/d$ have a definite limit as $d$ is indefinitely decreased, this limit may be quite different from the linear content as defined in § 2. It is easy to see that the limit may be zero, even when the linear content is positive (for instance, when the set lies on a straight line, we only have to take the span of the regions in the direction perpendicular to the straight line to decrease without limit compared to $d$), but the limit may also be greater than the linear content as defined in § 2.

If the set lie on a straight line, it is easy to show that the regions may be so constructed that, if $I$ be the linear content as defined in § 2, we get for $\Phi(d)$ the limit $2dI$, but they cannot be constructed so as to get a greater

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* See footnote, p. 461.

limit. Supposing for simplicity the set to be a closed segment $AB$, we only have to divide $AB$ into two sets $G_1$ and $G_2$, both dense everywhere in $AB$, and describe circles round the points of $G_1$ with their centres on one of the parallels to $AB$ at a distance $\frac{1}{2}d-e$, where $e$ is as small as we please compared to $d$, and circles round the points of $G_2$ with their centres on the other parallel. Such sets of circles have a content differing by as little as we please from $2dI$.

On the other hand, *whatever set G we have in the plane, of linear content I, the content of a set of small regions of span d enclosing the points of the set cannot exceed* $2d(I+d')$, *where* $d'$ *vanishes with* $d$. For each such region lies within a circle of *radius* $d$ with its centre at any point internal to it. Therefore the content of the small regions is not greater than that of a set of circles of diameter $2d$ with their centres at the points of the set. By the definition of the linear content, the content of these circles is $2d(I+d')$, where $d'$ vanishes with $d$; which proves the statement made above.

*It is not, however, always possible so to choose the regions as to reach near the limit* $2dI$. Ex. 2 is a case in point; whatever form of region we take, we cannot get a content much, if at all, greater than $dI$. To prove this, consider that the regions of span $d$, however constructed, cannot possibly cover over the whole circumference of the $l$-th circle, where $l$ has the same meaning as in § 6, so that

$$\operatorname*{Lt}_{d=0} l^2 d = 1.$$

Thus we get an upper limit for the content of the small regions if we suppose the whole area of a circle of radius $(1/l+d)$ to be tiled over, and the remaining small regions to be non-overlapping. Since the area of a region of span $d$ is at most $\frac{1}{4}\pi d^2$, we get as upper limit

$$\begin{aligned}\pi\left(\frac{1}{l}+d\right)^2+\sum_{k=1}^{k=l}\tfrac{1}{4}\pi d^2 k' &< \pi\left(\frac{1}{l}+d\right)^2+\tfrac{1}{4}\pi d^2\sum_{k=1}^{k=l}\{2\pi(k+1)+1\} \\ &< d\left\{\frac{\pi}{l^2 d}+\tfrac{1}{4}\pi^2 d l^2+\ d\right\} \\ &< d\{\pi+\tfrac{1}{4}\pi^2+d'\} \\ &< d\{3{\cdot}1416+2{\cdot}467418+d'\} \\ &< d\{5{\cdot}609013+d'\} \\ &< d\{I+0{\cdot}071013+d''\},\end{aligned}$$

$d'$ and $d''$ being small quantities which vanish with $d$.

9. These considerations suggest that the following definition of linear content might be used instead of that given in § 2.

*Let small regions of span* $d$ *be described in any manner round the points of the given closed set. Let their content be* $\Phi(d)$, *and let* $\Phi(d)/2d$ *be denoted by* $\phi(d)$. *As* $d$ *is indefinitely decreased, let* $J$ *be the upper limit of the variation of* $\mathop{\mathrm{Lt}}\limits_{d=0} \phi(d)$ ; $J$ *is called the linear content of* $G$.

It is clear, from what has been pointed out, that, if $G$ lie on a straight line, or circle, or ordinary algebraic curve, $J$ will coincide with $I$. Ex. 2 shows, however, that $J$ does not always coincide with $I$, since by the calculation of § 6 we can get the limit of $\Phi(d)/2d = \frac{1}{2}I$, the regions being chosen to be the circles of § 6, but we cannot get a limit greater than $\frac{1}{2}I + 0{\cdot}035507$, by the preceding article, so that $J$ cannot differ much, if at all, from $\frac{1}{2}I$.

By this definition *the linear content* $J$ *of a closed set is always definite, and the content of a set of regions of span* $d$ *round the points of the set is less than* $2d(J+d')$, *where* $d'$ *vanishes with* $d$, *but can, by proper choice of regions, be made as near as we please to* $2dJ$.

10. Neither of the definitions of linear content given in §§ 2 and 9 affects the anomalous results alluded to in the introduction. The investigations of §§ 6 and 7 show that, whether $I$ or $J$ be taken as the linear content, a countable closed plane set may have positive or even infinite content, and § 4 shows that $J$ for the set of arcs of Ex. 1 cannot be less than $\frac{1}{2}(\frac{1}{2}+\frac{1}{2}\pi)$, and is therefore greater than $\frac{1}{2}$, which is the sum of the lengths of the arcs.

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**Weddle Quartic Surface.** *See* Surfaces, Special Algebraic.

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Printed by C. F. Hodgson & Son, 2 Newton Street, High Holborn, W.C.

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